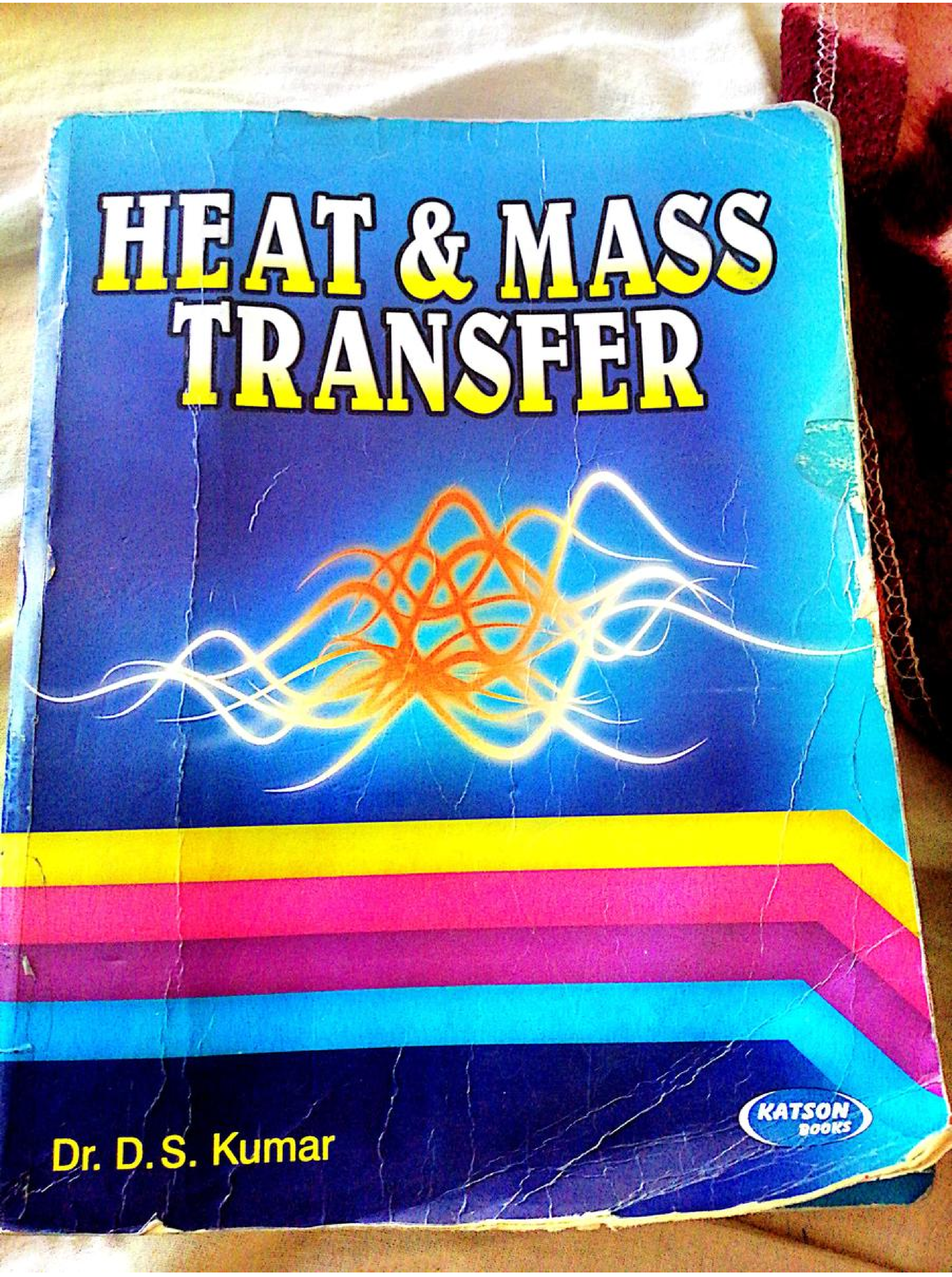
HEAT & MASS
TRANSFER
Dr. D.S. Kumar
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HEAT AND MASS TRANSFER

by

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Chapter

1

Definitions and Basic Concepts

Learning objectives : After a study of the subject matter presented in this chapter, the readers will be able to

- *state the fundamental difference between thermodynamics and heat transfer*
- *understand the different modes of heat transfer*
- *bring out the difference between steady and unsteady; one, two and three dimensional heat flows*
- *identify the different modes of heat transfer in a given system operation*
- *appreciate the importance of heat transfer in various fields of engineering*

Heat transfer is a subject of widespread interest to the students of engineering curriculum, practising engineers and technicians engaged in the design, construction, testing and operation of the many diverse forms of heat exchange equipment required in our scientific and industrial technology. While launching ourselves into the frontiers of heat transfer, let us recall the following basic concepts and definitions.

1.1. THERMODYNAMIC SYSTEM AND SURROUNDINGS

"A prescribed, identifiable and fixed collection of matter which is completely enclosed with in a definite region, and whose behaviour is being investigated"

The system may be a quantity of steam, a mixture of vapour and gas, or an internal combustion engine and its components. The term *surroundings* represent the environments (a combination of matter and space external to the system) which are affected by changes occurring within the system. The *boundary* separating a system from its surrounding may be real or imaginary, rigid and confined in the space or flexible, *i.e.*, it may change in shape, volume, position and/or orientation.

Fig. 1.1. shows the conventional representation of a system, its boundary and surroundings. The surroundings can also be regarded as a system and hence we are largely concerned with the interactions between systems.

When the system executes a process (undergoes a change from one state to another), there are mass and energy interactions between the system and the surroundings. For a *closed system* the same mass matter remains within the system and only work and heat energies cross the system boundaries. The physical nature and chemical composition of the matter can, however, change. Thus a liquid may evaporate, a gas may condense or a chemical reaction may take place between two or more constituents of a system. Examples of closed system are :

(*i*) gas expanding in a cylinder by displacing a piston

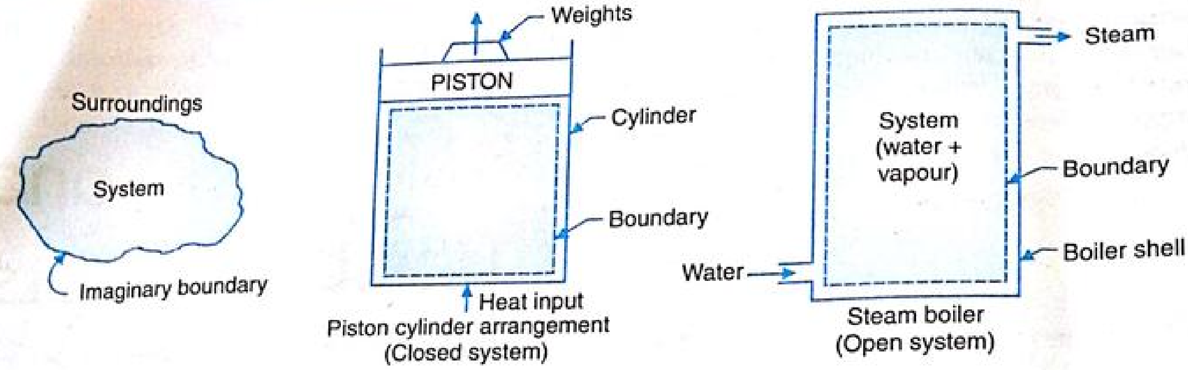


Fig. 1.1. Conventional representation of a system

(ii) mixture of water and steam in a closed vessel

(iii) Bomb calorimeter

When the mass flow of substance also takes place simultaneously with the transfer of energy, the system is known as an *open system*. Examples of an open system are :

(i) steam flowing through a turbine

(ii) water entering a boiler and leaving as steam

(iii) water wheel suspended in a water stream

1.2. THERMODYNAMIC PROPERTY

"Any characteristic which defines the overall condition or state of a system at any time; this characteristic depends only upon the state of the system and not upon the manner in which the state was reached."

Properties can be classified into two groups:

- Intensive, intrinsic or qualitative properties whose magnitude does not depend on the amount of matter considered. Examples are pressure, temperature and specific quantities like density, specific volume etc.
- Extensive, extrinsic or quantitative properties whose magnitude is proportional to the mass or extent of the system. Examples are volume, energy, weight etc.

The difference between these categories can well be appreciated by considering the property of a part of a system. Intensive properties retain their value whereas extensive properties get reduced in the corresponding ratio.

1.3. TEMPERATURE, HEAT AND THERMAL EQUILIBRIUM

"The very essence of heat is motion and nothing else".

Molecules comprising a system are interlinked by forces of mutual attraction and are in constant chaotic motion. The energy of molecular motion is called internal kinetic energy, and the energy of mutual attraction of molecules is termed as internal potential energy. The sum total of molecular kinetic and potential energy is referred to as *internal energy*, and it manifests itself in the motion and arrangement of molecules and the vibrations within the molecules. For a fixed mass, the amount of internal energy depends upon the temperature level of the system. Higher the temperature, greater would be the internal energy. Accordingly, temperature is a measure of velocity and hence kinetic energy of molecules of the system. When all the molecular motion ceases and there is no motion energy within the molecules, the temperature of the molecules is absolute zero.

Thermal energy is transferred whenever a temperature gradient exists, and the free flow of heat energy is always from a higher temperature to a lower temperature in

accordance with the second law of thermodynamics. Temperature gradients are inherent in thermodynamic systems comprising power generation, propulsion and transportation, heating and refrigeration, chemical and metallurgical processing etc.

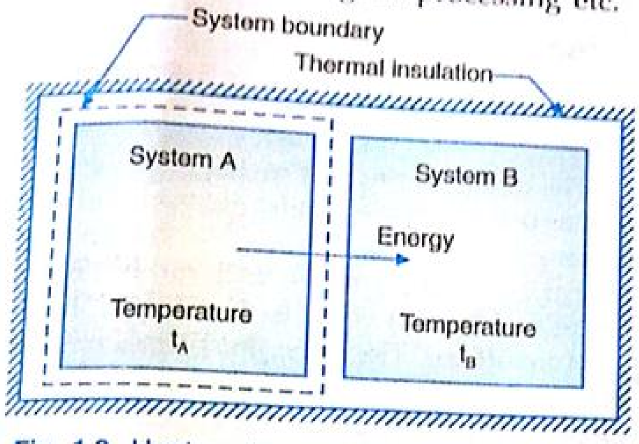


Fig. 1.2. Heat exchange between two systems

Consider system A at uniform temperature t_A and system B at another uniform temperature $t_B (t_A > t_B)$. Let the two systems be brought into contact and be thermally insulated from their surroundings but not form each other as depicted in Fig. 1.2. Energy will flow from system A to system B because of temperature difference $(t_A - t_B)$; greater the temperature imbalance the higher would be the rate of energy transfer. This will cause a reduction in the molecular activity (and hence internal energy) of system A and corresponding increase in molecular activity (and hence internal energy) of system B. This energy flow continues till both systems attain the same temperature level at some intermediate value between their original temperatures. The two system are then said to be in a state of *thermal equilibrium*.

Energy flow due to temperature difference is called heat; and the study of heat transfer deals with the rate at which such energy is transferred. Heat is thus the energy in transit between systems which occurs by virtue of their temperature difference when they communicate. Obviously conditions of temperature difference and communication must be fulfilled simultaneously for heat interaction between systems to occur. The finite temperature difference existing between the systems makes the process of heat exchange irreversible, *i.e.*, flow of heat cannot be reversed.

Whether a system has received or rejected heat is indicated by a change in its temperature; the temperature grows with the receipt of heat and diminishes with heat rejection. The quantity of heat received or rejected by a system during the process of heating and cooling is measured in terms of the product of mass, specific heat and temperature changes,

$$Q = mc\,(t_2 - t_1) \qquad \ldots(1.1)$$

where Q is the heat gained by the system in kJ, m is the mass of the substance in kg, $(t_2 - t_1)$ is the temperature rise in degree centigrade or Kelvin, and c is an experimental factor called *specific heat* and expressed in kJ/kg-deg. The product $(m \times c)$ is separately termed as *heat capacity* or *water equivalent* of the system.

The specific heat for liquids and solids varies primarily with temperature and does not differ much with nature of the process. In case of gases, specific heat is also dependent on the path of the thermodynamic process which may be carried out isochorically (at constant volume), isobarically (at constant pressure) or polytropically. For heat addition at constant pressure and at constant volume, the specific heats for a gas are respectively referred to as specific heat at constant pressure c_p, and specific heat at constant c_v. The specific heats are properties of gas, and are really not constant but vary with both pressure and temperature. However for moderate changes in pressure and temperature they are assumed constant.

For water :

$$c = 1.0 \text{ kcal/kg-deg}$$
$$= 4.186 \text{ kJ/kg-deg}$$

For air :

$$c_p = 0.24 \text{ kcal/kg-deg}$$
$$= 1.005 \text{ kJ/kg-deg}$$

c_v = 0.17 kcal/kg-deg
= 0.71 kJ/kg-deg

Obviously 1.0 kcal of heat
= 4.186 kJ of heat

1.4. THERMODYNAMICS VERSUS HEAT TRANSFER

Following remarks would illustrate the fundamental difference that exists between thermodynamics and heat transfer:

• Thermodynamics is concerned with the equilibrium states of matter, and precludes the existence of a temperature gradient. For heat exchange, temperature gradient must exist and as such heat transfer is inherently a non-equilibrium process.

• Thermodynamics helps to determine the quantity of work and heat interactions when a system changes from one equilibrium state to another. The analysis, however, does not provide any information on the nature of interactions and the time rate at which interactions occur. It simply describes how much heat is to be exchanged during a process without caring to explain how that could be achieved.

• Heat transfer helps to predict he temperature distribution, which may be the function of both spatial co-ordinates and time within regions of matter. Heat transfer also helps to determine the rate at which energy is transferred across a surface of interest due to temperature gradients at the surface, and temperature difference between the different surfaces.

The difference between thermodynamics and heat transfer can be well appreciated by considering the cooling of a hot steel bar which is placed in a water bath. Thermodynamic analysis would help to predict the final equilibrium temperature of the composite system comprised by steel bar-water combination. Analysis, however, fails to predict how long it will take to reach the equilibrium condition or what would be the temperature of the bar after a certain length of time before the equilibrium condition is attained. Heat transfer does help to predict the temperature of both the bar and the water as a function of time. That is, the temperature at all points of interest within the bar or temperature at any specific point (such as at the centre of the bar where it is the highest) at any time can be predicted. Also, the instantaneous heat transfer rate can be predicted from all or from any part of the surface of the bar at any time.

The design of the heat exchange equipment such as boilers, heaters, refrigerators, and heat exchangers depends not only on the amount of heat to be transmitted but rather on the rate at which heat is to be transferred under given conditions. The discipline of heat transfer seeks to quantify the rate at which heat transfer occurs in terms of the degree of non-equilibrium. This is accomplished through the rate equations for the different modes of heat transfer. The rate equations, when combined with energy balances and thermodynamic state equations, yield equations from which the temperature distribution and heat transfer rates can be worked out. Heat transfer theory is then essentially ***"thermodynamics with rate equations added"***.

Towards closure, it is worthwhile to summarise that :

(*i*) In thermodynamics, no consideration is given to time or temperature difference required to bring about the transfer of heat energy, and whether or not there is uniform temperature within the thermodynamic system.

(*ii*) The subject of heat transfer seeks to provide answer to the questions such as

• possibility of removal or addition of heat at a desired rate,

• temperature distribution existing within the system,

• amount of heat to be transferred,

• time taken (duration of heating and cooling) for a certain duty and surface area required to accomplish that duty.

1.5. MODES AND BASIC LAWS OF HEAT TRANSFER

The literature on heat transfer generally recognises three distinct modes of heat transmission; *conduction*, *convection* and *radiation*. These three modes are similar in that a temperature differential must exist and the heat exchange is in the direction of decreasing temperature. Each method has, however, different physical picture and different controlling laws.

1.5.1. Conduction

Thermal conduction is a mechanism of heat propagation from a region of higher temperature to a region of low temperature within a medium (solid, liquid or gaseous) or between different mediums in direct physical contact. Conduction does not involve any movement of macroscopic portions of matter relative to one another. The thermal energy may be transferred by means of electrons which are free to move through the lattice structure of the material. In addition, or alternatively, it may be transferred as vibrational energy in the lattice structure. Irrespective of the exact mechanism, the observable effect of conduction is an equalization of temperature.

Consider the flow of heat along a metal rod, one end of which is placed adjacent to a flame. The elementary particles (molecules, atoms, electrons) composing the rod, and which are in immediate vicinity of the flame, get heated. Because of the resulting temperature growth, their kinetic energy increases and this puts them in a violent state of agitation, and they start vibrating about their mean positions. Consequently, these more active particles collide with less active molecules lying next to them. During collision, the less active particles also get excited, *i.e.*, thermal energy is imparted to them. The process is repeated for layer after layer of molecules until the other end of the rod is reached. Each layer of molecules is at a slightly higher temperature than the preceding one, *i.e.*, a temperature gradient exists along the length of the rod. The rate of heat flow between the two ends depends upon the length of the rod, temperature difference between the two ends, and the physical and chemical composition of the bar material.

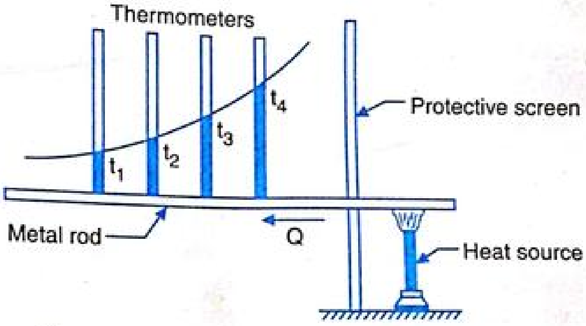


Fig. 1.3. Conduction heat flow along a rod

Since conduction is essentially due to random molecular motion, the concept is termed as *microform* of heat transfer and is usually referred to as *diffusion* of energy. The rate equation for one-dimensional steady flow of heat by conduction is prescribed by the Fourier Law :

$$Q = -\,kA\frac{dt}{dx} \qquad ...(1.2)$$

where Q is the heat transfer rate, A is the area of heat transfer surface, dt is the temperature difference for a short perpendicular distance dx, and the thermal conductivity k is a characteristic of the surface material. Since the temperature gradient is negative in the positive x-direction, the minus sign in the equation gives positive heat flow.

