Analytic Functions and Complex Integration
UNIT-III

ANALYTIC FUNCTIONS

INTRODUCTION
In this Chapter we continue the study about the complex numbers. When x and y are real variables,z=x+iy is called a complex variable. Let u (x, y) and $v(x, y)$ be two function of the variables x and y. Then
$w = f(z) = u(x, y) + iv(x, y)$ is a function of the complex variable $z = x + iy$.

If w gives a unique value corresponding to a value of z then w is called a single valued function of z and if it gives more than one value corresponding to a given value of z then it is called a multiple valued function of z.

Definition
A single valued function $f(z)$ is said to have a limit w o as $z \rightarrow z_0$ if

(i) $f(z)$ is defined in a neighborhood of z_0

(ii) and if for every $\in > 0$, we can find a positive number δ such that $|f(z) - w_0| < \in \ \forall z$ given by 0 $< |z - z_0| < \delta.$

We Write $\lim\limits_{z \rightarrow z_0} f(z) = w_{0.}$

Analytic Function:

Definition
A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and $\lim\limits_{z \rightarrow z_0} f(z) = f(z_0).$

Definition
A function $f(z)$ is said to be differentiable at a fixed point z if $\lim\limits_{\Delta z \rightarrow z_0} \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$ exists.

This limit is called the derivative of $f(z)$ at the point z and it is denoted by $f(z)$.

Note
Here $z + \Delta z$ is a neighbourhood point of z with small distance Δz, can be chosen in any side of z. This is an important aspect in the above definition.

Definition
A function $f(z)$ is said to be analytic **at a point** $z = a$ in a region R if

(i) $f(z)$ is differentiable at $z = a$

(ii) $f(z)$ is differentiable at all points for some neighbourhood of $z = a$.

Definition
A function $f(z)$ is said to be **analytic in a region R** if $f(z)$ is analytic at all points in the region R.

Note
Instead of the term analytic in R, the terms **holomorphic** I R and **regular** in R are also used.

CAUCHY-RIEMANN EQUATIONS

We shall now obtain the necessary conditions for a complex function $f(z) = u(x, y) + iv(x, y)$ to be analytic.

Let $f(z) = u + iv$ be analytic in a region R in z plane.

$\Rightarrow f(z)$ is differentiable in R.

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \text{ exists in R.} \quad \ldots (1)$$

Let $z = x + iy$

$\therefore z = \Delta x + i\Delta y$

$\therefore z + \Delta z = (x + iy) + (\Delta x + i\Delta y)$

$= (x + \Delta x) + i(y + \Delta y)$

Now $f(z) = u(x, y) + iv(x, y)$

$$\therefore f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \quad \ldots. (2)$$

Case (i): Choose $z + \Delta z$ along the horizontal line through z. Then $\Delta y = 0$

$\therefore \Delta z = \Delta x$

$\therefore \Delta z \to 0 \Rightarrow \Delta x \to 0$

Now From (1) and (2),

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x + \Delta x, y) - u(x, y)] + i[v(x + \Delta x, y) - v(x, y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$f'(z) = \frac{du}{dx} + i\frac{dv}{dx} \qquad \ldots(3)\text{s}$$

Case (ii)

Choose $z + \Delta z$ along the vertical line through z. Then $\Delta x = 0$

$$\Rightarrow \Delta z = i\Delta y$$

$$\therefore \Delta z \to 0 \Rightarrow \Delta y \to 0$$

From (1) and (2),

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{\Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i}\frac{du}{dy} + i\frac{dv}{dy}$$

$$f'(z) = -i\frac{du}{dy} + i\frac{dv}{dy} \quad \ldots (4)$$

From (3) and (4), we get

$$\frac{du}{dx} + i\frac{dv}{dx} = -i\frac{du}{dy} + i\frac{dv}{dy}$$

Equating the real and imaginary parts, we get

$$\frac{du}{dx} = \frac{dv}{dy} = \frac{du}{dy} = -\frac{dv}{dx}$$

The above equations are known as Cauchy-Riamann equations (or) C-R equations.

SUFFICIENT CONDITION FOR A FUNCTION $f(z)$ TO BE ANALYTIC

Continuous single valued function $f(z) = u + iv$ is analytic in a region R if

(i) the four partial derivatives $\frac{du}{dx}, \frac{du}{dy}, \frac{dv}{dx}, \frac{dv}{dy}$ exists and all are continuous

(ii) $\frac{du}{dx}=\frac{dv}{dy},\frac{du}{dy}=-\frac{dv}{dx}$

Proof

By Taylor's series

$$f(x+h,y+k)=f(x,y)+\left(h\frac{d}{dx}+k\frac{d}{dy}\right)f(x,y)+\frac{1}{2!}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^2 f(x,y)+\ldots \quad \ldots (1)$$

Let $z=x+iy \Rightarrow \Delta z=\Delta x+i\Delta y$

$$z+\Delta z=u(x+iy)+(\Delta x+i\Delta y)=(x+\Delta x)+i(y+\Delta y)$$

$$f(z+\Delta z)=u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)$$

$$=u(x,y)+\left(\Delta x\frac{d}{dx}+\Delta y\frac{d}{dy}\right)u(x,y)+\frac{1}{2!}\left(\Delta x\frac{d}{dx}+\Delta y\frac{d}{dy}\right)^2 u(x,y)$$

$$+\ldots+i\left[v(x,y)+\left(\Delta x\frac{d}{dx}+\Delta y\frac{d}{dy}\right)v(x,y)+\cdots\right]\{by(1)\}$$

$$=u(x,y)+\Delta x\frac{du}{dx}+\Delta y\frac{du}{dy}+i\left(v(x,y)+\Delta x\frac{dv}{dx}+\Delta y\frac{dv}{dy}\right)+\cdots$$

$$=u(x,y)+iv(x,y)+\Delta x\left(\frac{du}{dx}+i\frac{dv}{dx}\right)+\Delta y\left(\frac{du}{dx}+i\frac{dv}{dy}\right)$$

[Omitting the higher powers of $\Delta x,\Delta y$]

$$=f(z)+\Delta x\left(\frac{du}{dx}+i\frac{dv}{dx}\right)+\Delta y\left(-\frac{dv}{dx}+i\frac{du}{dx}\right) \quad \text{[by CR equations]}$$

$$=f(z)+\Delta x\left(\frac{du}{dx}+i\frac{dv}{dx}\right)+\Delta y\left(i^2\frac{dv}{dx}+i\frac{du}{dx}\right) \quad \text{[Since } i^2=\text{-1]}$$

$$=f(z)+\Delta x\left(\frac{du}{dx}+i\frac{dv}{dx}\right)+i\Delta y\left(\frac{du}{dx}+i\frac{dv}{dx}\right)$$

$$=f(z)+\left(\frac{du}{dx}+i\frac{dv}{dx}\right)(\Delta x+i\Delta y)$$

$$f(z+\Delta z)=f(z)+\left(\frac{du}{dx}+i\frac{dv}{dx}\right)\Delta z$$

$$\Rightarrow f(z+\Delta z)-f(z)=\left(\frac{du}{dx}+i\frac{dv}{dx}\right)\Delta z$$

$$\Rightarrow \frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{du}{dx}+i\frac{dv}{dx}$$

$$\Rightarrow \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{du}{dx}+i\frac{dv}{dx}$$

$$\Rightarrow f'(z)=\frac{du}{dx}+i\frac{dv}{dx}=\frac{dv}{dy}-i\frac{du}{dy} \quad \text{(by C-R equations)}$$

$\Rightarrow f'(z)$ exists and $\frac{du}{dx}, \frac{du}{dy}, \frac{dv}{dx}, \frac{dv}{dy}$ are all continuous

***Example*: 1 Test whether the function $f(z) = z^3 + z$ is analytic or not.**

***Solution*:** Let $f(z) = z^3 + z$

$\Rightarrow u + iv = (x+iy)^3 + (x+iy)$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)3 + x + iy$

[since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

$= x^3 + i3x^2y - 3xy^2 - iy^3 + x + iy$

[Since i^2=-1, $i^3 = -i$]

$= (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y)$

Equating the real and imaginary parts, we get

$u = x^3 - 3xy^2 + x,$ $v = 3x^2y - y^3 + y$

$\Rightarrow \frac{du}{dx} = 3x^2 - 3y^2 + 1;$ $\frac{dv}{dx} = 6xy - 0 + 0 = 6xy$

$\frac{du}{dy} = 0 - 3x(2y) + 0;$ $\frac{dv}{dy} = 3x^2(1) - 3y^2 + 1$

$= -6xy;$ $= 3x^2 - 3y^2 + 1$

$\Rightarrow \frac{du}{dx} = \frac{dv}{dy};$ $\frac{du}{dy} = \frac{-dv}{dx}$

$\Rightarrow$ C-R equations are satisfied.

$\Rightarrow$ $f(z) = z^3 + z$ is analytic.

***Example: 2* *Define an analytic function. Determine whether the function* $w = 2xy + i(x^2 - y^2)$ *is analytic*]**

Solution: A function $f(z)$ is said to be an analytic function at $z = a$ in a region R if

(i) $f(z)$ is differentiable at $z = a$

(ii) $f(z)$ is differentiable at all points for some neighbourhood of $z = a$

A function $f(z)$ is analytic in a reigion R if $f(z)$ is analytic at all points in the region R.

Now $w = 2xy + i(x^2 - y^2)$

$\Rightarrow u + iv = 2xy + i(x^2 - y^2)$

Equating the real and imaginary parts, we get

$$u = 2xy; \qquad v = x^2 - y^2$$

$$\Rightarrow \frac{du}{dx} = 2y; \qquad \frac{dv}{dx} = 2x$$

$$\frac{du}{dy} = 2x; \qquad \frac{dv}{dx} = 2y$$

$$\Rightarrow \frac{du}{dx} \neq \frac{dv}{dy}; \qquad \frac{du}{dy} \neq \frac{-dv}{dx};$$

$\Rightarrow$ C-R equations are not satisfied.

$\Rightarrow w$ is not analytic.

<u>Example:3</u> Find the constants a,b and c if $f(z) = x + ay + i(bx + cy)$ ***is analytic***

Solution: Let $f(z) = x + ay + i(bx + cy)$

$$\Rightarrow u + iv = (x + ay) + i(bx + cy)$$

Equating the real and imaginary parts, we get

$$u = x + ay; \qquad v = bx + cy;$$

$$\Rightarrow \frac{du}{dx} = 1; \qquad \frac{dv}{dx} = b$$

$$\frac{du}{dy} = a; \qquad \frac{dv}{dy} = c$$

Since $f(z)$ is analytic, we have

$$\frac{du}{dx} = \frac{dv}{dy}; \quad \frac{du}{dy} = -\frac{dv}{dx}$$

$\Rightarrow c = 1$ and $a = -b$

$\Rightarrow c = 1$ and $a = -b, b$ may be any value.

<u>Example4</u>: Verify whether $f(z) = \sinh z$ ***is analytic using C-R equations.*** **[AU June 2001] [MUOCTOBER 2001]**

Solution: Let $f(z) = \sinh z$

$$\Rightarrow u + iv = \sinh(x + iy)$$

$$= \frac{1}{i}\sin i(x + iy) \qquad [\text{since } \sin i\theta = i\sinh\theta]$$

$$= \frac{1}{i}\sin\left(ix + i^2 y\right)$$

$$= \frac{1}{i}\sin\left(ix - y\right) \qquad \text{[Since } i^2 = -1\text{]}$$

$$= \frac{1}{i}\{\sin ix \cos y - \cos ix \sin y\}$$

$$[\text{since } \sin(a-b) = \sin a \cos b - \cos a \sin b]$$

$$= \frac{1}{i}\{i \sinh x \cos y - \cosh x \sin y\}$$

$$[\text{since } \sin i\theta = i \sinh \theta, \qquad \cos i\theta = i \cosh \theta]$$

$$= \sinh x \cos y - \frac{1}{i}\cosh x \sin y$$

$$= \sinh x \cos y + i \cosh x \sin y \qquad [\text{since } 1/i = -i]$$

Equating the real and imaginary parts, we get

$$u = \sinh x \cos y; \qquad v = \sin y \cos x$$

$$\frac{du}{dx} = \cosh x \cos y; \qquad \frac{dv}{dx} = \sin y \sinh x$$

$$\frac{du}{dy} = -\sin y . \sin x; \qquad \frac{dv}{dy} = \cos y \cosh x$$

$$\Rightarrow \frac{du}{dx} = \frac{dv}{dy}; \qquad \frac{du}{dy} = -\frac{dv}{dx}$$

$\Rightarrow$ C-R equations are satisfied.