If δ is the path length in the direction of heat flow and $(t_1 - t_2)$ is the temperature difference, then

$$Q = \frac{kA\,(t_1 - t_2)}{\delta} \qquad ...(1.3)$$

The *heat flux* q is the heat conducted per unit time per unit area and is given by

$$q = \frac{Q}{A} = \frac{k(t_1 - t_2)}{\delta} \qquad ...(1.4)$$

Heat transfer in metal rods, in heat treatment of steel forgings and through the

walls of heat exchange equipment are some practical examples of heat conduction.

EXAMPLE 1.1.

A 7.5 cm thick side wall of an oven is primarily made of insulation with a thermal conductivity of 0.04 W/mK. Conditions on the inside of wall fix the temperature on that side at 420 K. The electric coils within the oven dissipate 36.5 watts of electrical energy to make up for the heat loss through the wall. Calculate the wall surface area, perpendicular to heat flow, so that temperature on the other side of the wall does not exceed 310 K.

Solution : Under the stipulations of one-dimensional steady state heat conduction, the electrical energy dissipation rate within the oven must equal the conduction heat transfer rate across the wall. That is:

$$Q = \frac{kA\,(t_1 - t_2)}{\delta}$$

$$36.5 = \frac{0.04\,A\,(420 - 310)}{0.75} = 5.87\,A$$

Hence, the required wall surface area,

$$A = \frac{36.5}{5.87} = 6.218 \text{ m}^2$$

EXAMPLE 1.2.

A plane wall of 10 cm thickness and 3 m² area is made of a material whose conductivity is 8.5 W/mK. The temperatures of the wall surfaces are steady at 100°C and 30°C respectively. Find the temperature gradient and heat flow across the wall.

Solution : The temperature gradient in the direction of heat flow is

$$\frac{dt}{dx} = \frac{t_2 - t_1}{\delta}$$

$$= \frac{30 - 100}{0.1} = -700°\text{C/m}$$

(*b*) Heat flow across the wall is given by Fourier's heat conudctiom equation

$$Q = -kA\frac{dt}{dx}$$

$$= -8.4 \times 3 \times (-700)$$

$$= 17850 \text{ W} \quad \text{or} \quad 17.85 \text{ kW}$$

EXAMPLE 1.3.

To effect a bond between two metal plates, 2.5 cm and 15 cm thick, heat is uniformly applied through the thinner plate by a radiant heat source. The bonding epoxy must be held at 320 K for a short time. When the heat source is adjusted to have a steady value of 43.5 kW/m², a thermocouple installed on the side of the thinner plate next to source indicates a temperature of 345 K. Calculate the temperature gradient for heat conduction through thinner plate and thermal conductivity of its material.

Solution : $t_1 = 345$ K ; $t_2 = 320$ K

$\delta = 2.5$ cm $= 0.025$ m

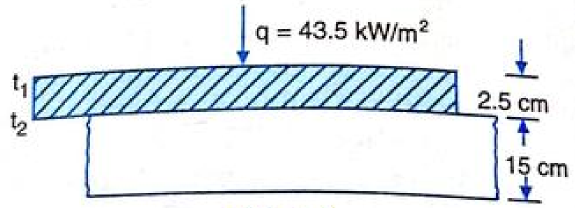


Fig. 1.4.

Temperature gradient,

$$\frac{dt}{dx} = \frac{t_2 - t_1}{\delta}$$

$$= \frac{320 - 345}{0.025} = -1000°\text{C/m}$$

Invoking Fourier law, the heat conduction equation for this one-dimensional case can be written as

$$\frac{q}{A} = -\frac{k\,(t_2 - t_1)}{\delta}$$

$$43.5 \times 10^3 = -\frac{k\,(325 - 345)}{0.025}$$

∴ Thermal conductivity for the material of thin plate is,

$$k = \frac{43.5 \times 10^3 \times 0.025}{(345 - 320)}$$

$$= 43.5 \text{ W/m-deg}$$

1.5.2. Convection

Thermal convection is a process of energy transport affected by the circulation or mixing of a fluid medium (gas, liquid or a powdery substance). Convection is possible only in a

fluid medium and is directly linked with the transport of medium itself. Macroscopic particles of a fluid moving in space cause the heat exchange, and thus convection constitutes the *macroform* of the heat transfer. The effectiveness of heat transfer by convection depends largely upon the mixing motion of the fluid.

With respect to origin, two types of convection are distinguished; forced and natural or free convection.

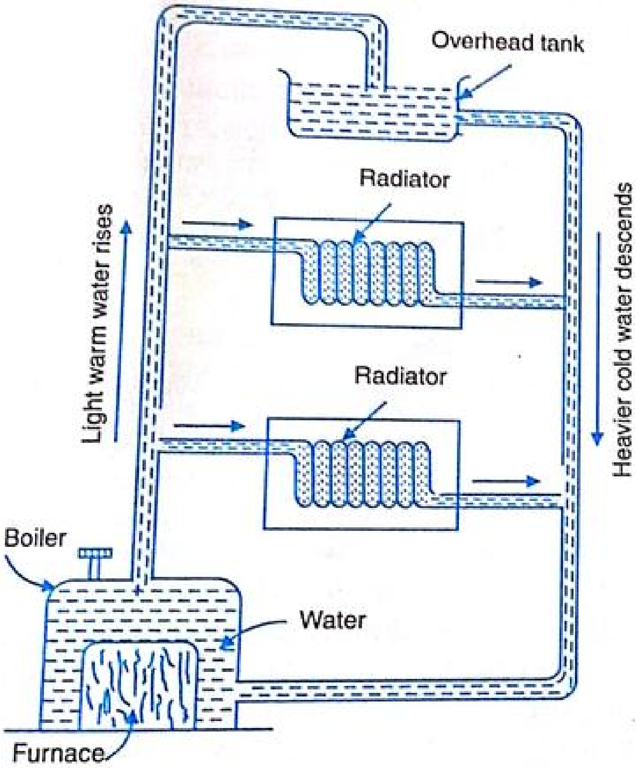


Fig. 1.5. Hot water heating system : free convection

In *natural* or *free convection*, the circulation of the fluid medium is caused by buoyancy effects, *i.e.*, by the difference in the densities of the cold and heated particles. Consider heat flow from a hot plate to atmosphere. The stagnant layer of air in the immediate vicinity of the plate gets thermal energy by conduction. The energy thus transferred serves to increase the temperature and internal energy of the air particles. Because of temperature rise these particles become less dense (and therefore lighter) than the surrounding air. The lighter air particles move upwards to a region of low temperature where they mix with and transfer a part of their energy to the cold particles. Simultaneously the cold air particles descend downwards to fill the space vacated by the hot air particles. The circulation pattern, upward movement of the warm air and the downward movement of the cold air, is called the *convection currents.*

A similar effect can also be demonstrated by a hot-water heating system (Fig. 1.5) where water serves as the medium for carrying heat to all parts of the building. Water is heated in the boiler installed at the base of the building. The hot water becomes lighter, rises up in the left hand vertical pipes and passes through the radiators fitted in different rooms of the building. The radiators get heated and dissipate heat to the rooms. After losing heat to the radiators, the water gets cooled and returns back to the boiler through the pipe on the right. Convection currents are setup and the building is kept warm continuously at a constant temperature. In this way, a constant circulation of water through the pipes and through the radiator is maintained.

Some other examples of free convection are :

- chilling effect of cold wind on a warm body,
- heat flow from a hot pavement to surrounding atmosphere and heating of air in a room by a stove,
- cooling of billets in the atmosphere ,
- heat exchange on the outside of cold and warm pipes.

In *forced convection*, the flow of fluid is caused by a pump, fan or by atmospheric winds. These mechanical devices provide a definite circuit for the circulating currents and that speeds up the heat transfer rate. Example of forced convection are :

- flow of water in condenser tubes,
- fluid passing through the tubes of a heat exchanger,
- cooling of internal combustion engine,

- air conditioning installation and nuclear reactors.

Regardless of the particular nature, the appropriate rate equation for the convective heat transfer between a surface and an adjacent fluid is prescribed by *Newton's law of cooling*.

$$Q = hA\,(t_s - t_f) \quad ...(1.5)$$

where, Q is the convective heat flow rate, A is the area exposed to heat transfer, t_s and t_f are the surface and fluid temperatures respectively. The heat transfer co-efficient h depends upon the thermodynamic and transport properties (*e.g.* density, viscosity, specific heat and thermal conductivity of the fluid), the geometry of the surface, the nature of fluid flow, and the prevailing thermal conditions.

Convection mechanism involving phase changes leads to the important fields of boiling (evaporation) and condensation.

EXAMPLE 1.4.

An oil cooler in a high performance engine has an outside surface area 0.12 m² and a surface temperature of 65°C. The air rushes over the surface of the cooler at a temperature of 30°C and gives rise to a surface coefficient of heat transfer equal to 45.4 W/m²K. Calculate the heat transfer rate from the cooler.

Solution : The conditions described imply a convective process, that is, heat transfer from a solid surface (the oil cooler) to an adjacent moving fluid (the air passing over the cooler). The rate of heat transfer by convection from oil cooler to the air is then,

$$Q = hA\,(t_s - t_f)$$
$$= 45.4 \times 0.12\,(65 - 30)$$
$$= 190.68 \text{ W}$$

EXAMPLE 1.5.

A wire 10 cm long and 1 mm in diameter is held taut between two conducting supports in a water tank and is submerged. A controlled amount of current is made to pass through the wire until the temperature of water becomes 100°C and it starts boiling. Make calculations for the steady temperature of wire if 23.5 watts of electric power is consumed. Take convective heat transfer coefficient to be 5000 W/m²-deg.

Solution : When steady state is reached, the power supplied to the wire equals the convective heat loss which is given by

$$Q = h\,A\,\Delta t = h\,(\pi dl) \times (t_s - t_w)$$

or $$23.5 = 5000 \times (\pi \times 0.001 \times 0.1) \times (t_s - 100)$$

$$= 1.57\,(t_s - 100)$$

∴ Surface temperature of wire is,

$$t_s = \frac{23.5}{1.57} + 100 \cong 115°\text{C}$$

EXAMPLE 1.6.

A 120 W heater has been employed to maintain a plate of 0.25 m² area at a temperature of 60°C when the surroundings are at 20°C temperature. What fraction of heat supplied is lost by natural convection ? It may be presumed that convection coefficient conforms to the relation

$$h = 2.5\,(\Delta T)^{0.25} \text{ W/m}^2\text{ K}$$

Solution : Convective heat transfer coefficient,

$$h = 2.5\,(\Delta T)^{0.25}$$
$$= 2.5\,(60 - 20)^{0.25}$$
$$= 6.287 \text{ W/m}^2\text{ K}$$

Heat lost by convection $= h\,A\,\Delta t$

$$= 6.287 \times 0.25 \times (60 - 20)$$
$$= 62.87 \text{ W}$$

Heat lost by convection as fraction of heat supplied,

$$= \frac{62.87}{120} = 0.5239 \text{ or } 52.39\%$$

The remaining 47.61% would be lost to the surroundings by radiation.

1.5.3. Radiation

Thermal radiation is the transmission of heat in the form of radiant energy or wave motion from one body to another across an intervening space. Unlike heat transfer by conduction and convection, transport of thermal radiation does not necessarily affect the material medium between the heat source and the receiver. An intervening medium is not even necessary and the radiation can be

affected through vacuum or a space devoid of any matter. Radiation exchange, in fact, occurs most effectively in vacuum. A material present between the heat source and the receiver would either reduce or eliminate entirely the propagation of radiation energy.

The mechanism of the heat flow by radiation consists of three distinct phases:

(i) *Conversion of thermal energy of the hot source into electromagnetic waves* : All bodies above absolute zero temperature are capable of emitting radiant energy. Energy released by a radiating surface is not continuous but is in the form of successive and separate (discrete) packets or quanta of energy called *photons*. The photons are propagated through the space as rays; the movement of swarm of photons is described as the electromagnetic waves.

(ii) *Passage of wave motion through intervening space*. The photons, as carriers of energy, travel with unchanged frequency in straight paths with speed equal to that of light.

(iii) *Transformation of waves into heat*: When the photons approach the cold receiving surface, there occurs reconversion of wave motion into thermal energy which is partly absorbed, reflected or transmitted through the receiving surface.

Thermal radiation is limited to range of wavelength between 0.1 and 100 μm of the electro-magnetic spectrum. Thermal radiations thus include the entire visible and infrared, and a part of ultra violet spectrum. It is to be recognized that thermal radiation is the transfer of energy by disorganized photon propagation. In contrast, an organized photon energy such as radio transmission can be macroscopically identified and is not considered heat. Further, emission of thermal radiations is associated with thermally excited conditions which depend upon the temperature and nature of the surface.

The most vivid evidence of radiation heat transfer is that represented by solar energy which passes through inter-stellar space (conditions close to that for perfect vacuum) on its way to the earth surface. Solar radiation plays an important part in the design of heating and ventilating systems. Heat transfer by radiation is encountered in boiler furnaces, billet reheating furnaces and other types of heat exchange apparatus. The design and construction of engines, gas turbines, nuclear reactors and solar collectors is also significantly influenced by the radiation heat transfer.

The basic rate equations for radiation heat transfer are based on ***Stefan-Boltzman Law*** :

$$E_b = \sigma_b A T^4 \qquad ...(1.6)$$

where, E_b is the energy radiated per unit time, T is the absolute temperature of the surface, and σ_b is the Stefan-Boltzman constant.

$$\sigma_b = 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$$
$$= 4.86 \times 10^{-8} \text{ kcal/m}^2 \text{ hr K}^4$$

Equation 1.6 is essentially valid for an ideal radiator or a black body-suffix b designates a black surface. The radiant energy emitted by a real surface is less than that for an ideal emitter and is given by

$$E = \varepsilon \sigma_b A T^4 \qquad ...(1.7)$$

where, ε is a radiative property of the surface and is called ***emissivity***; its value depends upon surface characteristics and temperature. It indicates how effectively the surface emits radiations compared to an ideal or black body radiator. Normally a body radiating heat is simultaneously receiving heat from other bodies as radiation. Consider that surface 1 at temperature T_1 is completely enclosed by another black surface 2 at temperature T_2. The net radiant heat transfer is

$$Q = \sigma_b A_1 (T_1^4 - T_2^4) \qquad ...(1.8)$$

Likewise, the net rate of heat transfer between the real surface (called gray surface) at temperature T_1 to a surrounding black surface at temperature T_2 is

$$Q = \sigma_b A_1 \varepsilon_1 (T_1^4 - T_2^4) \qquad ...(1.9)$$

The net exchange of heat between the two radiating surfaces is due to the fact that one at the higher temperature radiates more and receives less energy for its absorption. An isolated body which remains at constant temperature emits just as much energy by radiation as it receives.

EXAMPLE 1.7.

A radiator in a domestic heating system operates at a surface temperature of 60°C. Calculate the heat flux at the surface of the radiator if it behaves as a black body.

Solution : The heat flux at the surface is the rate at which radiant energy leaves the surface per unit area.

$$q = \frac{Q}{A} = \sigma_b T^4$$
$$= 5.67 \times 10^{-8} (273 + 60)^4$$
$$= \mathbf{697.2 \ W/m^2}$$

EXAMPLE 1.8.

A cylindrical rod, 1.5 m long and 2 cm in diameter, is heated electrically and positioned in a vacuum furnace which has interior walls at 800 K temperature. A controlled amount of current is passed through the rod and its surface is maintained at 1000 K. Calculate the power supplied to the heating rod if its surface has an emissivity of 0.9.

Solution : For steady state conditions, the electric power supplied to the rod equals the radiant heat loss from it. Further, since the walls of the furnace completely enclose the heating rod, all the radiant energy emitted by the surface of the rod is intercepted by the furnace walls. Thus

$$Q = \sigma_b A \varepsilon (T_1^4 - T_2^4)$$
$$= \sigma_b (\pi dl) \varepsilon (T_1^4 - T_2^4)$$
$$= 5.67 \times 10^{-8} (\pi \times 0.02 \times 1.5) \times (0.9) (1000^4 - 800^4)$$
$$= \mathbf{2838 \ W}$$

Thus the rate of electrical input to the rod must equal 2838 W.

1.6. STEADY AND UNSTEADY HEAT TRANSFER

Any physical phenomenon generally involves a change in space of its physical properties. Likewise heat exchange is also accompanied by space-time variation of temperature, and the analytical computations for the amount of heat exchange lie in determining a mathematical relation for the temperature field prescribed as

$$t = f(x, y, z, \tau) \quad ...(1.10)$$

Equation 1.10 refers to the entire set of temperature at all points of space studied at any instant of time τ.

Heat exchange between two systems may take place under steady (stable) thermal conditions or under unsteady (unstable) thermal conditions. Steady state implies that temperature at each point of the system remains constant in the course of time, and it is a function only of space co-ordinates

$$t = f(x, y, z) \; ; \frac{dt}{d\tau} = 0 \quad ...(1.11)$$

Steady state results in a constant rate of heat exchange (heat influx equals heat efflux), and there is no change in the internal energy of the system during such a process. Typical examples of steady state heat transfer are :

- cooling of an electric bulb by the surrounding atmosphere,
- heat flow from the products of combustion to water in the tubes of a boiler, from the hot to cold fluid in a heat exchanger, and from a refrigerated space to cooling surface of the evaporator.

Under unsteady thermal conditions, temperature of the system changes continuously with time. Temperature is obviously a function of space and time co-ordinates.

$$t = f(x, y, z, \tau) \; ; \frac{dt}{d\tau} \neq 0 \quad ...(1.12)$$

Unsteady state results in heat transfer rate which changes with time. Further, a change in temperature indicates a change of internal energy of the system. Energy storage is thus a part and parcel of unsteady heat flow. Typical examples of unsteady heat transfer are :

- warm-up periods of furnaces
- boilers and turbines
- cooling of castings in a foundry
- heat treatment and stress relieving of metal castings.

A special kind of unsteady process is the transient state wherein the system is subjected

to cyclic variations in the temperature of its environment. The temperature at a particular point of the system returns periodically to the same value; the rate of heat flow and energy storage also undergo periodic variations. Examples are : Heating or cooling of the water of an IC engine; heating or cooling of the walls of a building during the 24-hours cycle of the day.

Further, the heat transfer in a system may be in one, two or more directions. In a one dimensional heat flow, there is a single predominant direction in which temperature differential exists and obviously the heat flow takes place; heat flow in the other two directions can be safely neglected. When the temperature is a function of two co-ordinates, heat flow is two-dimensional. A three-dimensional heat flow stipulates that temperature is a function of three co-ordinates, and consequently heat flow occurs in all three directions.

Dimensionality of temperature field for steady/unsteady heat flow can be mathematically expressed as :

Type of Heat Flow	Steady	Unsteady
One-dimensional heat flow	$t = f(x)$	$t = f(x, \tau)$
Two-dimensional heat flow	$t = f(x, y)$	$t = f(x, y, \tau)$
Three-dimensional heat flow	$t = f(x, y, z)$	$t = f(x, y, z, \tau)$

For simplicity, solutions to majority of heat transfer problems are obtained by the one-dimensional analysis.

1.7. SIGNIFICANCE OF HEAT TRANSFER

The discipline of the heat transfer encompasses a great many fascinating areas like :

- Design of steam generators, condensers, and other heat exchange equipments in power plant engineering; solar energy conversion for space heating and for electric power production.
- IC engines, refrigeration and air-conditioning units, superheaters and condensers and many other cooling and heating appliances in mechanical engineering. The operation of refrigeration and air-conditioning units depends greatly on the effective transfer of heat in condensers and evaporators.
- Design of cooling systems for electric motors, generators and transformers in electrical engineering so that the heat generated during the flow of current through the windings of these machines can be effectively dissipated. This is to avoid the conditions which will cause overheating and damage the equipment. Heat transfer is often the controlling factor in the miniaturization of electronic systems.
- Evaporation, condensation, heating and cooling of fluids in chemical operations. Hardly any chemical operation can be identified that does not involve heating or cooling of a material at some stage or the other.
- Construction of dams and other heavy structures, calculation of thermal expansion of suspension bridges and railway tracks, minimisation of building-heat losses by means of improved insulation techniques.
- Proper functioning of valves and other controls operated by temperature changes, thermal control of space vehicles. Usually the orientation of a satellite is kept fixed. Obviously, side of the satellite facing the sun gets very hot. Maintenance of an optimum temperature is necessary for proper functioning of the instruments mounted on the satellite. For that, proper heat transfer is necessary from the hot to the comparatively cold side of the satellite.
- Heat treatment of metals where diffusion rate of carbon in steel is required to be made to estimate the period for which the steel component must be exposed to carburizing atmosphere.
- Dispersion of atmospheric pollutants; problem of thermal pollution associated with the discharge of large amounts of waste heat

from a power plant to environment. Industrial exhaust gases laden with noxious pollutants are discharged high enough. This is to ensure that by the time pollutants diffuse downwards, their concentration falls below safe limits. An understanding of mass transfer is needed for accurate predication of concentration at ground of the pollutants discharged from the chimney.

An engineer utilises his knowledge of heat transfer either to transmit heat in the most effective/economic way, or to protect his equipment against excessive heat gains or losses. The various engineering problems involving heat transfer can be categorised into two groups.

(i) Heat flow situations where maximum heat transfer is desirable with minimum possible heat exchange area. Gas turbine blades, walls of IC engines and combustion chambers, outer surface of a space vehicle all depend for their durability on rapid removal of heat from their surfaces. The design of heat exchangers is considered to be optimum under specified temperature conditions when maximum heat transfer occurs with minimum surface area.

(ii) Heat flow situations where heat transfer is undesirable and its flow is to be prevented. The walls of centrally heated buildings, and the steam pipes in a steam power plant are properly insulated to restrict heat losses.

With few exceptions, engineering problems involve more than one of the three modes of heat transfer and this aspect results into a complicated heat exchange pattern. The significance of heat transfer and the simultaneous occurrence of different modes of heat transfer can be well-judged by citing the following examples:

(i) Closed container filled with hot coffee and kept in a room whose air and walls are at a fixed temperature. Fig. 1.6.

All three models of heat transfer contribute towards cooling of coffee, and different paths for energy transfer from coffee are :

— free convection from the coffee to the flask, q_1

— heat conduction through the flask, q_2

— free convection from the flask to the air, q_3

— radiation exchange between the outer surface of the flask and the inner surface of the cover, q_4

— free convection from air to the cover, q_5

— heat conduction through the cover, q_6

— free convection from the cover to the room air, q_7

— radiation exchange between the outer surface of the cover and the surroundings, q_8

(ii) Automobile engine with thermo-syphon cooling system. Here the relevant heat transfer processes are :

— free convection and radiation from hot combustion gases to cylinder walls

— conduction through cylinder walls

— free convection from cylinder walls to water and from water to radiator tubes

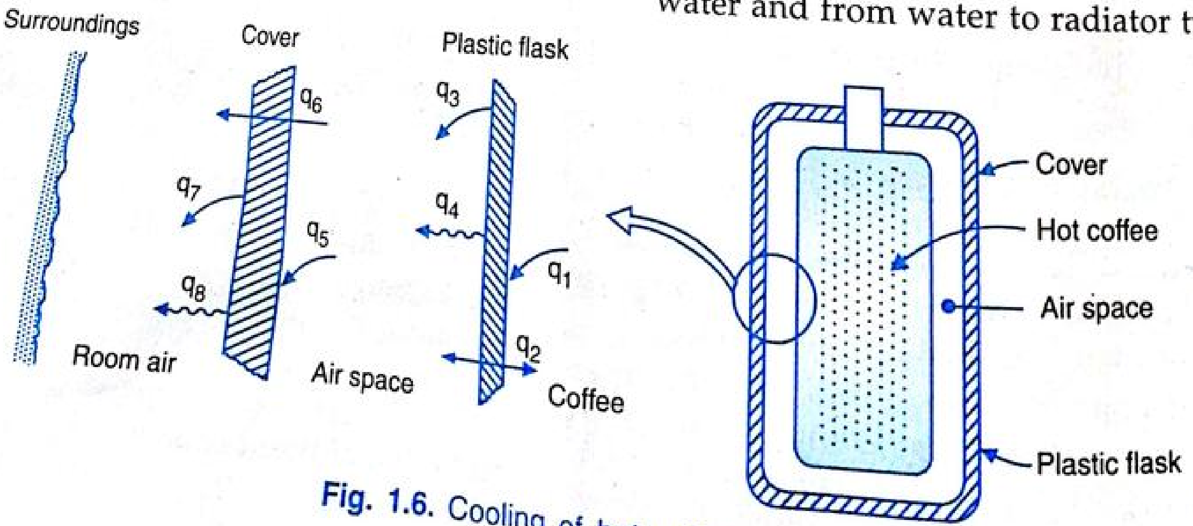


Fig. 1.6. Cooling of hot coffee

– conduction through walls of radiator tubes

– convection from radiator tubes to surrounding air

(*iii*) Heat flow through a wall consisting of two plates separated by vacuum (Fig. 1.7).

Heat is convected from the fluid at temperature t_{f1} to plate A, conducted through plate A, radiated from plate A to plate B, conducted through plate B, and finally convected from plate B to the fluid at temperature t_{f2}.

Likewise, in the process of steam generation, the boiler tubes receive heat from the products of combustion by all the three modes of heat transfer.

Undoubtedly, a study of the science of heat flow is must for an engineer so that he can understand the basic concepts of heat exchange and apply the same to heat flow situations encountered in engineering and physical problems.

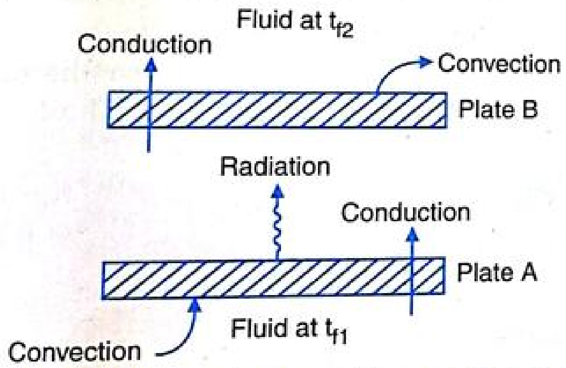


Fig. 1.7. Heat transfer through a composite wall

EXAMPLE 1.9.

(a) Cite an analogy that would be useful in fixing the concepts of heat conduction, convection and radiation. (b) Does convection strictly comply with the definition of heat transfer ?

Solution : (*a*) Consider that a house has caught fire which is being extinguished by people who carry water from well to the site of fire. Let the water be analogous to heat and the people be analogous to heat transfer medium. Then

- A person at the well delivers the water bucket to a person nearest to him who delivers the same to the next and so on till finally the bucket reaches the fire site. Here the water goes from the well to the site through the medium. This is analogous to heat conduction in solids.

- The person at the well becomes the single runner (represents the medium) and carries the water from the well to the site. This corresponds to heat convection in liquids and gases.

- The person, with the help of a hose, directs the water from the well to the house independently of the medium. The situation is reflected by thermal radiation in a vacuum or most gases.

(*b*) The mechanisms of heat conduction and radiation depend for their operation on the mere existence of temperature difference and as such constitute the fundamental physical mechanism of heat transfer. Convection does not strictly comply with this definition of heat transfer because it depends for its operation on mechanical mass transport also. Heat exchange between a solid wall and a fluid depends not only on temperature difference but also on mass transport of fluid. Convection may be likened to conduction through the film of fluid (gas or liquid) associated with transfer of energy due to mixing of fluid particles. Further, convection is strongly influenced by geometry of the surface, level of turbulence in the fluid and various fluid properties.

But since convection also accomplishes transmission of energy from regions of higher temperature to regions of lower temperature, the term "heat transfer by convection" has been generally accepted. Convection is a commonly occurring process and has been given the status of an independent mode.

EXAMPLE 1.10.

A 5 cm diameter steel pipe maintained at a temperature of 60°C is kept in a large room where the air and wall temperatures are 25°C. If the surface emissivity of the steel is 0.7, calculate the total heat loss per unit length of pipe if convective heat transfer coefficient is 6.5 W/m^2-deg. Comment on the result.

Solution : The pipe loses heat both by convection and radiation.

Heat loss by convection,

$$Q_{conv} = h\ (\pi dl)\ \Delta t$$
$$= 6.5 \times (\pi \times 0.05 \times 1)\ (60 - 25)$$
$$= 35.72 \text{ W}$$

Heat loss by radiation,

$$Q_{rad} = \varepsilon\ (\pi dl)\ \sigma_0\ (T_1^4 - T_2^4)$$
$$T_1 = 60 + 273 = 333 \text{ K};$$
$$T_2 = 25 + 273 = 298 \text{ K}$$
$$\therefore \quad Q_{rad} = 0.7 \times (\pi \times 0.05 \times 1) \times 5.67 \times 10^{-8}\ (333^4 - 298^4)$$
$$= 33.72 \text{ W}$$

Total heat loss

$$= Q_{conv} + Q_{rad}$$
$$= 35.72 + 33.72 = \mathbf{69.44 \text{ W}}$$

Comments: Both the convection and radiation heat loss are of almost equal amount. It would be a serious mistake to neglect either of the two.

EXAMPLE 1.11.

A surface at 475 K convects and radiates heat to the surroundings at 335 K. If the surface conducts this heat through a solid plate of thermal conductivity 12.5 W/m-deg, determine the temperature gradient at the surface in the solid. Take convective coefficient and radiation factor as 80 W/m²-deg and 0.9 respectively.

Solution : Under steady state conditions, the heat conducted through the plate equals the sum of convection and radiation heat losses. That is

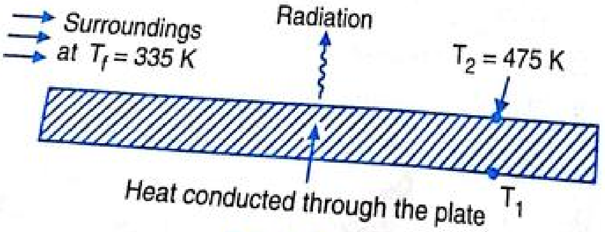


Fig. 1.8.

heat conducted through the plate
= convection heat losses
+ radiation heat losses

$$i.e., \ -kA\frac{dT}{dx} = hA\ (T_2 - T_f) + \varepsilon\ A\ \sigma_0\ (T_2^4 - T_f^4)$$

Taking unit area and substituting the relevant data, we have

$$-12.5\frac{dT}{dx} = 80\ (475 - 335) + 0.9 \times (5.67 \times 10^{-8})\ (475^4 - 335^4)$$
$$= 11200 + 1955 = 13155$$

$\therefore$ Temperature gradient through the plate,

$$\frac{dT}{dx} = -\frac{13155}{12.5} = \mathbf{-\ 1052.4°C/m}$$

EXAMPLE 1.12.

A surface at 200°C loses heat both by convection and radiation to the surroundings at 50°C. The convection coefficient is 75 W/m²K and the radiation factor due to geometric location and emissivity is 0.95. If the heat is conducted to the surface through a solid material of thermal conductivity 10 W/mK, determine the temperature gradient at the surface of the solid.

Solution : Under steady state conditions,

heat convected + heat radiated
= heat conducted

$$hA(T_1 - T_2) + \in A\ \sigma_b(T_1^4 - T_2^4) = -kA\frac{dT}{dx}$$

Given:

$$T_1 = 200 + 273 = 473 \text{ K}$$
and $$T_2 = 50 + 273 = 323 \text{ K}$$

Considering unit area and substituting the appropriate values, we obtain

$$75 \times 1(473 - 323) + 0.95 \times 1 \times 5.67 \times 10^{-8}\ (4734 - 323^4) = -10 \times 1 \times \frac{dT}{dx}$$

or $$11250 + 2110 = -10 \times 1 \times \frac{dT}{dx}$$

Thus, the temperature gradient at the surface of solid is

$$\frac{dT}{dx} = -\frac{11250 + 2110}{10} = \mathbf{-\ 1336 \text{ K/m}}$$

SALIENT POINTS

1. Thermodynamics helps to determine the quantity of work and heat interactions when a system changes from one equilibrium state to another.

   Heat transfer is concerned with the estimation of rate at which heat is transferred with in a medium, across an interface or from one surface to another. It also deals with temperature distribution in the medium, the duration of heating and cooling for a certain heat duty and the surface area required to accomplish that duty.

2. The three basic modes of heat transfer are:
   - *Conduction* is the transfer of heat within a medium or between different mediums which are in direct physical contact. Heat conduction occurs without appreciable displacement of the molecules constituting the medium.
   - *Convection* is the process of energy transport affected by circulation or mixing of one portion of the fluid with another. Convection is possible only in a fluid medium (gas, liquid or a powdery substance) and is directly linked with the transport of medium itself.

     Depending on the nature of forces which cause the mixing, convection heat transfer is classified as free or natural and forced convection.
   - *Radiation* is the transmission of heat in the form of radiant energy or wave motion without affecting the material medium between the heat source and the receiver. Radiation exchange requires no intervening space or medium, and, in fact, occurs most effectively in vacuum.

3. The basic laws which govern the heat transfer are:

   (*i*) Fourier's law of heat conduction

   $$Q = -kA\frac{dt}{dx}$$

   where Q is the heat transfer rate, A is the surface area perpendicular to the direction of heat flow, dt is the temperature difference for short perpendicular distance dx, and the thermal conductivity k is a characteristic of the surface material.

   (*ii*) Newton's law for the convective heat transfer between a surface and an adjacent fluid

   $$Q = hA\,(t_s - t_f)$$

   where Q is the convective heat flow rate, A is the area exposed to heat transfer, t_s and t_f are the surface and fluid temperatures respectively, and h is the coefficient of heat transfer.

   (*iii*) Stefon-Boltzman law of thermal radiation

   $$E_b = \sigma_b\, A\, T^4$$

   where E_b is the energy radiated per unit time, T and A are the absolute temperature and area of the radiating surface and σ_b is the Stefan-Boltzman constant.

4. Heat exchange is accompanied by space-time variation of temperature :

   $$t = f\,(x, y, z, \tau)$$

   Steady state implies that temperature at each point of a system remains constant in the course of time, *i.e.*, $\frac{dt}{d\tau} = 0$. Further, the heat transfer in a system may be in one, two or three dimensions.

5. The discipline of heat transfer encompasses a great many fascinating areas like thermal power plants, thermal control of chemical reactors, refrigeration and air-conditioning systems, design of cooling systems for electric motors-generators and transformers, construction of dams and other heavy structures, heat treatment of metals etc.

6. An engineer utilizes his knowledge of heat transfer either to transmit heat in the most effective/economic way, or to protect his equipment against excessive heat gains or losses.

7. With few exceptions, engineering problems involve more than one of the three modes of heat transfer and this aspect results into a complicated heat exchange pattern.

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. Comment on the validity of the statement : Transfer of heat is thermodynamically an irreversible process.
2. How does heat transfer differ from thermodynamics ? Is it true to say that heat transfer is essentially thermodynamics with rate equations added ?
3. What is the driving force for heat transfer ?

   (Ans: The temperature difference between two points in the same medium, or between two mediums which are in thermal contact represents the driving potential as applied to heat transfer problems.)
4. Distinguish between the conduction, convection and radiation modes of heat transfer.
5. Discuss the different modes by which heat can be transferred. Give suitable examples to illustrate your answer.
6. Cite few examples where conduction plays a major role.
7. What is the difference between natural and forced convection ? Does any convection process involve conduction to some extent ? Explain.
8. A person who sits in front of a fireplace feels warm. Through what process or processes of heat transfer does he receive heat ?
9. What will be your response to a person who states that heat can not be transferred in a vacuum ?
10. State the following basic laws which givern the heat transfer:

    Fourier's law, Newton's law of cooling and Stefan's Boltzman law.
11. State by giving illustrations that in practice the transfer of heat is the combined effect of conduction, convection and radiation.
12. Identify the different modes of heat transfer in the following systems/operations :
    - (*i*) steam raising in a steam boiler
    - (*ii*) air/water cooling of an IC engine cylinder
    - (*iii*) condensation of steam in a condenser
    - (*iv*) heat loss from a thermos flask
    - (*v*) heating of water in a bucket with an immersion heater
    - (*vi*) heat transfer from a room heater
    - (*vii*) thermo-regulator system of human body
13. Write the rate equations for the three modes of heat transfer. Define the symbols used and give the units for each.
14. Define and distinguish between (*i*) steady state, (*ii*) unsteady state, and (*iii*) transient state of heat transfer.
15. Clearly bring out the difference between one, two-, and three-dimensional temperature fields and heat flows.
16. Write some examples to illustrate the importance of heat transfer in various fields of engineering.
17. Determine the steady state heat transfer rate through a wall, 5 m long × 4 m high × 0.25 m thick, with its two faces maintained at uniform temperature of 100°C and 30°C. The wall is made of fire brick having thermal conductivity equal to 0.7 W/m-deg.

    (Ans. 3920 W)
18. A solar pane, 1 m × 1.25 m, receives solar radiation 1500 watts. Calculate surface temperature of the pane if the ambient temperature is 25°C and the convective heat transfer coefficient of the air film over the surface of pane is 12.5 W/m^2-deg.