$\Rightarrow f(z) = \sinh z$ is analytic.

Example:5 ***Find the analytic region of*** $f(z) = (x-y)^2 + 2i(x+y)$

Solution: Let $f(z) = (x-y)^2 + 2i(x+y)$

(i.e.) $u + iv = (x-y)^2 + i2(x+y)$

$u = (x-y)^2$ and $v = 2(x+y)$

$u_x = 2(x-y) \qquad v_x = 2$

$u_y = -2(x-y) \qquad v_y = 2$

The C-R equations will satisfy only if $x - y = 1$. Hence the function is analytic only on the line $x - y = 1$.

Example6: verify if the function $e^{-2x}\cos 2y$ ***can be real /imaginary part of an analytic function***

Solution: Let $f = e^{-2x}\cos 2y$

$$f_x = -2e^{-2x}\cos 2y$$
$$f_{xx} = 4e^{-2x}\cos 2y$$
$$f_y = -2e^{-2x}\sin 2y$$
$$f_{yy} = -4e^{-2x}\cos 2y$$

$f_{xx} + f_{yy} = 0$ $\therefore f$ is harmonic

In a simply connected domain, every harmonic function is the real part or the imaginary part of some analytic function.

$\therefore$ Given is a real or imaginary part of an analytic function.

PROPERTIES OF ANALYTIC FUNCTIONS

Property: 1

Show that an analytic function with constant real part is constant. **[Anna UQ]**

Solution: Let $f(z) = u + iv$ be analytic

$$\frac{du}{dx} = \frac{dv}{dy}; \quad \frac{du}{dy} = -\frac{dv}{dx}$$

Given that u =constant $= c_1$ (say)

$$\Rightarrow \frac{du}{dx} = 0, \qquad \frac{du}{dy} = 0$$

$$\Rightarrow \frac{dv}{dy} = 0, \qquad \frac{dv}{dx} = 0 \qquad \text{[by (1)]}$$

$\Rightarrow v$ is independent of x and y

$\Rightarrow v$ is constant

$\Rightarrow v = c_2, c_2$ is constant (say)

$\Rightarrow f(z) = u + iv = c_1 + ic_2$ is a constant.

Property: 2

Show that an analytic function with constant modulus is constant.

Solution: Let $f(z) = u + iv$ be analytic

$$\frac{du}{dx} = \frac{dv}{dy}; \qquad \frac{du}{dy} = -\frac{dv}{dx}$$

Given that $|f(z)| =$ Constant.

$\Rightarrow \sqrt{u^2 + v^2} = c_1, c_1$ is a constant.

$\Rightarrow u^2 + v^2 = c^2{}_1$

Differentiate partially with respect to x, y we get

$$2u\frac{du}{dx}+2v\frac{dv}{dx}=0, \qquad 2u\frac{du}{dy}+2v\frac{dv}{dy}=0,$$
$$\Rightarrow u\frac{du}{dx}+v\frac{dv}{dx}=0, \qquad u\frac{du}{dy}+v\frac{dv}{dy}=0,$$

$$\Rightarrow u\frac{du}{dx}+v\frac{dv}{dx}=0, \qquad -u\frac{dv}{dx}+v\frac{du}{dx}=0, \qquad \text{[by (1)]}$$
$$\Rightarrow u\frac{du}{dx}+v\frac{dv}{dx}=0, \qquad v\frac{du}{dx}-u\frac{dv}{dx}=0,$$

Now $\begin{vmatrix} u & v \\ v & -u \end{vmatrix} = u^2 - v^2 = -\left(u^2+v^2\right)$

$$= -c_1^{\,2} \neq 0$$

$\therefore$ The above equations have trivial solution.

$$\therefore \quad \frac{du}{dx}=0, \qquad \frac{dv}{dx}=0$$
$$\Rightarrow \frac{dv}{dy}=0, \qquad \frac{du}{dy}=0 \qquad \text{[by (1)]}$$

Hence $\frac{du}{dx}=\frac{du}{dy}=0, \qquad \frac{dv}{dx}=\frac{dv}{dy}=0$

$\Rightarrow u, v$ is independent of x and y

$\Rightarrow u$ and v are constants

$\Rightarrow f(z)=u+iv$ is a constant.

Property: 3

Show that an analytic function with constant imaginary part is constant.

Solution: Let $f(z)=u+iv$ be analytic

$$\Rightarrow \frac{du}{dx}=\frac{dv}{dy}, \qquad \frac{du}{dy}=-\frac{dv}{dx} \qquad \ldots (1)$$

Given that $v = \text{Constant} = c_1(say)$

$$\Rightarrow \frac{dv}{dx}=0, \qquad \frac{dv}{dy}=0,$$
$$\Rightarrow \frac{du}{dy}=0, \qquad \frac{du}{dx}=0, \qquad \text{[by (1)]}$$

$\Rightarrow u$ is independent of x and y

$\Rightarrow u = \text{Constant} = c_2(say)$

$\Rightarrow f(z)=u+iv=c_2+ic_2=$ Constant.

Property: 4

If $f(z)$ is a regular function prove that

$$\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

[MU April 2002, April 2000, AU Nov 2003, April 2004]

Solution: $\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right]|f(z)|^2 = 4\frac{d^2}{d_z d_{\bar z}}|f(z)|^2$

[By Property 6]

$$= 4\frac{d^2}{d_z d_{\bar z}}\left[f(z).\overline{f(z)}\right] \quad \text{[Since } \overline{z}z = |z|^2 \text{]}$$

$$= 4\frac{d^2}{d_z d_{\bar z}}\left[f(z).f(\bar z)\right] \quad \text{[Since } f(\bar z) = f(\bar z)\text{]}$$

$$= 4\frac{d}{dz}\left(\frac{d}{dz}\right)\left[f(z).f(\bar z)\right]$$

$$= 4\frac{d}{dz}\left[f(z).f'(\bar z)\right]$$

$$= 4f'(\bar z).f'(z)$$

$$= 4f'(z).\overline{f'(z)} = 4|f'(z)|^2$$

Property: 5

If $f(z)$ is an analytic function prove that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\log|f(z)| = 0$$ **(Anna UQ)**