    (Ans. 121°C)
19. A piece of aluminium is placed in a vacuum jar. The incident radiant energy from the sun is 950 W/m^2 and the aluminium absorbs 10% of the incident solar energy. If in the steady state 50% of absorbed energy is conducted to the surroundings and the remaining 50% is re-radiated to space, estimate the temperature of aluminium if its emissivity is 0.05.

    Hint : Energy re-radiated

$$= \frac{10}{100} \times 950 \times \frac{50}{100} = 47.5 \text{ W/m}^2$$

$$47.5 = \varepsilon \, \sigma_0 \, T_s^4$$

$$= 0.05 \times (5.67 \times 10^{-8}) \times T_s^4$$

$$T_s = \left[\frac{47.5}{0.05 \times 5.67 \times 10^{-8}}\right]^{0.25}$$

$$= 359.78 \text{ K}$$

The surface of steel plate measuring 0.9 m long × 0.6 m wide × 0.025 m thick is maintained at a uniform temperature of 300°C, and the plate loses 250 watt by radiation. If air at 15°C temperature and 20 W/m²-deg convective heat transfer coefficient blows over the plate, calculate the temperature on inside surface of the plate. Take thermal conductivity of plate as 45 W/m-deg.

Hint : $Q_{cond} = Q_{conv} + Q_{rad}$

$$-kA\frac{dt}{dx} = hA(t_s - t_f) + Q_{rad}$$

$$-45 \times (0.9 \times 0.6) \times \frac{300-t}{0.025}$$

$$= 20 \times (0.9 \times 0.6) \times (300 - 15) + 250$$

t = **303.42°C**

21. A black metal plate, 1 m in diameter and 1 cm thick, is exposed to sun's rays. The plate heats up to such a temperature that the rate at which it receives solar energy on one face equals the rate at which it loses heat by radiation and by convection from both surfaces. Presuming the following data, calculate the heat transfer by convection.

Solar energy flux = 200 W/m²
Net radiation heat flux leaving the plate = 80 W/m²
Plate temperature = 310 K,
and surrounding air temperature = 300 K

B. Fill in the blanks with appropriate word/words :

1. Heat transfer takes place according to .... law of thermodynamics.
2. The material medium between the heat source and receiver is not affected during the heat transmission by ....
3. The observable effect of conduction between different mediums in direct physical conatct is the equalization of ....
4. .... is possible only in a fluid medium and is directly linked with the transport of medium itself.
5. The expression $Q = -kA\frac{dt}{dx}$ prescribes the one-dimensional steady flow of heat by ....
6. The appropriate rate equation for convective heat transfer between a surface and adjacent fluid is prescribed by ....
7. The radiation energy emitted by a surface depends upon .... of its absolute temperature.
8. The process of heat transfer is thermodynamically an .... process.
9. .... implies that the properties of a system at any specified location are independent of time and there is no change in the .... of the system.

Answers : 1. second; 2. radiation; 3. temperature; 4. convection; 5. conduction; 6. Newton's law of cooling; 7. fourth-power; 8. irreversible; 9. steady state.

C. Multiple choice questions :

1. Heat transfer takes place according to .... law of thermodymics
   (*a*) Zeroth (*b*) First
   (*c*) Second (*d*) Third
2. The essential condition for the transfer of heat from one body to another is
   (*a*) both bodies must be in physical contact
   (*b*) heat content of one body must be more than that of the other
   (*c*) one of the bodies must have a high value of thermal conductivity
   (*d*) there must exist a temperature difference between the bodies
3. Identify the wrong statement
   (*a*) the process of heat transfer is thermodynamically an irreversible process
   (*b*) a material medium is always necessary for heat transmission
   (*c*) for heat exchange, a temperature gradient must exist
   (*d*) heat flow is always from a higher temperature to a lower temperature in accordance with second law of thermodynamics
4. Heat transmission is directly linked with the transport of medium itself,. *i.e.*, there is actual motion of heated particles during
   (*a*) conduction only
   (*b*) convection only
   (*c*) radiation only
   (*d*) conduction as well as radiation

5. The material medium between the heat source and receiver is not affected during the process of heat transmission by
   (*a*) conduction
   (*b*) convection
   (*c*) radiation
   (*d*) conduction as well as convection

6. Heat transfer in liquids and gases is essentially due to
   (*a*) conduction
   (*b*) convection
   (*c*) radiation
   (*d*) conduction and radiation put together

7. A satellite in space exchanges heat with the surroundings essentially by
   (*a*) conduction
   (*b*) convection
   (*c*) radiation
   (*d*) conduction and convection put together

8. A pipe carrying steam at 215°C traverses a room and heat is lost to the surrounding air at 27°C. The major fraction of heat loss to the surroundings will be essentially due to
   (*a*) conduction
   (*b*) convection
   (*c*) radiation
   (*d*) conduction and radiation put together

9. For quick warmth during a cold winter season, a person prefers to sit near a fire. Which of the following modes of heat transfer provides him the maximum heat ?
   (*a*) conduction from the fire
   (*b*) convection will be better if he is near the fire
   (*c*) combined effect of conduction and convection will be better near the fire
   (*d*) direct unimpeded radiation will provide quick warmth

10. All the three modes of heat transmission are involved in
    (*a*) melting of ice
    (*b*) cooling of a small metal casting in a quenching bath
    (*c*) heat flow through the walls of a refrigerator
    (*d*) automobile engine equipped with a thermo-syphon cooling system

11. Steady state heat flow implies
    (*a*) negligible flow of heat
    (*b*) no difference of temperature between the bodies
    (*c*) constant heat flow rate, *i.e.*, heat flow rate independent of time
    (*d*) uniform rate in temperature rise of a body

12. Most unsteady heat flow occurs
    (*a*) through the walls of a refrigerator
    (*b*) during annealing of castings
    (*c*) through the walls of a furnace
    (*d*) through lagged (insulated) pipes carrying steam

Answers :

1. (*c*)	2. (*d*)	3. (*b*)	4. (*b*)	5. (*c*)
6. (*b*)	7. (*c*)	8. (*c*)	9. (*d*)	10. (*d*)
11. (*c*)	12. (*b*)			

HINTS AND COMMENTS

3(*b*):

Radiation can be affected through vacuum or space devoid of any matter. Radiation exchange, in fact, occurs most effectively in vacuum.

Chapter

2

Fourier Equation and Thermal Conductivity

Learning objectives : The subject matter presented in this chapter will enable the readers to

- *understand the salient features of Fourier's law of heat conduction*
- *define thermal conductivity, thermal resistance and appreciate the analogy between heat conduction and flow of electricity through ohmic resistance*
- *appreciate the variation in thermal conductivity for different materials and under different conditions*
- *set up the general heat conduction equation in cartesian, cylindrical and spherical co-ordinates*
- *specify homogenous and isotropic material and define thermal diffusivity of a material.*

2.1. FOURIER EQUATION

Conduction is primarily a molecular phenomenon requiring temperature gradient as the driving force. Experimental evidence does indicate that steady-state one-dimensional flow of heat by conduction through a homogeneous material is given by the *Fourier Law*

$$Q = -k A \frac{dt}{dx}$$

$$q = \frac{Q}{A} = -k\frac{dt}{dx} \qquad \text{...(2.1)}$$

The heat flux q (heat conducted per unit time per unit area) flows along normal to area A in the direction of decreasing temperature. The units on each term are :

Q : rate of heat flow, kcal/hr or kJ/hr

A : area perpendicular to the direction of heat flow, m^2

dx : thickness of material along the path of heat flow, m

dt : temperature difference between the two surfaces across which heat is passing, degree kelvin K or degree centigrade C.

The ratio dt/dx represents the change in temperature per unit thickness, *i.e.*, the temperature gradient. The negative sign indicates that the heat flow is in the direction of negative temperature gradient, and that serves to make the heat flow positive. The proportionality factor k is called the ***heat conductivity* or *thermal conductivity*** of the material through which the heat propagates. Thermal conductivity of a material is one of its transport properties. Others are the viscosity associated with the transport of momentum, and the diffusion coefficient associated with the transport of mass. Thermal conductivity provides an indication of the rate at which heat energy is transferred through a medium by the diffusion (conduction) process. For a prescribed temperature gradient and geometric

parameters, the heat flow rate increase with increasing thermal conductivity.

The Fourier law is essentially based on the following assumptions :

- steady state conduction which implies that the time rate of heat flow between any two selected points is constant with time. This also means that the temperature of the fixed points within a heat conducting body does not change with time : $t \neq f(\tau)$.
- one-directional heat flow; only one space co-ordinate is required to describe the temperature distribution within the heat conducting body; $t = f(x)$. The surfaces in the y-and z-directions are perfectly insulated.
- bounding surfaces are isothermal in character, *i.e.*, constant and uniform temperatures are maintained at the two faces.
- isotropic and homogeneous material, *i.e.*, thermal conductivity has a constant value in all the directions.
- constant temperature gradient and a linear temperature profile.
- no internal heat generation.

Some essential features of the Fourier relation are enumerated below :

- Fourier law predicts how heat is conducted through a medium from a region of high temperature to a region of low temperature
- Fourier law is valid for all matter regardless of its state: solid, liquid or gas.
- Fourier law is a vector expression indicating that heat flow rate is normal to an isotherm and is in the direction of decreasing temperature.
- Fourier law cannot be derived from first principles; it is a generalization based on experimental evidence.
- Fourier law helps to define the transport property k, *i.e.*, the thermal conductivity of the heat conducting medium.

Assuming $dx = 1\text{m}$; $A = 1\text{m}^2$ and $dt = 1°$, we obtain

$$Q = k$$

Hence thermal conductivity may be defined as *the amount of heat conducted per unit time across unit area and through unit thickness, when a temperature difference of unit degree is maintained across the bounding surfaces.* The magnitude of thermal conductivity tells us how well a material transports energy by conduction.

The units of thermal conductivity are worked out from the Fourier law written in the form:

$$k = -\frac{Q}{A}\,\frac{dx}{dt}$$

$$\text{Thus, } [k] = \frac{\text{kcal}}{\text{hr}}\,\frac{1}{\text{m}^2}\,\frac{\text{m}}{\text{deg}}$$

$$= \text{kcal/m-hr-deg}$$

$$[k] = \frac{\text{kJ}}{\text{hr}}\,\frac{1}{\text{m}^2}\,\frac{\text{m}}{\text{deg}} = \text{kJ/m-hr-deg}$$

The unit kJ/m-hr-deg could also be specified as J/m-s-deg or W/m-deg and this is actually done while quoting the numerical values of thermal conductivity.

Note: While quoting the numerical value of thermal conductivity, the temperature difference of unit degree maintained across the boundary surfaces may be prescribed in degree Kelvin or degree centigrade. Since the temperature difference of 1 degree Kelvin equals temperature difference of 1 degree centigrade, the numerical value of thermal conductivity will be same when stated as kJ/m-hr-°C or kJ/m-hr-K. However, while using SI. units, the recommended practice is to specify thermal conductivity as kJ/m-hr-deg or W/m-deg.

Following conversion factors help to convert the thermal conductivity from MKS system of units into SI units.

$$1.0 \text{ kcal/m-hr-deg}$$

$$= 4.186 \text{ kJ/m-hr-deg}$$

$$= 1.163 \text{ W/m-deg}$$

2.2. THERMAL RESISTANCE

Observations indicate that in systems involving flow of fluid, heat and electricity, the flow quantity is directly proportional to the driving

potential and inversely proportional to the flow resistance. In a hydraulic circuit, the pressure along the path is the driving potential and roughness of the pipe is the flow resistance. The current flow in a conductor is governed by the voltage potential and electrical resistance of the material. Likewise, temperature difference constitutes the driving force for heat conduction through a medium.

From Ohm's law

$$\text{current } (i) = \frac{\text{voltage potential } (dV)}{\text{electrical resistance } (Re)}$$

and from Fourier's law

heat flow rate (Q)

$$= \frac{\text{temperature potential } (dt)}{\text{thermal resistance } (dx\,/\,kA)}$$

Obviously there is a one-one correspondence between the flow of electric current and heat, *i.e.*,

— electric current (amperes) is analogous to thermal heat flow rate (kJ/hr).

— electric voltage (volts) corresponds to thermal temperature difference (degree Kelvin).

— electric resistance (ohms) is analogous to quantity dx/kA. This quantity is called ***thermal resistance.***

Thermal resistance, $R_t = (dx/kA)$, is expressed in the units hr-deg/kcal or s-deg/J or deg/W. The reciprocal of thermal resistance is called ***thermal conductance*** and it represents the amount of heat conducted through a solid wall of area A and thickness dx when a temperature difference of unit degree is maintained across the bounding surfaces.

Sometimes the heat conducting capacity of a given physical system is expressed in terms of unit thermal resistance r_t and unit thermal conductance c

$$r_t = \frac{dx}{k} \quad \text{and} \quad c = \frac{1}{r_t} = \frac{k}{dx}$$

$$\therefore \quad Q = -kA\frac{dt}{dx} = kA\,\frac{(t_1 - t_2)}{dx}$$

$$Q = \frac{A\,(t_1 - t_2)}{r_t} = cA\,(t_1 - t_2) \qquad ...(2.1a)$$

The concept of thermal resistance is advantageously applied while making computations for heat flow.

2.3. THERMAL CONDUCTIVITY OF MATERIALS

Thermal conductivty is a property of the material and it depends essentially upon the material structure (chemical composition, physical state and texture), moisture content and density of the material, and operating conditions of pressure and temperature. The value of thermal conductivity may range from

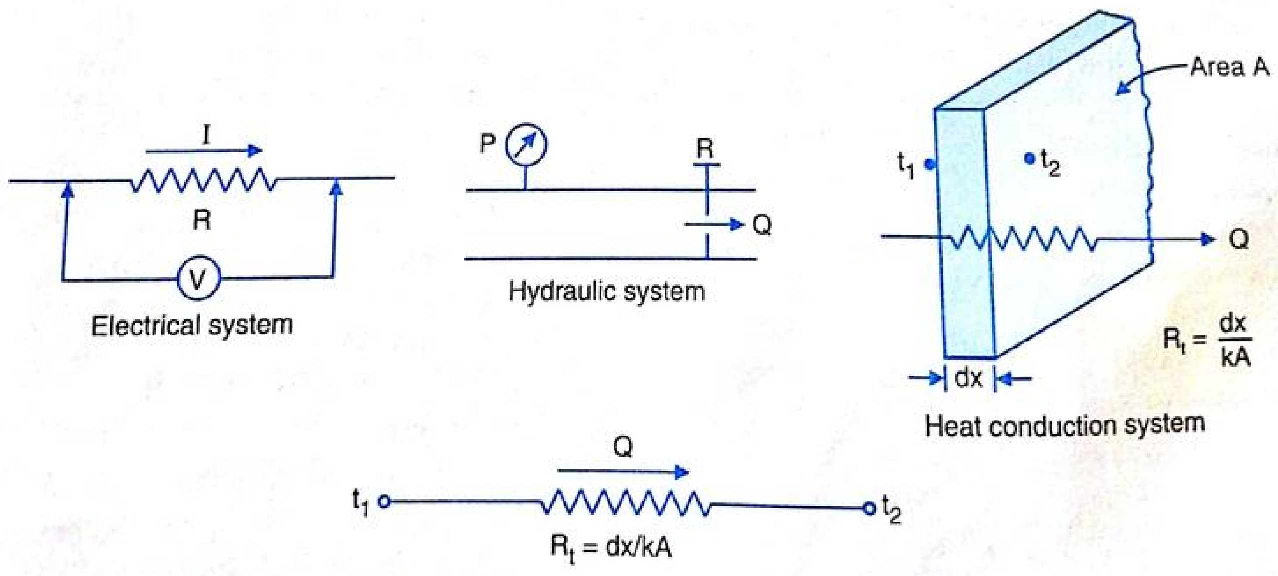


Fig. 2.1. Concept of thermal resistance

0.0083 W/m-deg for gases such as Freon-12 to as great as 410 W/m-deg for metals such as silver. Apparently, the pure silver has conductivity almost 50,000 times as great as that of Freon-12. Average values of thermal conductivity at normal pressure and temperature for selected materials of technical importance have been listed in Table 2.1.

Following remarks apply to the thermal conductivity and its variation for different materials and under different conditions :

1. A material is considered to be comprised of (*a*) free electrons, and (*b*) atoms which are bound in a periodic arrangement called lattice.

Accordingly, thermal conductivity of a material is the outcome of migration of free electrons and lattice vibrational waves. In metals the molecules are closely packed; molecular activity is rather small and so thermal conductivity is substantially due to the flow of free electrons. In fluids, the free electron movement is negligibly small and therefore thermal conductivity results primarily form the frequency of interactions between the lattice atoms.

Further, metals are the best conductors while liquids are generally poor conductors. Probably the disordered structure of the liquids

Table 2.1. Average Values of Thermal Conductivity at Normal Pressure and Temperature

Materials	Thermal Conductivity k	
	W/m-deg	kcal/m-hr-deg
Metals		
Aluminium	225	193
Brass	107	92
Copper	385	331
Cast iron	55 - 65	47 - 56
Steel	20 - 45	17 - 38
Silver	410	352
Construction Materials		
Concrete	1.20	1.03
Brick (masonary)	0.65	0.56
Brick (first clay)	0.75 - 1.75	0.5 - 1.5
Earth	0.138	0.119
Furnace or boiler slag	0.30	0.26
Glass (window)	0.75	0.65
Plaster	0.75 - 0.95	0.65
Insulating Materials		
Asbestos sheet	0.17	0.146
Cork, felt	0.05 - 0.10	0.043 - 0.086
Glass wool	0.03	0.032
Saw dust	0.07	0.06
Wood (Balsa)	0.052	0.045
Miscellaneous		
Air	0.024	0.021
Ash	0.12	0.103
Ice (ρ = 925 kg/m^3)	2.25	1.935
Water	0.55 - 0.7	0.047 - 0.6
Freon	0.0083	0.0096

and so of the gases is not conducive for transmitting molecular vibrations.

2. Thermal conductivity is always higher in the purest form of a metal. Alloying of metals and presence of other impurities cause an appreciable decrease in thermal conductivity. For instance, thermal conductivity of pure copper is 385 W/m-deg and that of nickel is 93 W/m-deg. Monel metal, an alloy of 30% nickel and 70% copper, has thermal conductivity of only 24 W/m-deg. Again for the same copper containing traces of arsenic, thermal conductivity reduces to 142 W/m-deg.

3. Mechanical forming (*i.e.*, forging, drawing and bending) or heat treatment of metal cause considerable variation in thermal conductivity. For instance, thermal conductivity of hardened steel is lower than that of annealed steel.

4. At elevated temperatures, thermal vibration of the lattice becomes higher and that retards the motion of free electrons. Consequently thermal conductivity of most metals decreases with temperature growth; aluminium and uranium are the exceptions. Thermal conductivity of aluminium stays almost constant within the temperature range of 130°C to 370°C. Most of the outer electrons of the uranium atoms are tied up in covalent bonds and as such the contribution of free electrons to the conduction process is small. Conduction of heat within uranium depends mainly on the vibration of atoms. The vibrational tendency increases with temperature rise and so does the thermal conductivity of uranium.

5. Heat transfer by conduction in gases occurs through transport of the kinetic energy of molecular motion resulting from the random movement and collisions of the molecules. From the concept of kinetic theory, mean travel velocity $\bar{V}$ of the gas molecules is prescribed by the relation :

$$\bar{V} = \sqrt{\frac{3\,GT}{M}} \qquad \text{...(2.2)}$$

where G is the universal gas constant, M is the molecular weight of the gas and T is the absolute temperature. Thermal conductivity is worked out from the relation

$$k = \frac{1}{3}\bar{V}c_v(l\rho) \qquad \text{...(2.3)}$$

where l is the mean free path (average distance travelled by a molecule before experiencing collision), c_v is the specific heat at constant volume and ρ is the mass density. Experimental investigations show that with increase in pressure the gas density increases; while the mean free path diminishes in the inverse proportion. However the product (ρl) practically remains constant indicating there by that thermal conductivity of gases does not depend upon pressure.

An increase in thermal conductivity of gases with rise in temperature may be attributed to an increase in mean travel velocity and specific heat at elevated temperatures. The increased agitations of gaseous molecules at elevated temperatures also results in greater frequency contact and an attendant increase in molecular exchange rates.

Equations 2.2 and 2.3 also stipulate that gases with higher molecular weight have small thermal conductivity than those with lower molecular weight. For example :

k for hydrogen (mol wt = 2)
= 0.190 W/m-deg

k for oxygen (mol wt = 32)
= 0.0272 W/m-deg

6. For liquids, the thermal conductivity is governed by the relation

$$k = \frac{A\,c_p\,\rho^{4/3}}{M^{1/3}} \qquad \text{...(2.4)}$$

The parameter A depends not on the nature of the liquid but on the temperature; the quantity (Ac_p) is nearly constant for all liquids. Thermal conductivity for liquids diminishes with rising temperature due to decrease in density with temperature increase.

7. Thermal conductivity is only very weakly dependent on pressure for solids and

for liquids as well, and essentially independent of pressure for gases at pressure near standard atmospheric.

8. For most materials, the dependence of thermal conductivity on temperature is almost linear

$$k = k_0\ (1 + \beta t) \qquad ...(2.5)$$

where, k_0 is the thermal conductivity at 0°C temperature, and β is a constant whose value depends upon the material. This constant may be positive or negative depending on whether thermal conductivity increases or decreases with temperature. The co-efficient β is usually positive for non-metals and insulation materials (exception magnesite bricks) and negative for metallic conductors (exceptions are aluminium and certain non-ferrous alloys).

The value of thermal conductivity increases with temperature for gases while it tends to decrease with temperature for most of liquids; water being notable exception.

9. Non-metallic solids do not conduct heat as efficiently as metals. For many of the building and insulating materials (concrete, stone, brick, glass wool, cork etc.) the thermal conductivity may vary from sample to sample due to variations in structure, composition, density and porosity.

Thermal conductivity of porous materials depends upon the type of gas or liquid existing in the voids. Presence of air filled pores and cavities reduce thermal conductivity because then the heat has to be transferred across many air spaces and air is known to be a poor heat conductor.

Thermal conductivity of a damp material is considerably higher than the thermal conductivity of the dry material and water taken individually. For dry brick $k = 0.35$ W/m-deg and for water $k = 0.60$ W/m-deg but for a damp brick $k = 1.0$ W/m-deg. This behaviour may be attributed to

(i) capillary movement of water with in the pores which results in convectional heat transfer,

(ii) properties of the absorbed moisture are different form those of free moisture.

Density is another parameter that affects the thermal conductivity of material; thermal conductivity increases with density growth. At densities 400 and 8000 kg/m^3, thermal conductivity values of asbestos are 0.105 and 0.248 W/m-deg respectively. Thermal conductivity of snow is also proportional to its density. With a density of 99.8 kg/m^3 thermal conductivity is 0.081 W/m-deg, and with density 598 kg/m^3 thermal conductivity increases to 0.627 W/m-deg. Since density of ice increases with decrease in temperature the variation of thermal conductivity with temperature would also follow the same pattern. This information helps to estimate the rate of ice formation on a lake or elsewhere.

Materials having a crystalline structure have a high value of thermal conductivity than the substances in amorphous form

For quartz (a solid with crystalline structure)

$$k = 30.5 \text{ W/m-deg at } -100°\text{C and}$$
$$= 10.4 \text{ W/m-deg at } +100°\text{C}$$

For pyrex (a substance of amorphous form)

$$k = 1.02 \text{ W/m-deg at } 0°\text{C and}$$
$$= 1.73 \text{ W/m-deg at } 500°\text{C}$$

Irregular arrangement of the atoms in case of amorphous solids inhibits the effectiveness of heat transfer by molecular impact.

10. Majority of engineering materials are isotropic, that is, their properties and constitution in the neighbourhood of any point are invariant with direction from that point. However, some materials exhibit some non-isotropic conductivities due to a directional preferences caused by a fibrous structure (as in the case of wood, asbestos etc.). Thermal conductivity of most types of wood is large in the direction parallel to the grain compared to that in a direction across the grain. Other materials with such characteristics include crystalline substances, laminated plastics and laminated metals. Conducting materials having this property are called anisotropic materials.

11. Based on experimental results, Wiedemann and Franz made the following

observations concerning thermal and electrical conductivities of a material.

"The ratio of the thermal and electrical conductivities is same for all metals at the same temperature; and that the ratio is directly proportional to the absolute temperature of the metal."

Let k and σ be the thermal and electrical conductivities of metal at a temperature T degrees absolute, then

$$\frac{k}{\sigma} \propto T$$

or $$\frac{k}{\sigma T} = \text{constant for all metals}$$

The constant is referred to as Lorenz number with the value

$$L_0 = 2.45 \times 10^{-8} \text{ W ohms/K}^2$$

This law holds good for a large number of metals between – 100°C and 100°C. At low temperatures the ratio k/σ decreases and the value tends to be zero at absolute zero. With decrease in temperature, the thermal and electrical conductivities of the metal increase. But the increase in the electrical conductivity is higher and its value tends to infinity at absolute zero. This corresponds to super conductivity state of the metal.

The Wiedemann and Franz law does suggest that materials that are good electrical conductors (pure metals viz copper and silver) are good conductors of heat too.

Since it is easier to measure electrical conductivity than thermal conductivity, use of this relation is often made to work out thermal conductivity of a material from the known value of its electrical conductivity.

12. Materials with large thermal conductivity are called thermal conductors, and those with small thermal conductivity are called thermal insulators. Insulating materials are used for obstructing the flow of heat between an enclosure and its surroundings.

(*a*) Low temperature insulation (cork, rock wool, glass wool, cattle hair, slag wool and thermocole etc.) are used when the enclosure is at a temperature lower than the ambient temperature and it is desired to prevent the enclosure from gaining heat.

(*b*) High temperature insulations (asbestos, diatomaceous earth, magnesia etc.) are used when it is desired to prevent an enclosure at a temperature higher than the ambient from losing heat to its surroundings.

(*c*) Super insulators include powders, fibres or multi-layer materials that have been evacuated of all air.

The low conductivity of insulating materials is due primarily to air (a poorly conducing gas) that is contained in the pores rather than the low conductivity of the solid substance.

Substances under low temperature conditions that have exceedingly high thermal conductivity are known as super conductors. For example thermal conductivity of aluminium reaches a value of 20000 W/m-deg at 10°K and this is over 100 times as large as the value that occurs at room temperature.

13. Pure metals possess the highest thermal conductivity, (k = 10 to 400 W/m-deg). Heat insulating and building materials have a comparatively low thermal conductivity, (k = 0.023 to 2.9 W/m-deg) and it ranges from 0.2 to 0.5 W/m-deg for liquids. Still gases and vapours possess the lowest thermal conductivity within the range 0.006 to 0.05 W/m-deg. Evidently the ratio of conductivity values between metals and good insulators is about 10^4 and it causes heat loss to be an important factor in thermal systems. In contrast, the ratio of electrical conductivity between good and poor conductors is about 10^{24} and therefore the current loss through insulation in electrical net works is negligible.

Range of thermal conductivity for various states of matter at normal temperature and pressure has ben depicted in Fig. 2.2.

14. Thermal conductivity of different materials decreases in the following order :

(*i*) Pure metals

(*ii*) Alloys

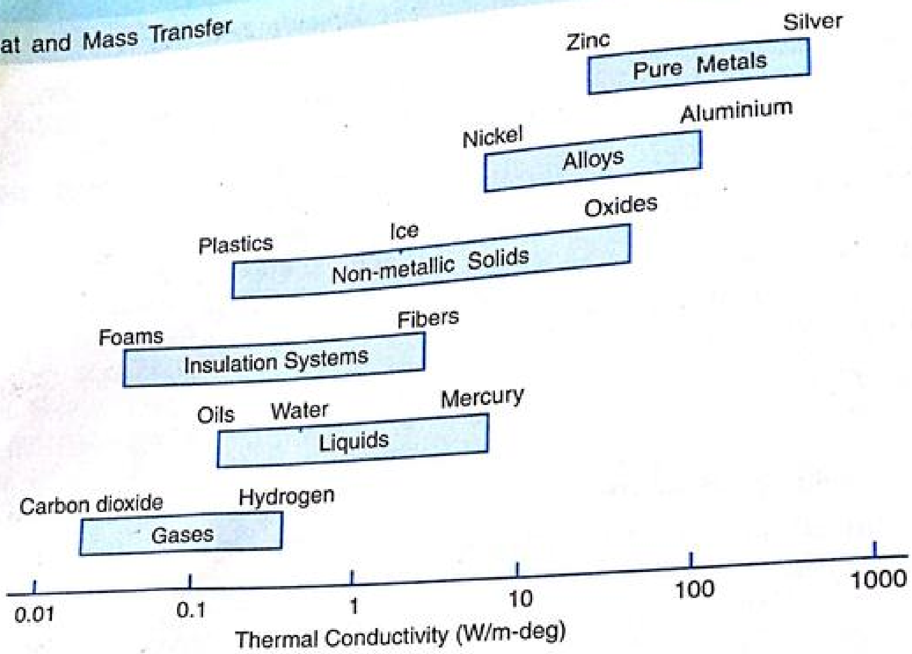


Fig. 2.2. Range of thermal conductivity for various states of matter

(*iii*) Non-metallic crystalline and amorphous substances

(*iv*) Liquids and

(*v*) Gases

2.4. GENERAL HEAT CONDUCTION EQUATION

The objective of conduction analysis is two fold :

(*i*) to determine the temperature distribution, *i.e.*, variation of temperature with time and position, and

(*ii*) to make computations for heat transfer etc.

Fourier law of heat conduction is essentially valid for heat flow under uni-directional and steady state conditions. However in many practical cases the temperature may be a function of space co-ordinate as well as time. Recourse is then made to three-dimensional heat flow equations which consider both non-uniformity of temperature and any irregularity in the boundary of the surface. To accomplish this task an elemental volume is taken, the relevant energy transfer processes are identified and the appropriate rate equations are introduced. Solution of the resulting differential equations yields the temperature distribution. Fourier rate equation is then invoked to workout the heat transfer rate through the conducting medium.

2.4.1. Cartesian Co-ordinates

Consider the flow of heat through an infinitesimal volume element oriented in a three-dimensional co-ordinate system (Fig. 2.3). The sides dx, dy and dz have been taken parallel to the x, y and z axis respectively.

Let t represent the temperature at the left face of the differential control volume. Since area of this face can be made arbitrarily small, the temperature t may be assumed uniform over the entire surface. The temperature changes along the x-direction and the rate of change is given by $\partial t/\partial x$. Then change of temperature through distance dx will be $(\partial t/\partial x)dx$. This temperature change has been graphically illustrated in Fig. 2.4. Therefore the temperature on the right face, which lies at a distance dx from the left face will be $[t + (\partial t/\partial x)dx]$.

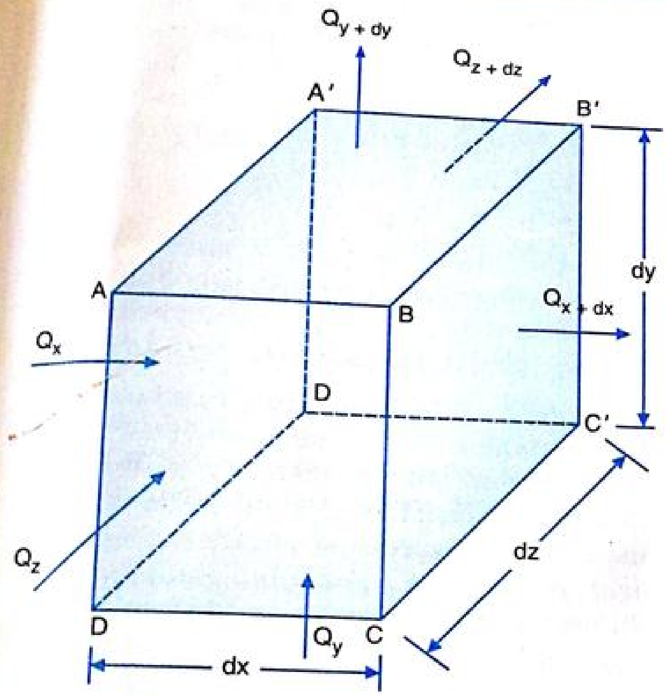


Fig. 2.3. Conduction analysis in cartesian co-ordinates

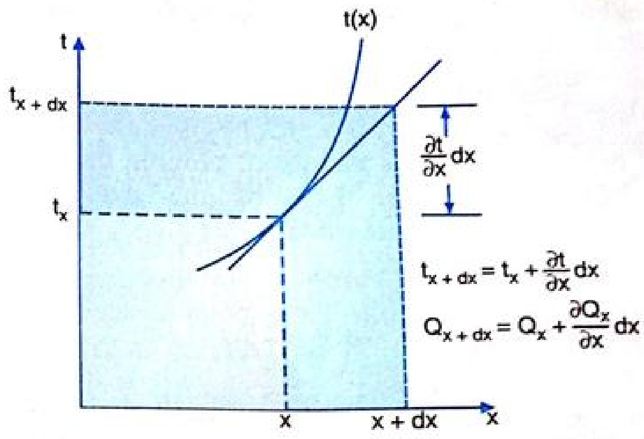


Fig. 2.4. Change in temperature as a function of distance

For non-isotrophic materials there will also be a change in thermal conductivity as heat flows through the control volume.

The general conduction equation can be set up by applying Fourier equation in each cartesian direction, and then applying the energy conservation requirement. If k_x represents the thermal conductivity at the left face, then quantity of heat flowing into the control volume through this face during time interval $d\tau$ is given by :

Heat influx Q_x

$$= -k_x\,(dy\,dz)\frac{\partial t}{\partial x}\,d\tau \qquad \text{...(2.6)}$$

During the same time interval the heat flow out of the right face of the control volume will be,

Heat efflux, Q_{x+dx}

$$= Q_x + \frac{\partial}{\partial x}\,(Q_x)dx \qquad \text{...(2.7)}$$

Equation 2.7 simply states that the x-component of heat transfer rate at $(x + dx)$ is equal to value of this component at x plus the amount by which it changes with respect to x times dx.

Accumulation of heat in the elemental volume due to heat flow in the x-direction is given by the difference between heat influx and heat efflux. Thus the heat accumulation due to heat flow in x-direction is

$$dQ_x = Q_x - \left[Q_x + \frac{\partial}{\partial x}(Q_x)\,dx\right]$$

$$= -\frac{\partial}{\partial x}\,(Q_x)\,dx$$

$$= -\frac{\partial}{\partial x}\left[-k_x\,(dy\,dz)\frac{\partial t}{\partial x}\,d\tau\right]dx$$

$$= \frac{\partial}{\partial x}\left[k_x\,\frac{\partial t}{\partial x}\right]dx\,dy\,dz\,d\tau \qquad \text{...(2.8)}$$

Likewise the heat accumulation in the control volume due to heat flow along the y- and z-directions will be :

$$dQ_y = \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) dx\, dy\, dz\, d\tau \qquad ...(2.9)$$

$$dQ_z = \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right) dx\, dy\, dz\, d\tau \qquad ...(2.10)$$

Sum of heat accumulations as prescribed by equations 2.8, 2.9 and 2.10 gives the total heat stored in the elemental volume due to heat flow along all the co-ordinate axes.

Total or net accumulation of heat is equal to

$$\left[\frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right)\right] dx\, dy\, dz\, d\tau \qquad ...(2.11)$$

There may be heat sources inside the control volume due to nuclear fission, flow of electric current in the coils of electric motors and generators, and ohmic heating of the material. If q_g is the heat generated per unit volume and per unit time, then the the total heat generated in the control volume equals to

$$q_g\, dx\, dy\, dz\, d\tau \qquad ...(2.12)$$

The total heat accumulated in the lattice due to heat flow along all the co-ordiante axes (Eq. 2.11) and the heat generated within the lattice (Eq. 2.12) together serve to increase the thermal energy of the lattice. This increase in thermal energy is reflected by the time rate of change in the heat capacity of the control volume and is given by :

$$\rho\, (dx\, dy\, dz)\, c \frac{\partial t}{\partial \tau} d\tau \qquad ...(2.13)$$

where ρ is the density and c is the specific heat of the material. Thus from energy balance considerations :

$$\left[\frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right)\right] \times dx\, dy\, dz\, d\tau + q_g\, dx\, dy\, dz\, d\tau$$

$$= \rho\, dx\, dy\, dz\, c \frac{\partial t}{\partial \tau} d\tau$$

Dividing both sides by $dx\, dy\, dz\, d\tau$,

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right) + q_g = \rho c \frac{\partial t}{\partial \tau} \qquad ...(2.14)$$

or, using the vector operator ∇,

$$\nabla \cdot (k \nabla t) + q_g = \rho c \frac{\partial t}{\partial \tau} \qquad ...(2.14a)$$

Equation 2.14 represents a volumetric heat balance which must be satisfied at each point for self generating, unsteady state three-dimensional heat flow through a non-isotropic material. This expression, known as the ***general heat conduction equation***, establishes in differential form the relationship between the time and space variation of temperature at any point of the solid through which conduction takes place. It should be noted that the heat generation term q_g may be a function of position or time, or both.