Solution: $\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\log|f(z)| = 4\frac{d^2}{d_z d_z}\log\left[f(z)\overline{f(z)}\right]^{1/2}$

[Since $|z| = (\overline{z}z)^{1/2}$]

$$= 4\frac{d^2}{d_z d_{\bar z}}\frac{1}{2}\log\left[f(z).f(\bar z)\right] \quad \text{[Since } \log a^x = x\log a\text{]}$$

$$= 2\frac{d^2}{d_z d_{\bar z}}\left[\log f(z) + \log f(\bar z)\right] \quad \text{[Since } \log mn = \log m + \log n\text{]}$$

$$= 2\frac{d}{dz}\left(\frac{d}{dz}\right)\left[\log f(z) + \log f(\bar z)\right]$$

$$= 2\frac{d}{dz}\left[0 + \frac{1}{f(\bar z)}f'(z)\right] = 2(0) = 0$$

Property: 6

If $f(z)$ is an analytic function, prove that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$$

Solution: $\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)|f(z)|^p = 4\frac{d^2}{d_z \overline{d}_z}|f(z)|^p$

$$= 4\frac{\partial^2}{\partial_z \partial_{\bar{z}}}\left[f(z).\overline{f(z)}^{1/2}\right]^p \qquad [\text{Since} |z| = (\overline{zz})^{1/2}]$$

$$= 4\frac{\partial^2}{\partial_z \partial_{\bar{z}}} f(z).f(\bar{z})^{p/2} \qquad \left[Since \frac{(a^m)^n = a^{mn}}{f(z)=f(\bar{z})}\right]$$

$$= 4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}(f(z))^{p/2}.(f(z))^{p/2}\right)$$

$$= 4\frac{\partial}{\partial z}(f(z))^{p/2}\frac{p}{2}\left(f(\bar{z})\right)^{p/2-1}f'(\bar{z})$$

$$= 2p\frac{d}{dz}\left[\left(f(z)^{p/2}.f(\bar{z})\right)^{\frac{p-2}{2}}.f'(\bar{z})\right]$$

$$= 2pf(\bar{z})^{\frac{p-2}{2}}.f'(\bar{z})\frac{d}{d_z}f(z)^{p/2}$$

$$= 2pf(\bar{z})^{\frac{p-2}{2}}.f'(\bar{z}).\frac{p}{2}(f(z))^{\frac{p}{2}-1}f'(z)$$

$$= p^2\left(f(\bar{z})\right)^{\frac{p-2}{2}}.f'(\bar{z})^{\frac{p-2}{2}}f'(z)$$

$$= p^2\left(f(\bar{z}).f'(z)\right)^{\frac{p-2}{2}}.f'(z).f(\bar{z}) \qquad [\text{Since } a^m.b^m = (ab)^m]$$

$$= p^2\left[f(z)\overline{f'(z)}^{1/2}\right]^{p-2}\left(f'(z)\overline{f'(z)}\right)$$

$$= p^2|f(z)|^{p-2}.|f'(z)|^2.$$

POLAR FORM OF CAUCHY RIEMANN EQUATIONS

Let $z = re^{i\theta}$ and $f(z) = p(r,\theta)+iQ(r,\theta)$

Then $p(r,\theta)+iQ(r,\theta) = f(re^{i\theta})$

Differentiating (1) partially w.r.t to r, we get

$$\frac{\partial P}{\partial r}+i\frac{\partial Q}{\partial r} = f'(re^{i\theta})e^{i\theta}$$

Differentiating (1) partially w.r. to θ, we get

$$\frac{\partial P}{\partial \theta}+i\frac{\partial Q}{\partial \theta} = f'(re^{i\theta}).r.e^{i\theta}$$

$$= ri\left[f'(re^{i\theta})e^{i\theta}\right]$$

$$= ri\left[\frac{\partial p}{\partial r} + i\frac{\partial Q}{\partial r}\right]$$

$$= ir\frac{\partial p}{\partial r} + i^2 r\frac{\partial Q}{\partial r}$$

$$= ir\frac{\partial p}{\partial r} - r\frac{\partial Q}{\partial r} \qquad \text{[Since } i^2 = -1\text{]}$$

Equating the real and imaginary parts, we get

$$\frac{\partial p}{\partial \theta} = -r\frac{\partial Q}{\partial r}; \quad \frac{\partial Q}{\partial \theta} = r\frac{\partial p}{\partial r}$$

$$\Rightarrow \frac{\partial Q}{\partial r} = \frac{-1}{r}\frac{\partial p}{\partial \theta}; \quad \frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial Q}{\partial \theta}$$

The above equations are called Cauchy Riemann equations in polar form.

Show that $f(z) = z^n$ is differentiable and hence find its derivative.

Solution: Let $f(z) = z^n$

$$\Rightarrow P + iQ = \left[re^{i\theta}\right]^n$$

$$= r^n e^{in\theta} = r^n\left[\cos n\theta + i\sin n\theta\right]$$

$$= r^n \cos n\theta + ir^n \sin n\theta$$

Equating the real and imaginary parts, we get

$$P = r^n \cos n\theta; \qquad Q = r^n \sin n\theta$$

$$\frac{\partial P}{\partial r} = nr^{n-1}\cos n\theta; \qquad \frac{\partial Q}{\partial r} = nr^{n-1}\sin n\theta$$

$$\frac{\partial P}{\partial \theta} = r^n(-n\sin n\theta); \qquad \frac{\partial Q}{\partial \theta} = r^n.n\cos n\theta$$

$$\Rightarrow \frac{\partial P}{\partial r} = nr^{n-1}\cos n\theta = \frac{nr^n \cos n\theta}{r} = \left(\frac{\partial Q}{\partial r}\right)\frac{1}{r}$$

$$\frac{\partial Q}{\partial r} = nr^{n-1}\sin n\theta = \frac{nr^n \sin n\theta}{r} = \left(-\frac{\partial Q}{\partial r}\right)\frac{1}{r}$$

$\Rightarrow$ C-R equations are satisfied

$\Rightarrow$ $f(z) = z^n$ is differentiable

Now $f'(z) = e^{-i\theta}\left[\frac{\partial P}{\partial r} + i\frac{\partial Q}{\partial r}\right]$

$$= e^{-i\theta}\left[nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta\right]$$

$$= e^{-i\theta}\ \left(= e^{-i\theta}\right)\left(\cos n\theta + i\sin n\theta\right)$$

$$= e^{-i\theta}\ nr^{n-1}.e^{n-1}$$

$$= e^{-i\theta + in\theta}.nr^{n-1} \qquad \text{[Since } a^m.a^n = a^{m+n}\text{]}$$

$$= nr^{n-1}e^{i\theta(n-1)}$$

$= n\left[re^{i\theta}\right]^{n-1}$ [Since $(ab)^m = a^m b^m$]

$= nz^{n-1}$

$\Rightarrow f'(z) = nz^{n-1}$

Example: 3 Show that** $r^n = a\sec n\theta, r^n = b\cos ec n\theta$ **interested orthogonally where is an integer, a and b are constants.