Homogeneous and isotropic material: A homogeneous material implies that the properties, *i.e.*, density, specific heat and thermal conductivity of the material are same everywhere in the material system. Isotropic means that these properties are not directional characteristics of the material, *i.e.*, they are independent of the orientation of the surface. Therefore for an isotropic and homogeneous material, thermal conductivity is same at every point and in all directions. In that case

$$k_x = k_y = k_z = k$$

and the differential equation 2.14 takes the form :

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{\rho c}{k}\frac{\partial t}{\partial \tau} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau} \qquad ...(2.15)$$

The quantity $\alpha = k/\rho c$ is called the ***thermal diffusivity***, and it represents a physical property of the material of which the solid

element is composed. Thermal diffusivity is an important characteristic quantity for unsteady conduction situations. By using the Laplacian operator ∇^2, the equation 2.15 may be written as:

$$\nabla^2 t + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau} \quad ...(2.15a)$$

Equation 2.15 governs the temperature distribution under unsteady heat flow through a homogeneous and isotropic material.

Different cases of particular interest are :

(*i*) In many situations there is no dependence of temperature on time. Conduction then occurs in the steady state, and the heat flow equation reduces to :

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = 0$$

$$\text{or } \nabla^2 t + \frac{q_g}{k} = 0 \quad \textbf{(Poisson's equation)} \quad ...(2.16)$$

In the absence of internal heat generation or release of energy within the body, equation 2.16 further reduces to:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0$$

$$\text{or} \quad \nabla^2 t = 0 \quad \textbf{(Laplace equation)} \quad ...(2.17)$$

(*ii*) Unsteady state heat flow with no internal heat generation gives:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

$$\text{or} \quad \nabla^2 t = \frac{1}{\alpha}\frac{\partial t}{\partial \tau} \quad \textbf{(Fourier equation)} \quad ...(2.18)$$

(*iii*) For one-dimensional and steady state heat flow with no internal heat generation, the general conduction equation takes the form :

$$\frac{d}{dx}\left(k\frac{dt}{dx}\right) = 0; \quad \frac{d^2 t}{dx^2} = 0 \quad ...(2.19)$$

Solution of these equations for any specific boundary conditions will yield the temperature distribution in the conducting material.

Thermal diffusivity: The following reflections can be made with regard to this physical property of the conducting material:

(*i*) Thermal diffusivity α of a material is the ratio of its thermal conductivity k to the thermal storage capacity ρc. The heat storage capacity essentially represents thermal capacitance or thermal inertia of the material, *i.e.*, its sluggishness to conduct heat. A high value of thermal diffusivity could result either from a high value of thermal conductivity or from low value of thermal capacity. Liquids have a low thermal conductivity, high thermal inertia and hence a small thermal diffusivity. Metals possess high thermal conductivity, low thermal inertia and hence a large thermal diffusivity.

(*ii*) Thermal diffusivity indicates the rate at which heat is distributed in a material, and this rate depends not only on the conductivity but also on the rate at which heat energy can be stored. An insight into equation 2.15 would reveal that larger the thermal diffusivity, higher would be the rate of change of temperature at any point of the material. Equalisation of temperature would then proceed at a higher rate in materials having large thermal diffusivity.

(*iii*) Temperature distribution in the unsteady state is being governed both by thermal conductivity as well as by thermal storage capacity. In contrast, during steady state heat condition (Eq. 2.16), thermal conductivity is the only property of the medium which influences the temperature distribution.

(*iv*) The relative magnitude of thermal diffusivity is a measure of the rapidity with which the material responds to sudden temperature changes in the surrounding. Metals and gases have relatively large value of α and their response to temperature changes is quite rapid. The non-metallic solids and liquids respond slowly to temperature changes because of their relatively small value of thermal diffusivity.

2.4.2. Cylindrical Co-ordinates

When heat conduction occurs through systems having cylindrical geometries (*e.g.*, conduction through rods and pipes) it is considered more convenient to work in the cylindrical co-ordiantes. The general heat equation can be set up by considering an infinitesimal cylindrical volume element

$$dV = (dr\ r d\phi\ dz)$$

and writing energy balance equations in the radial, tangential and axial directions.

Assumptions :

(*i*) thermal conductivity k, density ρ and specific heat c for the material do not vary with position.

(*ii*) uniform heat generation at the rate of q_g per unit volume per unit time,

(*a*) Radial direction ($x - \phi$ plane)

Heat influx, Q_r

$$= -k\,(r d\phi\ dz)\,\frac{\partial t}{\partial r}\,d\tau$$

Heat efflux, Q_{r+dr}

$$= Q_r + \frac{\partial}{\partial r}\,(Q_r)\,dr$$

Heat stored in the element due to flow of heat in the radial direction

$$\begin{aligned}
dQ_r &= Q_r - Q_{r+dr} \\
&= -\frac{\partial}{\partial r}\,(Q_r)\,dr \\
&= -\frac{\partial}{\partial r}\left[-k\,(r d\phi\ dz)\,\frac{\partial t}{\partial r}\,d\tau\right]dr \\
&= k\,(dr\ d\phi\ dz)\,\frac{\partial}{\partial r}\left(r\,\frac{\partial t}{\partial r}\right)d\tau \\
&= k\,(dr\ d\phi\ dz)\left(r\,\frac{\partial^2 t}{\partial r^2} + \frac{\partial t}{\partial r}\right)d\tau \\
&= k\,(dr\ r d\phi\ dz)\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\,\frac{\partial t}{\partial r}\right)d\tau
\end{aligned} \qquad ...(2.20)$$

(*b*) Tangential direction (r-z plane)

Heat influx Q_ϕ

$$= -k\,(dr\ dz)\,\frac{\partial t}{r\,\partial \phi}\,d\tau$$

Heat efflux $Q_{\phi+d\phi}$

$$= Q_\phi + \frac{\partial t}{r\,\partial \phi}\,(Q_\phi)\ r d\phi$$

Heat stored in the element due to flow in the tangential direction,

$$\begin{aligned}
dQ_\phi &= Q_\phi - Q_{\phi+d\phi} \\
&= -\frac{\partial t}{r\,\partial \phi}\,(Q_\phi)\ r\,d\phi
\end{aligned}$$

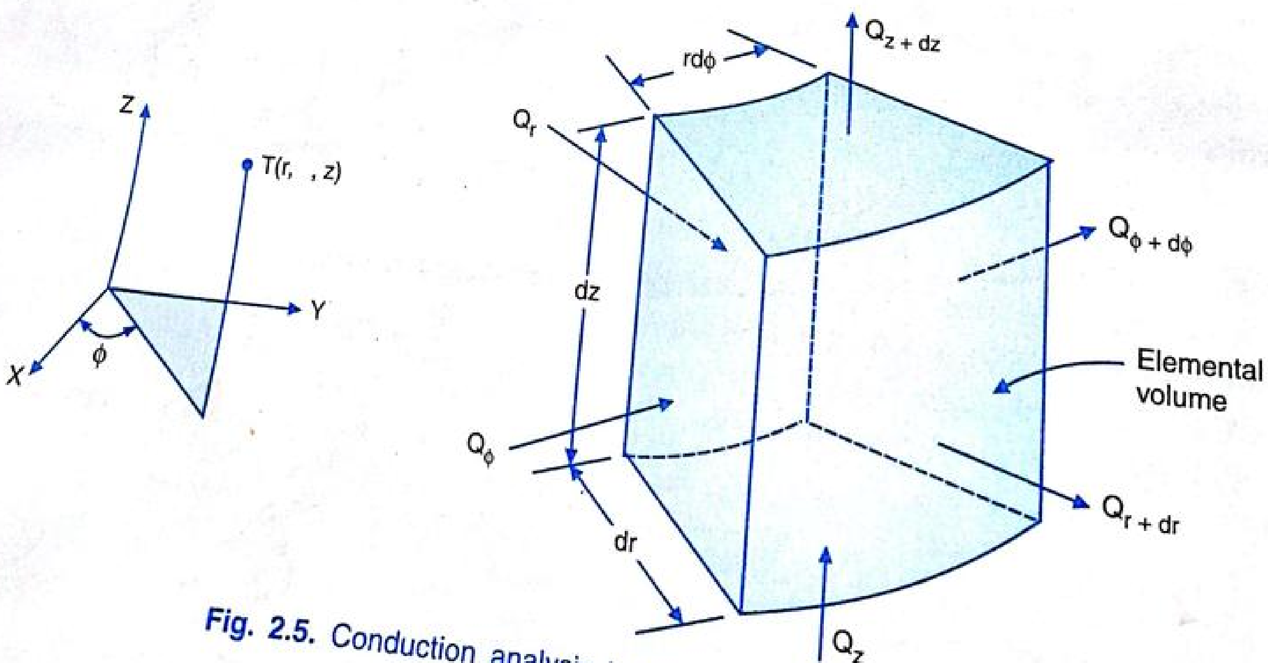


Fig. 2.5. Conduction analysis in cylindrical co-ordinates

$$= -\frac{\partial}{r\partial\phi}\left[-k\,(dr\,dz)\frac{\partial t}{r\partial\phi}d\tau\right]rd\phi$$

$$= k\,(dr\,d\phi\,dz)\,\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial t}{\partial\phi}\right)d\tau$$

$$= k\,(dr\,rd\phi\,dz)\frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2}\,d\tau \qquad ...(2.21)$$

(c) Axial direction ($r - \phi$ plane)

Heat influx Q_z

$$= -k\,(rd\phi\,dr)\frac{\partial t}{\partial z}\,d\tau$$

Heat efflux Q_{z+dz}

$$= Q_z + \frac{\partial}{\partial z}\,(Q_z)\,dz$$

Heat stored in the element due to heat flow in axial direction,

$$dQ_z = Q_z - Q_{z+dz}$$

$$= -\frac{\partial}{\partial z}\,(Q_z)\,dz$$

$$= -\frac{\partial}{\partial z}\left[-k\,(rd\phi\,dr)\frac{\partial t}{\partial z}d\tau\right]dz$$

$$= k\,(dr\,rd\phi\,dz)\frac{\partial^2 t}{\partial z^2}\,d\tau \qquad ...(2.22)$$

(d) Heat generated within the control volume

$$= q_g\,dV\,d\tau \qquad ...(2.23)$$

(e) Rate of change of energy within the control volume

$$= \rho\,dV\,c\frac{\partial t}{\partial\tau}\,d\tau \qquad ...(2.24)$$

From energy balance considerations, the rate of change of energy within the control volume equals the total heat storage plus the heat generated. Therefore,

$$k\,dV\left[\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2}+\frac{\partial^2 t}{\partial z^2}\right]d\tau + q_g\,dV\,d\tau$$

$$= \rho\,dV\,c\frac{\partial t}{\partial\tau}\,d\tau$$

Dividing both sides by $dV\,d\tau$

$$k\left[\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2}+\frac{\partial^2 t}{\partial z^2}\right] + q_g = \rho\,c\,\frac{\partial t}{\partial\tau}$$

or
$$\left[\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2}+\frac{\partial^2 t}{\partial z^2}\right] + \frac{q_g}{k} = \frac{\rho c}{k}\frac{\partial t}{\partial\tau} = \frac{1}{\alpha}\frac{\partial t}{\partial\tau} \qquad ...(2.25)$$

which is the general heat conduction equation in the cylindrical co-ordinates.

The heat conduction equation in cylindrical co-ordinates could also be obtained by doing the following co-ordinate transformation.

$$x = r\cos\phi;\quad y = r\sin\phi \quad\text{and}\quad z = z$$

Using the chain rule

$$\frac{\partial t}{\partial r} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial t}{\partial x}\cos\phi + \frac{\partial t}{\partial y}\sin\phi$$

or
$$\cos\phi\frac{\partial t}{\partial r} = \cos^2\phi\frac{\partial t}{\partial x} + \sin\phi\cos\phi\frac{\partial t}{\partial y} \qquad ...(i)$$

Also,
$$\frac{\partial t}{\partial\phi} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial\phi}$$

$$= \frac{\partial t}{\partial x}(-r\sin\phi) + \frac{\partial t}{\partial y}(r\cos\phi)$$

or
$$\frac{\sin\phi}{r}\frac{\partial t}{\partial\phi} = -\sin^2\phi\frac{\partial t}{\partial x} + \sin\phi\cos\phi\frac{\partial t}{\partial y} \qquad ...(ii)$$

From expressions (i) and (ii)

$$\frac{\sin\phi}{r}\frac{\partial t}{\partial\phi} = -\sin^2\phi\frac{\partial t}{\partial x} + \left(\cos\phi\frac{\partial t}{\partial r} - \cos^2\phi\frac{\partial t}{\partial x}\right)$$

$$= -\frac{\partial t}{\partial x} + \cos\phi\,\frac{\partial t}{\partial r}$$

$$\therefore \quad \frac{\partial t}{\partial x} = \cos\phi \frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\frac{\partial t}{\partial \phi}$$

Upon differentiation with respect to x,

$$\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\right) = \frac{\partial}{\partial x}\left[\cos\phi \frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\frac{\partial t}{\partial \phi}\right]$$

$$\frac{\partial^2 t}{\partial x^2} = \cos\phi \frac{\partial}{\partial r}\left(\frac{\partial t}{\partial x}\right) - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\frac{\partial t}{\partial x}\right)$$

$$= \cos\phi \frac{\partial}{\partial r}\left(\cos\phi \frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\frac{\partial t}{\partial \phi}\right)$$

$$- \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\frac{\partial t}{\partial \phi}\right)$$

$$= \cos^2\phi \frac{\partial^2 t}{\partial r^2} + \frac{\cos\phi\sin\phi}{r^2}\frac{\partial t}{\partial \phi}$$

$$+ \frac{\sin^2\phi}{r}\frac{\partial t}{\partial \phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 t}{\partial \phi^2}$$

$$+ \frac{\sin\phi\cos\phi}{r^2}\frac{\partial t}{\partial \phi} \quad ...(iii)$$

Similarly,

$$\frac{\partial^2 t}{\partial y^2} = \sin^2\phi \frac{\partial^2 t}{\partial r^2} + \frac{\cos^2\phi}{r}\frac{\partial t}{\partial r}$$

$$- \frac{\cos\phi\sin\phi}{r^2}\frac{\partial t}{\partial \phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 t}{\partial \phi^2}$$

$$- \frac{\cos\phi\sin\phi}{r^2}\frac{\partial t}{\partial \phi} \quad ...(iv)$$

Summation of expressions (*iii*) and (*iv*) gives

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2}$$

Substituting it into equation 2.15, we get the following general heat conduction equation in cylindrical coordinates system for a constant thermal conductivity material.

$$\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

which is same as equation 2.25.

For steady-state uni-directional heat flow in the radial direction, and with no internal heat generation, equation 2.25 reduces to

$$\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right) = 0$$

$$\text{or} \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dt}{dr}\right) = 0$$

Since, $\frac{1}{r} \neq 0$

$$\frac{d}{dr}\left(r\frac{dt}{dr}\right) = 0$$

$$\text{or} \quad r\frac{dt}{dr} = \text{constant} \quad ...(2.26)$$

2.4.3. Spherical Co-ordinates

The general heat conduction equation in spherical co-ordinates can be set up by considering an infinitesimal spherical volume element

$$dV = (dr \times r d\theta \times r \sin\theta\, d\phi)$$

and writing the heat balance equation for the r, θ and ϕ directions.

Assumptions : (*i*) Thermal conductivity k, density ρ and specific heat c for the material do not vary with position, (*ii*) uniform heat generation at the rate of q_g per unit volume per unit time.

(*a*) Heat flow through r-θ plane; ϕ-direction

Heat influx Q_ϕ

$$= -k\,(dr \times r d\theta)\frac{\partial t}{r\sin\theta\,\partial\phi}\,d\tau$$

Heat efflux $Q_{\phi + d\phi}$

$$= Q_\phi + \frac{\partial}{r\sin\theta\,\partial\phi}(Q_\phi)\, r\sin\theta\, d\phi$$

Heat stored in the elemental volume due to heat flow in the ϕ-direction

$$dQ_\phi = Q_\phi - Q_{\phi + d\phi}$$

$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(Q_\phi)\, r\sin\theta\, d\phi$$

$$= -\frac{1}{r\sin\theta\,\partial\phi}\frac{\partial}{\partial\phi}\left[-k\,(dr \times r d\theta) \times \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}d\tau\right] r\sin\theta\, d\phi$$

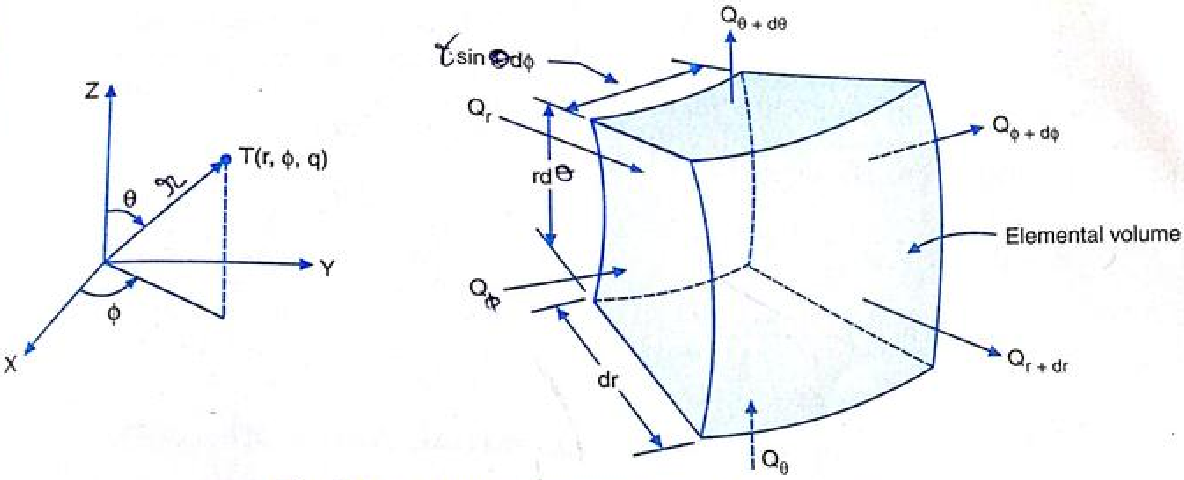


Fig. 2.6. Conduction analysis in spherical co-ordinates

$$= k\,(dr \times rd\theta \times r\sin\theta\, d\phi)\;\frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}\,d\tau \quad ...(2.27)$$

(*b*) Heat flow through r-ϕ plane; θ direction

Heat influx Q_θ

$$= -k\,(dr \times r\sin\theta\, d\phi)\,\frac{\partial t}{r\partial\theta}d\tau$$

Heat efflux $Q_{\theta + d\theta}$

$$= Q_\theta + \frac{\partial}{r\partial\theta}(Q_\theta)\, rd\theta$$

Heat stored in the elemental volume due to heat flow in the θ-direction

$$dQ_\theta = Q_\theta - Q_{\theta + d\theta}$$

$$= -\frac{\partial}{r\partial\theta}(Q_\theta)\, rd\theta$$

$$= -\frac{\partial}{r\partial\theta}\left[-k\,(dr \times r\sin\theta\, d\phi)\frac{\partial t}{r\,\partial\theta}d\tau\right] rd\theta$$

$$= \frac{k}{r}\,\frac{dr \times d\phi \times rd\theta}{r}\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right)d\tau$$

$$= k\,(dr \times rd\theta \times r\sin\theta\, d\phi) \times \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right)d\tau \quad ...(2.28)$$

(*c*) Heat flow in the θ–ϕ plane ; r-direction

Heat influx Q_r

$$= -k\,(rd\theta \times r\sin\theta\, d\phi)\frac{\partial t}{\partial r}\,d\tau$$

Heat efflux $Q_{r + dr}$

$$= Q_r + \frac{\partial}{\partial r}(Q_r)\, dr$$

Heat storage in the elemental volume due to heat flow in the r-direction

$$dQ_r = Q_r - Q_{r + dr}$$

$$= -\frac{\partial}{\partial r}(Q_r)\, dr$$

$$= -\frac{\partial}{\partial r}\left[-k\,(rd\theta \times r\sin\theta\, d\phi)\frac{\partial t}{\partial r}d\tau\right] dr$$

$$= k\, d\theta\,\sin\theta\, d\phi\, dr\,\frac{\partial}{\partial r}\left[r^2\frac{\partial t}{\partial r}\right]d\tau$$

$$= k\,(dr \times rd\theta \times r\sin\theta\, d\phi) \times \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial t}{\partial r}\right]d\tau \quad ...(2.29)$$

(*d*) Heat generated within the control volume

$$= q_g\, dV\, d\tau \quad ...(2.30)$$

(*c*) Rate of change of energy within the control volume

$$= \rho \, dV \, c \frac{\partial t}{\partial \tau} \, d\tau \quad ...(2.31)$$

From energy balance consideration, the rate of change of energy within the control volume equals the total storage plus the internal heat generation. Therefore,

$$k \, dV \left[\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \times \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) \right] d\tau + Q_g \, dV \, d\tau$$

$$= \rho \, dV \, c \frac{\partial t}{\partial \tau} \, d\tau$$

Dividing sides by $k \, dV \, d\tau$

$$\left[\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \times \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) \right] + \frac{q_g}{k}$$

$$= \frac{\rho c}{k} \frac{\partial t}{\partial \tau}$$

$$= \frac{1}{\alpha} \frac{\partial t}{\partial \tau} \quad ...(2.31a)$$

which is the general heat conduction equation in spherical co-ordinates.

The heat conduction equation in spherical co-ordinates could also be obtained by utilizing the following inter-relation between the rectangular and spherical co-ordinates.

$$x = r \sin \theta \sin \phi$$

$$y = r \sin \theta \cos \phi$$

$$z = r \cos \theta$$

For steady-state, uni-direction heat flow in the radial direction for a sphere with no internal heat generation, equation 2.31 can be rewritten as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dt}{dr} \right) = 0 \quad ...(2.32)$$

2.4.4. General One-dimensional Conduction Equation

The one-dimensional time dependent heat conduction equation can be written more compactly as a simple equation

$$\frac{1}{r^n} \frac{\partial}{\partial r} \left[r^n \, k \frac{\partial t}{\partial r} \right] + q_g = \rho \, c \frac{\partial t}{\partial \tau} \quad ...(2.33)$$

where $n = 0, 1$ and 2 for rectangular, cylindrical and spherical co-ordinates respectively. Further, while using rectangular co-ordinates it is customary to replace the r-variable by the x-variable.

2.5. INITIAL AND BOUNDARY CONDITIONS

The general heat conduction equation is of first order in time co-ordinates, and of second order in spatial co-ordinates. For the solution of single integration one constant is required, and for the solution of each double integration two constants are required. Obviously for the complete solution of general conduction equation, seven integration constants are needed. These constants are evaluated through a set of initial and boundary conditions.

The **initial conditions** describe the temperature distribution in a medium at the initial moment of time, and these are needed only for the transient (time-dependent) problems. The initial conditions can be expressed as

$$\text{At } \tau = 0; \quad t = t\,(x, y, z) \quad ...(2.34a)$$

For a uniform initial temperature distribution, a simple but typical form of the above identity can be recast as

$$\text{At } \tau = 0; \quad t = t_0 = \text{constant} \quad ...(2.34b)$$

The **boundary conditions** refer to physical conditions existing at the boundaries of the medium, and specify the temperature or the heat flow at the surface of the body. The typical boundary conditions are :

(*i*) *Prescribed surface temperature* : The temperature distribution t_s is prescribed at a bounding surface for each moment of time

$$t_s = t(x, y, z, \tau) \qquad \text{...(2.35)}$$

For a boundary condition of this type pertaining to slab depicted in Fig. 2.7, we have

At $\quad x = 0 : t(x, y, \tau) = 0$

$x = a : t(x, y, \tau) = f_1(y)$

$y = 0 : t(x, y, \tau) = 0$

$y = b : t(x, y, \tau) = f_2(x)$

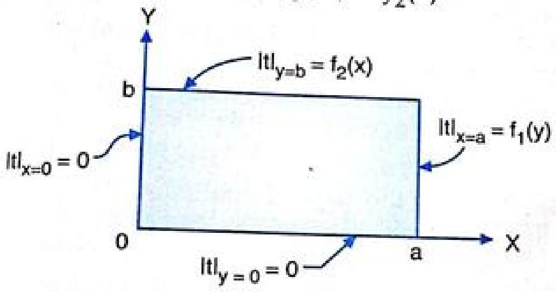


Fig. 2.7. Boundary condition-1st kind

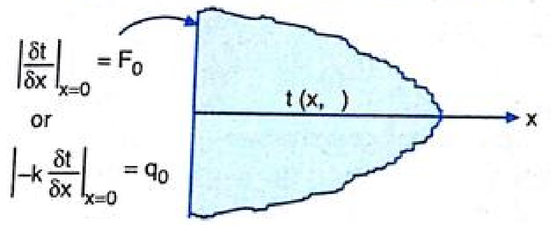


Fig. 2.8. Boundary condition-2nd kind

(*ii*) *Prescribed heat flux:* The heat flux is prescribed at the boundary surface, and is expressed as

$$-k \frac{\partial t(x, \tau)}{\partial x} = q_0 \quad \text{at} \quad x = 0$$

$$\text{or} \qquad \frac{\partial t}{\partial x} = -\frac{q_0}{k} = F_0 \quad \text{at} \quad x = 0 \qquad \text{...(2.36)}$$

Here $\frac{\partial t}{\partial x} = 0$ at $x = 0$ describes an insulated or adiabatic boundary. Such a condition can also exist at the plane of symmetry.

(*iii*) *Convective condition* : This condition is encountered at a solid boundary when there is equality between heat transfer to the surface by conduction and that leaving the surface convection (Fig. 2.9).

At $x = 0$:

$$h_1 [t_1 - t_{x=0}] = -k \left|\frac{\partial t}{\partial x}\right|_{x=0}$$

$$\text{or} \quad \left[k \frac{\partial t}{\partial x} + h_1 t\right]_{x=0} = h_1 t_1 = F_1 \qquad \text{...(2.37a)}$$

At $x = l$:

$$h_2 [t_{x=\delta} - t_2] = -k \left|\frac{\partial t}{\partial x}\right|_{x=\delta}$$

$$\text{or} \quad \left[k \frac{\partial t}{\partial x} + h_2 t\right]_{x=\delta} = h_2 t_2 = F_2 \qquad \text{...(2.37b)}$$

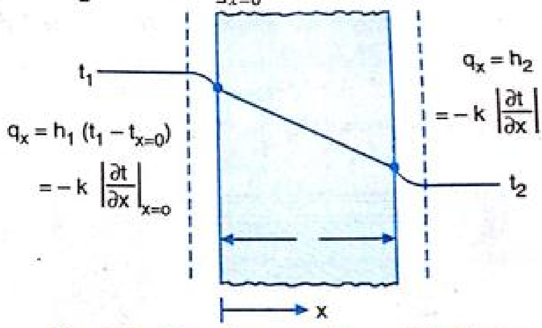


Fig. 2.9. Boundary condition -3rd kind

The convective boundary conditions described above can be expressed in the compact form as

$$\left[k \frac{\partial t}{\partial n_i} + h_i t\right]_{boundary} = F_i \qquad \text{...(2.38)}$$

where h_i is the specific convection coefficient, F_i is the prescribed function and $\partial t/\partial n_i$ is the temperature derivative at the bounding surface in the direction of outward normal.

It is rather difficult to achieve solution of general heat conduction in closed form due to complex nature of the problem. Finite difference and finite element types of numerical techniques are available and are used for the approximate solution of heat conduction problems.

2.6. MEASUREMENT OF THERMAL CONDUCTIVITY : GUARDED HOT PLATE METHOD

The guarded hot plate method has been recognized by scientists and engineers as most

dependable and reproducible for the measurement of thermal conductivity of insulating materials. It is a steady state absolute method suitable for materials which can be laid flat between two parallel plates and can be adopted for loose fill materials which can be filled between such plates.

Construction : An estimate of the thermal conductivity can be made by having a one-dimensional heat flow through the flat specimen, an arrangement for maintaining its faces at constant temperature, and some metering method to measure the heat flux through a known area.

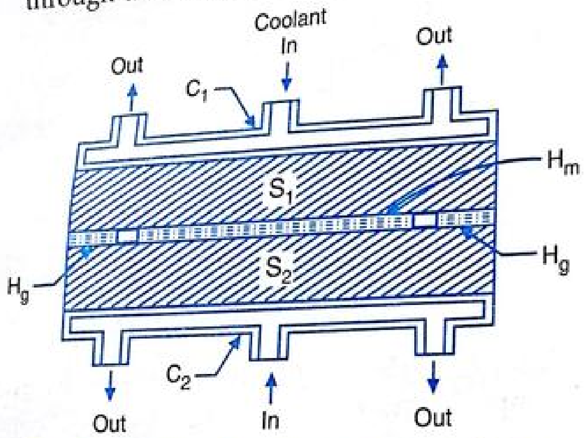


Fig. 2.10. Essential elements of guarded hot plate method

The essential elements of the experimental set-up, as indicated in Fig. 2.10 are :

(*i*) Main heater H_m placed at the centre of the unit. It is maintained at a fixed temperature by electrical energy which can be metered.

(*ii*) Guarded heater H_g which surrounds the main heater on its ends. The guarded heater is supplied electrical energy enough to keep its temperature same as that of the main heater. That ensures unidirectional heat flow and eliminates the distortion caused by edge losses. The entire input to the main heater by electrical input then leaves it only in a direction perpendicular to its axis.

The main and the guarded heater units are made up of mica sheets in which is wound closely spaced Nichrome wire. Quite often, the main and guarded heaters are surrounded longitudinally by copper surface plates with a view to provide even distribution of temperature.

(*iii*) Test specimens S_1 and S_2 which are placed on both sides of the heater or copper surface plates as the case may be.

(*iv*) Cooling unit plates C_1 and C_2 which communicate with the test specimens; water or some other cooling medium is circulated through them at a sufficiently high rate. This keeps them at a practically constant temperature; the temperature of the surface of these plates does not vary more than a fraction of degree.

(*v*) Thermocouples attached to the specimens at the hot and cold faces. The heater plate assembly together with cooling plates and the test specimens is held in position by vertical nuts and studs onto the base plate.

Desired measurements : From the Fourier rate heat equation for conduction we have :

$$Q = -kA\frac{dt}{dx} = \frac{kA}{\delta}(t_h - t_c)$$

or thermal conductivity k

$$= \frac{Q}{A}\frac{\delta}{(t_h - t_c)}$$

Apparently the following measurements need to be made :

(*i*) heat flow Q from the main heater through a test specimen; it will be half of the total electrical input to the main heater

(*ii*) thickness δ of the test-specimen

(*iii*) temperature drop across the specimen $(t_h - t_c)$; subscripts h and c refer to the hot and cold faces respectively

(*iv*) area A of heat flow; the area for heat flow is taken to be the area of main heater plus the area of one-half of air gap between it and the guarded heater

For specimens of different thicknesses, the respective temperatures at the hot and cold faces would be different and then the thermal conductivity is worked out from the relation:

$$k = \frac{Q}{A}\left[\frac{\delta_1}{(t_{h1} - t_{c1})} + \frac{\delta_2}{(t_{h2} - t_{c2})}\right]$$

where suffix 1 is for the upper specimen and suffix 2 is for lower specimen. Here Q is the total electrical input to the main heater.

EXAMPLE 2.1.

(a) Explain the mechanism of heat conduction in metals and insulators.

(b) List the various factors which influence the thermal conductivity of a substance. In what way, the conductivity is affected by the solid, liquid and gaseous phase of the substance?

(c) Mention some of the situations where poor conductivity of air helps to restrict the heat transmission by conducion.

Solution : In insulators (glass, wood, asbestos) conduction of heat takes place due to vibration of atoms about their mean positions. When heat is given to one part of an insulating substance, atoms belonging to that part are put in a violent state of agitation and start vibrating with greater amplitudes. Consequently these more active particles collide with less active atoms lying next to them. During collision, the less active atoms also get excited *i.e.*, thermal energy is imparted to them. The process is repeated layer after layer of molecules/atoms until the other part of the insulator is reached.

In metals besides atomic vibrations, there are large number of free electrons which also participate in the process of heat conduction. When a temperature difference exists between the different parts of the metal, a general drift of these free electrons occurs in the direction of decreasing temperature. It is this drift of free electrons which makes the metals so much better as conductors than other solids. These free electrons account for the observed proportionality between the thermal and electrical conductivities of pure metals.

The electrons do not contribute to heat conductivity in insulators. Electrons in an insulators are not free but fixed in the valence band. According to band structure of solids, the energy gap between valence band and conduction band is quite large and electrons cannot move to conduction band and contribute towards heat as well as electrical conductivity.

Insulators have low value of thermal conductivity due to their porosity, which may contain air.

(*b*) Many factors are known to influence the thermal conductivity of a material such as :

- chemical composition of the substance or substances of which it is composed,
- gaseous, liquid and solid phase in which the substance exists,
- crystalline, amorphous and porous structure of the substance,
- temperature and pressure to which the substance is subjected,
- homogeneous or non-homogeneous character of the material.

The factors with the greatest influence are the chemical composition, phase change and temperature. For a particular material, only the temperature effect has to be accounted for.

Generally a liquid is a better conductor than a gas, and a solid is a better conductor than a liquid. This aspect can be best illustrated by considering the three phases of mercury.

— Mercury is solid at 193°C and has a thermal conductivity of 48 W/m-deg (55.8 kcal/m-hr-deg).

— At 0°C, mercury becomes liquid and its thermal conductivity drops to 8.0 W/m-deg (9.3 kcal/m-hr-deg).

— Mercury acquires the gaseous phase at 200°C and then has a thermal conductivity as low as 0.034 W/m-deg (0.0395 kcal/m-hr-deg).

A partial explanation to this aspect stems from the fact that while in a gaseous state, the molecules of a substance are spaced relatively far apart and their motion is random. Obviously then the energy transfer by molecular impact is much more slow than in

the case of a liquid where the motion is still random but in which the molecules are more closely packed. The same is true concerning the difference between the thermal conductivity of the liquid and solid phases. However, many other factors become important when the substance transforms to solid state.

(c) Listed below are the applications where poor conductivity of air restricts the heat transmission by conduction:

(i) Eskimos make double walled glass houses; air is enclosed between the walls and that reduces the outflow of heat from the inside of houses.

(ii) Woollen fibres are rough and hence have fine pores filled with air. Both wool and air are bad conductors of heat and do not allow the body heat to flow to the atmosphere.

(iii) Two thin blankets are warmer than a single blanket of double the thickness because the two blankets enclose between them a layer of air. A simple blanket of double the thickness does not have air entrapped in it, and so it does not provide as good an insulation as the two thin blankets.

(iv) Birds often swell their features to enclose air and thus prevent the outflow of body heat.

SALIENT POINTS

1. Fourier law of heat conduction

$$Q = -kA\frac{dt}{dx} = -\frac{dt}{R_t}$$

The factor k is called the thermal conductivity of the material through which the heat propagates.

2. Thermal conductivity may be defined as the amount of heat conducted per unit time across unit area and through unit thickness, when a temperature difference of unit degree is maintained across the bounding surface.

3. The unit of thermal conductivity is specified as

kJ/m-hr-deg or J/m-s-deg or W/m-deg

4. The term $R_t = \frac{dx}{kA}$ is called thermal resistance and it is expressed in units of hr-deg/kJ or s-deg/J or deg/W.

The reciprocal of thermal resistance is called thermal conductance.

5. Thermal conductivity of different materials decreases in the following order:

(i) pure metals

(ii) alloys

(iii) non-metallic crystalline and amorphous substances

(iv) liquids and

(v) gases

6. For most materials, the dependence of thermal conductivity on temperature is almost linear

$$k = k_0(1 + \beta t)$$

The coefficient β is usually positive for non-metals and insulation materials and negative for metallic conductors. Further, thermal conductivity of a damp material is considerably higher than the thermal conductivity of dry material and water taken separately.

7. The general heat conduction equation in Cartesian co-ordinates is

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial x^2} + \frac{q_g}{k} = \frac{\rho c}{k}\frac{\partial t}{\partial \tau} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

By using the Laplacian operator ∇^2,

$$\nabla^2 t + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

This equation governs the temperature distribution under unsteady heat flow through a material which is homogeneous and isotropic.

(i) The quantity $\alpha = \frac{k}{\rho c}$ is called thermal diffusivity. Metals and gases have a large value of α and their response to temperature changes is quite rapid.