Solution: Given that

$r^n = a\sec n\theta;$ $r^n = b\cos ec n\theta$

$\Rightarrow \dfrac{r^n}{\sec n\theta} = a;$ $\dfrac{r^n}{\cos ec n\theta} = b$

$\Rightarrow r^n \cos n\theta = a;$ $r^n \sin n\theta = b,$

Consider $f(z) = r^n \cos n\theta + ir^n \sin n\theta$

Then $P = r^n \cos n\theta;$ $Q = r^n \sin n\theta$

$\dfrac{\partial P}{\partial r} = nr^{n-1}\cos n\theta;$ $\dfrac{\partial Q}{\partial r} = nr^{n-1}\sin n\theta$

$\dfrac{\partial P}{\partial \theta} = r^n(-n\sin n\theta);$ $\dfrac{\partial Q}{d\theta} = r^n.n\cos n\theta$

$\Rightarrow \dfrac{\partial P}{\partial r} = \dfrac{1}{r}\dfrac{\partial Q}{\partial \theta};$ $\dfrac{\partial Q}{\partial r} = \dfrac{-1}{r}\dfrac{\partial P}{\partial \theta}$ except at $r = 0$

$\Rightarrow f(z)$ is analytic except at $r = 0$

Also we know that if $f(z) = u + iv$ is analytic then the family of curves $u(x, y) = c_1, v(x, y) = c_2$ Insects orthogonally.

$\therefore r^n \cos n\theta = a,$ $r^n \sin n\theta = b,$ Intersects orthogonally.

$\Rightarrow r^n = a\sec n\theta,$ $r^n = b\cos ec n\theta$ Intersects orthogonally.

HARMONIC FUNCTIONS

Definition

An expression of the form $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y} = 0$ is called the **Laplace equation** in two dimensions.

Definition

Any function having continuous second order partial derivatives which satisfies the Laplace is called **harmonic function.**

Definition

Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic are called conjugate harmonic functions.

Note: If u and v are conjugate harmonic functions then u is conjugate harmonic to v is conjugate harmonic to u .

Property: 13

If $f(z) = u + iv$ is an analytic function then u is a harmonic function.

Solution: Let $f(z) = u + iv$ be analytic

$$\Rightarrow \frac{du}{dx} = \frac{dv}{dy}, \qquad \frac{du}{dy} = \frac{-dv}{dx}$$

Now $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{-\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore v$ is a harmonic function.

Example: **1 prove that $u = x^2 - y^2, v = \dfrac{-y}{x^2 - y^2}$ are harmonic**

but $u + iv$ is not a regular function.

Solution: Let $u = x^2 - y^2$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x; \qquad \frac{\partial u}{\partial y} = -2y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 2; \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\therefore u$ is harmonic

Let $v = \dfrac{-y}{x^2 + y^2}$

$$\frac{\partial v}{\partial x} = \frac{-\left[(x^2 + y^2)0 - y(2x)\right]}{(x^2 + y^2)^2}$$

$$= \frac{-(-2xy)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2(2y) - (2xy)2(x^2 + y^2)2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)\left[(x^2 + y^2)2y - 8x^2 y\right]}{(x^2 + y^2)^3}$$

$$= \frac{2x^2 y + 2y^3 - 8x^2 y}{(x^2 + y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-[(x^2+y^2)(1) - y(2y)]}{(x^2+y^2)^2}$$

$$= \frac{-[x^2+y^2-2y^2]}{(x^2+y^2)^2}$$

$$= \frac{-[x^2-y^2]}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)[(x^2+y^2)(2y) - 4y(y^2-x^2)]}{(x^2+y^2)^4}$$

$$= \frac{2x^2y + 2y^3 - 4y^3 + 4xy^2}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$= \frac{(2y^3 - 6x^2y)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3-6x^2y}{(x^2+y^2)^3} \frac{(2y^3-6x^2y)}{(x^2+y^2)^3} = 0$$

$\therefore v$ is harmonic

But $\frac{\partial u}{\partial x} = 2x;\ \frac{\partial u}{\partial y} = -2y$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2};\ \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2};$$

$$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y};\ \frac{\partial u}{\partial y} \neq \frac{-\partial v}{\partial x}$$

$\Rightarrow f(z) = u + iv$ is not analytic.

CONSTRUCTION OF CONJUGATE HARMONIC FUNCTIONS

Method 1

Suppose u is given

Then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are known.

Now by total derivative,

$$\partial v = \frac{\partial v}{\partial x}.\partial x + \frac{\partial v}{\partial y}\partial y$$

$$= \frac{-\partial u}{\partial y}\partial x + \frac{\partial u}{\partial x}.\partial y \qquad \left\{ \text{Since} \frac{+\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \right\}$$

Integrating on both sides. we get

$$\int \partial v = \int \left(\frac{-\partial u}{\partial y}\partial x + \frac{\partial u}{\partial x}.\partial y \right)$$

$$\Rightarrow v = \int \left(\frac{-\partial u}{\partial y}\partial x + \frac{\partial u}{\partial x}.\partial y \right)$$

Method: 2

Suppose v is given

Than $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are known

Now by total derivative,

$$\partial u = \frac{\partial u}{\partial x}.\partial x + \frac{\partial u}{\partial y}\partial y$$

$$= \frac{\partial v}{\partial x}.\partial x - \frac{\partial v}{\partial y}\partial y \quad \left\{ \text{Since} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \right\}$$

Integrating on both sides. we get

$$\int \partial u = \int \left(\frac{\partial v}{\partial y}\partial x + \frac{\partial v}{\partial x}.\partial y \right)$$

$$\Rightarrow u = \int \left(\frac{\partial v}{\partial y}\partial x + \frac{\partial v}{\partial x}.\partial y \right)$$

Example:* 1 *Find the real part of the analytic function whose imaginary part is $\left[e^{-x}(2xy\cos y) + \left(y^2 - x^2\right)\sin y\right]$

Solution: Given that $v = e^{-x}\left[2xy\cos y + \left(y^2 - x^2\right)\sin y\right]$

$$\frac{\partial v}{\partial x} = e^{-x}[2y\cos y(1) + \sin y(0 - 2x)]$$

$$+ \left[2xy\cos y + \left(y^2 - x^2\right)\sin y\right]\left(-e^{-x}\right)$$

$$= e_{-x}\left[2y\cos y - 2x\sin y - 2\cos y - y^2\sin y + x^2\sin y\right]$$

$$= e_{-x}\left[\cos y(2y - 2xy) + \sin y\left(x^2 - y^2 - 2x\right)\right]$$

$$\frac{\partial v}{\partial y} = e^{-x}\left[2x(y(-\sin y) + \cos y + \left(y^2 - x^2\right)\cos y + \sin y(2y - 0)\right]$$

$$= e^{-x}\left[-2xy - \sin y + 2x\cos y + \left(y^2 - x^2\right)\cos y + 2y\sin y\right]$$

$$= e^{-x}\left[\sin y(2y - 2xy) + \cos y\left(2x + y^2 - x^2\right)\right]$$

Now real part $u = \int \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$

$$= \int e^{-x} \left[\sin y(2y - 2xy) + \cos y\left(2x + y^2 - x^2\right)\right] dx$$

$$- \int e^{-x} \left[(\cos y(2y - 2xy) + \sin y\left(x^2 - y^2 - 2x\right)\right] dy$$

$$= \sin y \int e^{-x} \left[(2y - 2xy)dx + \cos y \int e^{-x} \left(2x + y^2 - x^2\right)\right] dx$$

$$- e^{-x} \int \cos y \left[(2y - 2xy)dy - e^{-x} \int \sin y\left(x^2 - y^2 - 2x\right)\right] dy$$

$$= \sin y\left[(2y - 2xy)\left(e^{-x}\right) - (0 - 2y)\left(e^{-x}\right)\right]$$

$$+ \cos y\left[\left(2x + y^2 - x^2\right)\left(e^{-x}\right) - (2 + 0 + 2x)\left(e^{-x}\right) + (-2)\left(-e^{-x}\right)\right]$$

$$- e^{-x}\left[(2y - 2xy)(\sin y) - (2 - 2x)(-\cos y)\right]$$

$$- e^{-x}\left[\left(x^2 - y^2 - 2x\right)(\cos y) - (0 - 2y - 0)(-\sin y) + (-2)(\cos y)\right] + C$$

$$= \sin y\left[-2ye^{-x} + 2xye^{-x} + 2ye^{-x}\right]$$

$$+ \cos y\left[-2xe^{-x} - y^2e^{-x} + x^2e^{-x} - 2e^{-x} + 2xe^{-x} + 2e^{-x}\right]$$

$$e^{-x}\left[2y\sin y - 2xy\sin y + 2\cos y - 2x\cos y\right]$$

$$- e^{-x}\left[-x^2\cos y + y^2\cos y + 2x\cos y - 2y\sin y - 2\cos y\right] + C$$

$$= \sin y\left(2xye^{-x}\right) + \cos y\left(x^2 - y^2\right)e^{-x} - 2ye^{-x}\sin y$$

$$+ 2xye^{-x}\sin y - 2e^{-x}\cos y + 2xe^{-x}\cos y$$

$$+ \left(x^2 - y^2\right)(\cos y)e^{-x} - 2x\cos ye^{-x} + 2ye^{-x}\sin y + 2xe^{-x}\cos y + C$$

$$= 2xye^{-x}\sin y + \left(x^2 - y^2\right)e^{-x}\cos y + C$$

$$\Rightarrow u(x, y) = e^{-x}\left[2xy\sin y + \left(x^2 - y^2\right)\cos y\right] + C$$

Example: 6 show that the function $u = \frac{1}{2}\log\left(x^2 + y^2\right)$ ***is harmonic and find its conjugate***

Solution: Let $u = \frac{1}{2}\log\left(x^2 + y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2 + y^2}\right)(2x) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\left(x^2 + y^2\right)(1) - x(2x - 0)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{1}{x^2 + y^2}\right)(2y) = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{\left(x^2 + y^2\right)1 - y(2y)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2} = \frac{-\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$

$$\text{Now } = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} = 0$$

$\therefore u$ is harmonic.

$$\text{Now } v = \int \frac{\partial u}{\partial y}dx + \frac{\partial u}{dx}dy$$

$$= \int \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \int \frac{xdy-ydx}{x^2+y^2} = \int \frac{xdy-ydx}{x^2\left(1+(y/x)^2\right)}$$

$$= \int \frac{d(y/x)}{1+(y/x)^2}$$

$$v = \tan^{-1}(y/x) + C$$

CONFORMAL TRANSFORMATION

Definition:

Conformal mapping

A mapping or transformation which preserves angles in magnitude and in direction between every pair of curves through a point is said to be conformal at that point.

Isogonal Transformation

The transformation preserves the angle in magnitude but not in direction between every pair of curves through a point is said to be a isogonal at that point.

Critical Point:

A point at which $f'(z) = 0$ is called a critical point of the transformation.

ie) At the critical point, the transformation $w = f(z)$ is not conformal.

Example: Find the critical points for the transformation $w^2 = (z-\alpha)(z-\beta)$

Solution: $w^2 = (z-\alpha)(z-\beta)$

$$2w\frac{dw}{dz} = (z-\alpha)+(z-\beta) = 2z-(\alpha+\beta)$$

$$\therefore w\frac{dw}{dz} = z - \frac{1}{2}(\alpha+\beta)$$

Critical points occur at $\frac{dw}{dz} = 0$

$$\therefore z - \frac{1}{2}(\alpha + \beta) = 0$$

Also $\frac{dw}{dz} = \frac{w}{z - \frac{1}{2}(\alpha + \beta)}$

The critical points occurs at $\frac{dw}{dz} = 0$

$$\therefore \frac{w}{z - \frac{1}{2}(\alpha + \beta)} = 0$$

$$\Rightarrow w = 0$$

$$\Rightarrow (z - \alpha)(z - \beta) = 0$$

$$\Rightarrow z = \alpha, z = \beta$$

$\therefore$ **The critical points occur at** $\therefore z = \frac{1}{2}(\alpha + \beta), \alpha \,\&\, \beta$

Simple Translations

(i)Translation:

The transformation $w = z + a$ **where a is a complex constant,represents a translation.**

Let $z = x + iy, w = u + iv$ **and** $a = a_1 + ia_2$,

Then $u + iv = (x + iy) + (a_1 + ia_2)$

$= (x + a_1) + i(y + a_2)$

$\therefore u = x + a_1$ and $v = y + a_2$ are the equations of transformation,

The image of the point (x,y) in the z-plane is the point $(x + a, y + b)$ **in the w-plane.**

$\therefore$ **The transformation** $w = z + a$ **translates every point (x , y) through a constant vector representing 'a' .**

Every point in any region of the z-plane is mapped upon the w-plane in the same manner. If the whole w-plane is superposed on the z-plane the figure is shifted through a distance given by the vector 'a'.

In particular this transformation maps circles into circles. Also the corresponding region in the z-plane and w-plane will have the same shape, size and orientation.

(ii) Magnification:

The transformation w =az, where a is a complete constant, represents magnification.

Now w = az

$\Rightarrow u + iv = a(x + iy)$

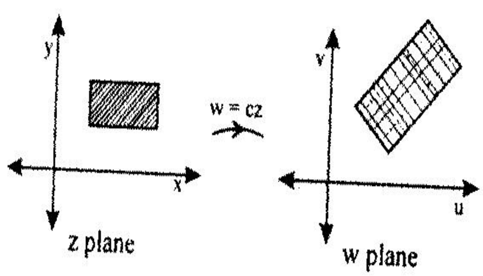
$\Rightarrow u = ax, v = ay$ **are the equators of transformation.**

It is clear that the image of any figure in the z-plane is magnified 'a' times. Without any changes in its shape and orientation. The transformation maps circles into circles.

(iii) Magnification and Rotation:

If we consider 'a' is a complex number it represents both magnification and rotation.

Let $z = re^{i\theta}, \alpha = \alpha e^{i\beta}$ **and** $w = \mathrm{Re}^{i\phi}$



Now w = az

$\Rightarrow \mathrm{Re}^{i\phi} = \alpha e^{i\beta} re^{i\theta}$

$\Rightarrow R = r\alpha, \phi = \beta + \theta$

The image θ **of any point p in z-plane is obtained from p by rotating op through an angle** $\alpha = \arg$ **'a' and magnifying op in the ration** $|a|$**.**

The transformation w = az corresponds to a rotation together with a magnification.

Example 1: **Find the image of the circle** $|z|$ **=c by the transformation w = 5z**

Solution: Given w = 5z

$\therefore u + iv = 5(x + iy)$

= 5x + i5y

$\Rightarrow u = 5x, v = 5y$

Given $|z|$**=C**

$\Rightarrow (x^2 + y^2)^{1/2} = C$

$\Rightarrow x^2 + y^2 = C^2$

$\Rightarrow \left(\frac{u}{5}\right)^2 + \left(\frac{v}{5}\right)^2 = C^2$

$\Rightarrow u^2 + v^2 = 25C^2$

$\Rightarrow u^2 + v^2 = (5C)^2$

$\therefore |z| = C$ **Maps to a circle in w-plane with centre at the origin and radius 3C.**

The Transformation $w = \frac{1}{z}$

The transformation $w = \frac{1}{z}$ **is conformal at all points of the z-plane except at z = 0.**

Put $z = x + iy$ **and** $w = u + iv$ **then**

$$u + iv = \frac{1}{x + iy}$$

$$\Rightarrow x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

Equating real and imaginary parts,

$$x = \frac{u}{u^2 + v^2} \;\&\; y = -\frac{v}{u^2 + v^2}$$

Also $x^2 + y^2 = \frac{1}{u^2 + v^2}$

Case (i)

The line x = 0 (ie) the imaginary axis in the z-plane.

When x = 0, u = 0 which is the imaginary axis in the w-plane.

$\therefore$ **The imaginary axis in the z-plane is mapped to the imaginary axis in the w-plane.**

Case (ii)

The line y = 0

When y = 0, v = 0

$\therefore$ **The real axis in the z-plane is mapped to the real axis in the w-plane.**

Case (iii)

The equation of any line parallel to x- axis in the z-plane (i.e.,) y = k

$$y = -\frac{v}{u^2 + v^2}$$

$$\therefore k = -\frac{v}{u^2 + v^2}$$

$$\Rightarrow u^2 + v^2 + \frac{v}{k} = 0$$

$$\Rightarrow u^2 + \left(v + \frac{1}{2k}\right)^2 = \left(\frac{1}{2k}\right)^2$$

This is a circle in the w-plane with centre at $\left(0, \frac{-1}{2k}\right)$ **and passes through the origin.**

$\therefore$ **The line parallel to real axis in the z-plane map into a family of circle in the w-plane passing through the origin and having centre on v-axis.**

Case (IV)

The equation of a line parallel to y-axis in the z-plane (ie) x = c

Given $x = \frac{u}{u^2 + v^2}$

$$c = \frac{u}{u^2 + v^2}$$

$$\Rightarrow u^2 + v^2 + \frac{u}{c} = 0$$

$$\Rightarrow \left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$$

This is a circle with centre at $\left(\frac{1}{2c}, 0\right)$ **and radius is** $\frac{1}{2c}$

Hence the line parallel to y-axis in the z-plane map onto the family of circles in the w-plane passing through the origin and having centre on axis.

Case (v)

Consider the line y-axis in the z-plane

Now $\frac{u}{v} = \frac{x}{y}$

Put y = mx $\Rightarrow \frac{u}{v} = -\frac{1}{m}\left(\because \frac{x}{y} = \frac{1}{m}\right)$

$\therefore V = -mu$ **, which is a straight line in the w-plane passing through the origin.**
Hence the line y = mx passes through the origin maps into the straight line v = - mu in the w-plane.

Case (vi)

The equation of a circle centre at the origin $x^2 + y^2 = r^2$

We know that $x^2 + y^2 = \frac{1}{u^2 + v^2}$

$\therefore u^2 + v^2 = \frac{1}{r^2}$ **Which is a circle at the origin in the w-plane.**

$\therefore$ **The circle** $x^2+y^2=r^2$ **maps into the circle** $u^2+v^2=\frac{1}{r^2}$

$\therefore$ **The circle** $x^2+y^2>1$ **in the z-plane maps into the region** $x^2+y^2<1$ **(ie) the exterior of unit circle** $|z|=1$ **maps into the interior of the unit circle** $|w|=1$**.Also the interior of the unit circle** $|z|=1$ **maps into the exterior.**

Example 2: Find the image of $|z-2i|=2$ **under the transformation** $w=\frac{1}{z}$

Solution: Given $|z-2i|=2$

$$|x+iy-2i|=2$$

$$|x+i(y-2)|=2$$

$$\sqrt{x^2+(y-2)^2}=2$$

$$x^2+(y-2)^2=4$$

$$x^2+y^2+4-4y=4$$

$$x^2+y^2-4y=0 \text{ ....(1)}$$

The given transformation is $w=\frac{1}{z}$

$$\Rightarrow x=\frac{u}{u^2+v^2}, y=-\frac{v}{u^2+v^2}$$

Substitute the values of x and y in (1)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left(\frac{-v}{u^2+v^2}\right)=0$$

$$\Rightarrow u^2+v^2+4v(u^2+v^2)=0$$

$$\Rightarrow (u^2+v^2)(1+4v)=0$$

$\Rightarrow (1+4v)=0$ **which is the straight line equation in the w-plane.**

Example 3: Find the image of the circle $|z-1|=1$ **in the complex plane under the mapping** $w=\frac{1}{z}$

Solution: Given $|z-1|=1$

$$|x+iy-1|=1$$

$$\sqrt{(x-1)^2+y^2}=1$$

$$\Rightarrow x^2-2x+1+y^2=1$$

$$\Rightarrow x^2-2x+y^2=0 \text{ ....(1)}$$

Given transformation is $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, y = -\frac{v}{u^2 + v^2}$$

Substitute in (1)

$$\left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 - 2\left(\frac{u}{u^2 + v^2}\right) = 0$$

$$\Rightarrow u^2 + v^2 - 2u\left(u^2 + v^2\right) = 0$$

$$\Rightarrow (u^2 + v^2)(1 - 2u) = 0$$

$$\Rightarrow (1 - 2u) = 0$$

Which is the straight line equation in the w-plane.

BILINEAR TRANSFORMATION

Definition: Bilinear transformation (or) Mobius transformation (or) linear fractional

The transformation $w = \frac{az + b}{cz + d}, ad - bc \neq 0$ **where a, b, c, d are complex constants is called the bilinear transformation.**

Fixed Point (or) Invariant Point:

A fixed of a mapping $w = f(z)$ **is a point z whose image is the same point. The fixed point or invariant points of the transformation** $w = f(z)$ **are obtained by solving** $z = f(z)$**.**

The fixed point of $w = \frac{az + b}{cz + d}$ **are obtained by** $z = \frac{az + b}{cz + d} \Rightarrow cz^2 - (a - d)z - b = 0$

This is a quadratic equation in z, giving two values of z unless a = d and b = c =0.
Hence, a bilinear transformation has (at most) two fixed points.

Cross Ratio:

If z_1, z_2, z_3, z_4 **are four complex numbers, then** $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ **is called the cross ratio of four points** z_1, z_2, z_3, z_4**.**

Properties of Bilinear transformation:

1. **The bilinear transformation always transforms circles into circles with lines as limiting cases.**
2. **The bilinear transformation preserves cross ratio of four points.**

Example: 1 **Find the fixed points for the transformation** $w = \frac{2z-5}{z+4}$

Solution: Fixed points are obtained from

$$z = \frac{2z-5}{z+4}$$
$$\Rightarrow z^2 + 4z - 2z + 5 = 0$$
$$\Rightarrow z^2 + 2z + 5 = 0$$
$$\therefore z = -\frac{2 \pm 4i}{2}$$

Example: 2. **find the bilinear transformation mapping the points** $z = 1, i, -1$ **into the points** $w = 2, i, -2$ **respectively.**

Solution: The bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$
$$\frac{(w-2)}{(w+2)} = \frac{(z-1)(i+1)(i-2)}{(z+1)(i-1)(i+2)}$$
$$= \frac{(z-1)(-3-i)}{(z+1)(i-3)}$$
$$= \frac{(z-1)(3+i)}{(z+1)(3-i)}$$
$$\frac{w-2+w+2}{w-2-w-2} = \frac{(z-1)(3+i)+(z+1)(3-i)}{(z-1)(3+i)-(z+1)(3-i)}$$
$$\frac{2w}{-4} = \frac{z(3+i+3-i)+3-i-3-i}{z(3+i-3-i)-3-i-3+i}$$
$$\frac{w}{-2} = \frac{6z-2i}{2iz-6}$$
$$w = \frac{-6z+2i}{iz-3}$$

Example: 3 **Find the mobius transformation that maps the points** $z = 0, 1, \infty$ **into the points** $z = -5, -1, 3$ **respectively. What are the invariant points of this transformation?**

Solution: The bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Since $z_3=\infty$, we simplify the transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)}{(z_2-z_1)}$$

$$\Rightarrow \frac{(w+5)(-4)}{(w-3)(4)}=\frac{(z-0)}{(1-0)}$$

$$\Rightarrow \frac{(w+5)}{(w-3)}=z$$

$$\Rightarrow w+5=3z-wz$$

$$\Rightarrow w(1+z)=3z-5$$

$\Rightarrow w=\dfrac{3z-5}{1+z}$ **is the required transformation.**

To get the invariant points, put w = z

$$\therefore z=\frac{3z-5}{z+1}$$

$$\Rightarrow z^2-2z+5=0$$

$$z=\frac{2\pm\sqrt{4-20}}{2}$$

$$=\frac{2\pm 4i}{2}$$

$$=1\pm 2i$$

$\therefore$ **The invariant points are** $z=1\pm 2i$

Example: 4. Find the bilinear transformation that maps the points $z=0,-1,i$ into the points $z=1,0,\infty$ respectively.

Solution: The bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Since $w_3=\infty$, we simply the transformation.

$$\frac{(w-w_1)}{(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)}{(0-1)}=\frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$w-1=\dfrac{-z(1+i)}{(z-i)}$ is the required transformation

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