(ii) When the system is in steady states, i.e.,

$$\frac{\partial t}{\partial \tau} = 0$$

$$\nabla^2 t + \frac{q_g}{k} = 0 \qquad \text{(Possion's equation)}$$

(*iii*) In the absence of internal heat generation

$$\nabla^2 t = \frac{1}{\alpha}\frac{\partial t}{\partial \tau} \qquad \text{(Fourier's equation)}$$

(*iv*) If the system is in steady state and there is no heat generation

$$\nabla^2 t = 0 \qquad \text{(Laplace equation)}$$

(*v*) For one-dimensional and steady state heat flow with no internal heat generation

$$\frac{d^2 t}{dx^2} = 0$$

8. General heat conduction in cylindrical co-ordinates is

$$\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

9. General heat conduction in spherical co-ordinates

$$\left[\frac{1}{r^2 \sin\theta}\frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial t}{\partial t}\right)\right] + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. (*a*) Write the Fourier rate equation for heat transfer by conduction. Give the units and physical significance of each term appearing in this equation.
   (*b*) Why there is a negative sign in the Fourier's law of heat conduction?
2. Define thermal conductivity, thermal resistance and thermal conductance. What is the approximate range of thermal conductivity of solids, liquids and gases?
3. How are Fourier's law and Ohm's law similar?
4. List some good conductors of heat; some poor conductors.
5. State the effect of impurities on the thermal conductivity of a metal.
6. How thermal conductivity is affected by the nature of solid state (crystalline or amorphous)?
7. Explain the mechanism of thermal conduction in gases, liquids and solids. Discuss the effect of temperature on thermal conductivity.
8. Name the various types of insulating materials used in engineering, and mention the specific purpose for which they are used.
9. Which aspect makes the thermal conductivity of insulating materials lower than that of metals?
10. Point out and explain the various factors which affect the thermal conductivity of a material.
11. Explain why :
    (*i*) Steel is a better conductor of heat than bricks.
    (*ii*) Quilt is better insulator than a woolen blanket of the same thickness.
    (*iii*) Marble floor appears colder than cemented floor in winter though both are at the same temperature.
    (*iv*) Eskimos make double walled houses of blocks of ice.
12. Define thermal diffusivity and explain its physical significance.
13. Comment upon the validity of following statements :
    - Good electrical conductors are always good conductors of heat.
    - Thermal conductivity is the ability of solids to conduct heat and thermal diffusivity is a measure of thermal inertia.
    - Thermal conductivity of a pure metal is always higher than that of its alloys.
    - Thermal conductivity of liquids is generally higher than that of gases and vapours.
14. Show that for an incompressible homogeneous fluid with no internal heat sources, the energy equation can be expressed by

$$\frac{Dt}{D\tau} = \alpha \nabla^2 t$$

where t, τ and α respectively denote temperature, time and thermal diffusivity, and ∇^2 represents the Laplacian operator.

15. Prove that the 3-dimensional conduction equation in Cartesian co-ordinates for a homogeneous material, steady state conditions

and without heat generation is given by the Laplace equation

$$\nabla^2 t = 0$$

where ∇^2 represents the Laplacian operator. Deduce therefrom an expression for one-dimensional steady state heat conduction through a slab.

16. Derive a 3-dimensional general conduction equation in cylindrical co-ordinates for a homogeneous material. Deduce therefrom an expression for unidirectional unsteady state system when heat is generated within it at the rate of q_g per m^3 of the material.

17. Establish the general differential equation in Cartesian co-ordinates for three-dimensional unsteady heat conduction by considering an infinitesimal volume element. Deduce therefrom the conduction equations for the following cases:
    (*i*) unsteady state two-dimensional flow with heat generation at uniform rate within the material.
    (*ii*) steady one-dimensional flow without heat generation.

18. Write down the general conduction equation of unsteady state of heat flow and with uniform heat generation for rectangular system of co-ordinates. Then transform this equation in (*i*) cylindrical and (*ii*) spherical system of co-ordinates.

19. Derive the general conduction equation for (*i*) cylindrical co-ordinates (*ii*) spherical co-ordinates, the system being with uniform heat generation and unsteady state.

20. Write a general conduction equation for one-dimensional heat flow with uniform heat generation and unsteady state for rectangular, cylindrical and spherical co-ordinate systems.

21. What are boundary and initial conditions? How many boundary conditions are needed to solve a second order differential equation for heat conduction.

22. State and explain the different types of boundary conditions applied to heat conduction problems.

B. Fill in the blanks with appropriate word/words :

1. Conduction is primarily a .... phenomenon requiring .... gradient as the driving forces.
2. Thermal conductivity represents the amount of heat conducted across unit .... through unit .... when a temperature difference of unit degrees is maintained across the bounding surface.
3. The reciprocal of thermal resistance is called ....
4. Thermal conductivity is always .... in the purest form of a metal.
5. Materials with small thermal conductivity are called thermal ....
6. For an .... material the properties and constitution in the neighbour hood of any point are invariant with direction from that point.
7. Materials having a crystalline structure have a .... value of thermal conductivity than the substance in the atmosphous form.
8. The ratio of the thermal conductivity and .... is same for all metals at the same temperature.
9. Thermal diffusivity of a material is the ratio of its thermal conductivity to thermal ....
10. For an isotropic and homogenous material, the expression $\nabla^2 t = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$ is the .... equation for unsteady state heat flow with no internal heat generation.

Answers : 1. molecular, temperature; 2. area, distance; 3. thermal conductance; 4. high; 5. insulators; 6. isotropic; 7. high; 8. electrical conductivity; 9. storage; 10. Fourier.

C. Multiple choice questions :

1. Consider the following statements :
   The Fourier heat conduction equation

$$Q = -k A \frac{dt}{dx}$$

   presumes
   1. Steady state conditions
   2. Constant value of thermal conductivity
   3. Uniform temperature at the wall surfaces
   4. One-dimensional heat flow.

   Which of these statements are correct ?
   (*a*) 1, 2 and 3 (*b*) 1, 2 and 4
   (*c*) 2, 3 and 4 (*d*) 1, 3 and 4

2. Which of the followings is a wrong statement with respect to the Fourier's heat conduction equation?
(a) Fourier law is valid for all matter regardless of its state
(b) Fourier law is a vector expression indicating heat flow in the direction of decreasing temperature
(c) Fourier law can be derived from first principles
(d) Fourier law helps to define thermal conductivity of the heat conducting medium

3. Thermal conductivity is defined as the heat flow per unit time
(a) when temperature gradient is unity
(b) when a unit temperature difference is maintained across the opposite faces of the wall
(c) through a unit thickness of the wall
(d) across unit area when temperature gradient is unity

4. The rate of heat transfer per unit area per unit thickness of wall when a unit temperature difference is maintained across the opposite faces of the wall is called
(a) thermal loading
(b) thermal conductivity
(c) thermal resistance
(d) heat flux

5. All of the followings are units of thermal conductivity except
(a) kcal/m-hr-°C (b) kJ/m-hr-K
(c) W/m-s-K (d) cal/cm -s-°C

6. In M-L-T-θ system (T being time and θ temperature), what is the dimension of thermal conductivity?
(a) $ML^{-1}T^{-1}\theta^{-3}$ (b) $MLT^{-1}\theta^{-1}$
(c) $ML\theta^{-1}T^{-3}$ (d) $ML\theta^{-1}T^{-2}$

7. The heat conducted through a wall of thickness δ is given by

$$Q = -kA\frac{dt}{dx} = -kA\frac{(t_2 - t_1)}{\delta}$$

Which amongst the followings is then not a correct statement ?
(a) the term (δ/kA) is called thermal resistance
(b) the term (kA/δ) is called thermal conductance
(c) the factor (Q/A) is called thermal loading
(d) the temperature gradient (dt/dx) is positive

8. Most metals are good conductor of heat because of
(a) energy transport due to molecular vibration
(b) migration of neutrons from hot end to cold end
(c) lattice defects such as dislocations
(d) presence of many free electrons and frequent collision of atoms
(e) capacity to absorb free energy of electrons

9. Molecular transmission of heat is smallest in case of
(a) gases (b) liquids
(c) solids (d) alloys

10. In which one of the following materials, the heat energy propagation due to conduction heat transfer will be minimum?
(a) lead (b) copper
(c) water (d) air

11. Heat conduction in gases is due to
(a) motion of electrons
(b) elastic impact of molecules
(c) mixing motion of the different layers of the gas
(d) electromagnetic waves

12. Indicate the metal with highest value of thermal conductivity
(a) steel (b) silver
(c) copper (d) saw dust

13. Mark the matter with least value of thermal conductivity
(a) air (b) water
(c) ash (d) window glass

14. Which of the following forms of water have the highest value of thermal conductivity
(a) boiling water (b) steam
(c) solid ice (d) melting ice

15. The average thermal conductivities of water and air conform to the ratio
(a) 5 : 1 (b) 15 : 1
(c) 25 : 1 (d) 50 : 1

16. Identify the very good insulator
(a) saw dust (b) glass wool
(c) cork (d) asbestos sheet

17. Identify the most widely used heat insulating material for pipelines carrying steam
(*a*) cotton
(*b*) asbestos
(c) saw dust
(*d*) 85% magnesia cement and glass wool

18. Cork is a good insulator because
(*a*) it is flexible and can be cast into rolls
(*b*) it can be powdered
(*c*) it is porous
(*d*) its density is low

19. Which amongst the following least affects the thermal conductivity of wood
(*a*) moisture
(*b*) density
(*c*) temperature
(*d*) grain orientation, *i.e.*, direction parallel to or across the grain

20. Which of the following is anisotropic, *i.e.*, exhibits change in thermal conductivity due to directional preferences
(*a*) wood (*b*) glass wool
(*c*) concrete (*d*) masonary brick

21. Choose the false statement
(*a*) thermal conductivity is always higher in the purest form of metal
(*b*) heat treatment causes considerable variation in thermal conductivity
(*c*) thermal conductivity of a damp material is considerably higher than the thermal conductivity of the dry material and water taken individually
(*d*) thermal conductivity decreases with increase in the density of the substance

22. Consider the following parameters
(1) composition (2) density
(3) porosity (4) structure
Then, thermal conductivity of glass wool varies from sample to sample because of variation in
(*a*) 1 and 2 (*b*) 1 and 3
(*c*) 1, 2 and 3 (*d*) 1, 2, 3 and 4

23. The thermal conductivity k and the electrical conductivity σ of a metal at absolute temperature T are related as
(*a*) $\frac{k}{\sigma}$ = constant (*b*) $\frac{k}{\sigma T}$ = constant
(*c*) $\frac{k\sigma}{T}$ = constant (*d*) $\frac{kT}{\sigma}$ = constant

24. Consider the following materials:
1. Carbon 2. Mica
3. Bakelite 4. Fibre glass
Which of these materials are good conductors of heat but bad conductors of electricity?
(*a*) 1 only (*b*) 2 only
(*c*) 2 and 3 (*d*) 3 and 4

25. The diffusion equations
$$\nabla^2 t + q_g = \frac{1}{\alpha}\frac{dt}{dr}$$
governs the temperature distribution under unsteady heat flow through a homogeneous and isotropic material. The Fourier equation follows from this expression when
(*a*) there is no dependence of temperature on time
(*b*) there is no internal heat generation
(*c*) steady state conditions prevail and there is no internal heat generation
(*d*) there is no internal heat generation but unsteady state conditions prevail

26. The relation $\nabla^2 t = 0$ is referred to as
(*a*) Fourier heat conduction equation
(*b*) Laplace equation
(*c*) Poisson's equation
(*d*) Lumped parameter solution for transient conduction

27. Choose the wrong statement about thermal diffusivity
(*a*) it represents a physical property of the material
(*b*) it is a dimensionless quantity
(*c*) it is an important characteristic for unsteady heat conduction
(*d*) it is the ratio of thermal conductivity to thermal storage capacity of a material

28. Which one of the following expresses the thermal diffusivity of a substance in terms of thermal conductivity k, mass density ρ and specific heat c?
(*a*) $\rho^2 k c$ (*b*) $\frac{1}{\rho k c}$

(c) $\frac{k}{\rho c}$ (d) $\frac{\rho c}{k^2}$

29. The unit of thermal diffusivity is
(a) m²/hr °C (b) kcal/m²-hr
(c) m/hr °C (d) m²/hr

30. In M-L-T-θ system, the dimensions of thermal diffusivity are
(a) $L^2 T^{-1}$ (b) $L T^{-1} \theta^{-1}$
(c) $M L^2 T^{-1}$ (d) $L^2 T^{-1} \theta^{-1}$

Answers :

1. (d)	2. (c)	3. (d)	4. (b)	5. (c)
6. (c)	7. (d)	8. (d)	9. (a)	10. (d)
11. (b)	12. (b)	13. (a)	14. (c)	15. (c)
16. (b)	17. (d)	18. (c)	19. (c)	20. (a)
21. (d)	22. (d)	23. (b)	24. (a)	25. (d)
26. (b)	27. (b)	28. (c)	29. (d)	30. (a)

HINTS AND COMMENTS

1(d):

A constant value of thermal conductivity is not the essential condition for the Fourier heat conduction equation.

2(c):

Fourier law cannot be derived from first principle; it is a generalisation based on experimental evidence.

12(b):

$$k_{silver} > k_{copper} > k_{sawdust} > k_{steel}$$

15(c):

$$k_{water} = 0.55 - 0.7 \text{ W/m-deg}$$
$$k_{air} = 0.024 \text{ W/m-deg}$$

16(b):

$$k_{glass\ wool} > k_{cork} > k_{sawdust} > k_{asbestos}$$

21(d):

Thermal conductivity increases with increase in density of the substance.

27(b):

Thermal diffusivity has units of m²/s and dimensions of L^2T^{-1}.

Chapter

3 Steady State Conduction

Learning objectives : A study of the subject matter included in this chapter will enable the readers to

- *analyse the steady state one-dimensional heat conduction for temperature distribution and rates of heat transfer through*
  - *a plane wall and a composite well in which thermal resistancy are connected in series as well as in parallel*
  - *a hollow cylinder and a multilayered cylinder*
  - *a spherical shell and a multilayered spherical vessel*
- *explain the heat conduction through a wall/cylinder/sphere separating two fluids, and the concept of overall heat transfer coefficient*
- *examine the effect of temperature-dependent thermal conductivity on temperature profile and heat flow rate*
- *appreciate the dependence of heat loss on thickness of insulation; define critical thickness of insulation and set up expressions for critical radius of insulation for a cylinder and a sphere*

This chapter concentrates on the use of diffusion rate equations to workout the steady state and one-dimensional heat transfer through bodies of simple geometries and with no internal heat generation. The term steady state refers to the condition which prevails in a heat conducting medium when temperatures at fixed points do not change with time. In one-dimensional heat flow, there is a single predominant direction in which temperature differential exists and obviously the heat flow takes place; heat flow in the other directions can be safely neglected. Evidently then only one space co-ordinate is required to describe the distribution of temperature within the heat conducting body. Though such situations rarely exist in real problems, fairly good estimates of heat conduction are made in the following cases by making the assumption of one-dimensional heat conduction.

- Heat flow through a plane wall at regions far removed from the edges. Edge effects are then neglected and the heat flow depends only on the co-ordinate measured normal to the plane of the wall.
- Heat flow through a very long hollow cylinder (such as a pipe) which is maintained at uniform temperatures on its inner and outer surfaces. Here the heat conduction is considered to depend only on radial distance as the co-ordinate.
- Heat flow from a very thin wire or rod whose ends are maintained at different temperatures. For a sufficiently thin wire, the

temperature can be taken as uniform over any cross-section.

The relevant differential heat equation is then suitably modified, and solved to obtain the temperature distribution by the use of the imposed boundary conditions. Subsequently, the Fourier law is applied to determine the heat flow rate.

3.1. CONDUCTION THROUGH A PLANE WALL

Consider one-dimensional heat conduction through a homogeneous, isotropic wall of thickness δ with constant thermal conductivity k and constant cross-sectional area A. The wall is insulated on its lateral faces and constant but different temperatures t_1 and t_2 are maintained at its boundary surfaces. Obviously temperature varies only in the direction normal to the wall and the temperature potential causes heat transfer in the positive x-direction.

Starting with general heat equation in cartesian co-ordinates

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial t}{\partial \tau} \quad ...(3.1)$$

With stipulations of

$$\frac{\partial t}{\partial \tau} = 0 \text{ (steady state)}$$

$$\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0 \text{ (one-dimensional)}$$

$$\frac{q_g}{k} = 0$$

(no internal generation of heat)

the conduction equation transforms to

$$\frac{\partial^2 t}{\partial x^2} = 0 \text{ or } \frac{d^2 t}{dx^2} = 0 \quad ...(3.2)$$

This second order differential equation can be twice integrated with respect to x to give

$$\frac{dt}{dx} = C_1 \text{ and } t = C_1 x + C_2 \quad ...(3.3)$$

The constants of integration are evaluated with regard to the boundary conditions relevant to the flow situation. Here the boundary conditions are the known temperatures. That is

$$t = t_1 \text{ at } x = 0 \text{ and } t = t_2 \text{ at } x = \delta$$

When these boundary conditions are applied to the equation 3.3 for temperature distribution,

$$t_1 = 0 + C_2 \text{ and } t_2 = C_1\delta + C_2$$

From these identities, the integration constants are obtained as

$$C_2 = t_1 \text{ and } C_1 = \frac{t_2 - t_1}{\delta}$$

Accordingly the expression for temperature profile becomes

$$t = C_1 x + C_2$$

$$= \left(\frac{t_2 - t_1}{\delta}\right)x + t_1 \quad ...(3.4)$$

or in the dimensionless form

$$\frac{t - t_1}{t_2 - t_1} = \frac{x}{\delta} \quad ...(3.4a)$$

The temperature distribution is thus linear across the wall. Since equation 3.4 does not involve thermal conductivity, a conclusion may be drawn that temperature distribution is independent of the material; whether it is steel, wood or asbestos.

Computations for heat flow can be made by substituting the value of temperature gradient into the Fourier equation

$$Q = -kA\frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{d}{dx}\left[t_1 + \frac{t_2 - t_1}{\delta}x\right] = \frac{t_2 - t_1}{\delta}$$

$$\therefore \quad Q = -kA\frac{t_2 - t_1}{\delta} = \frac{kA(t_1 - t_2)}{\delta} \quad ...(3.5)$$

Obviously the heat flow rate is a constant independent of x.

Alternatively, the Fourier rate equation may be used directly to determine the heat flow rate. Attention is focussed on an elementary strip of thickness dx located at a distance x from the reference plane. The temperature difference acorss the strip is

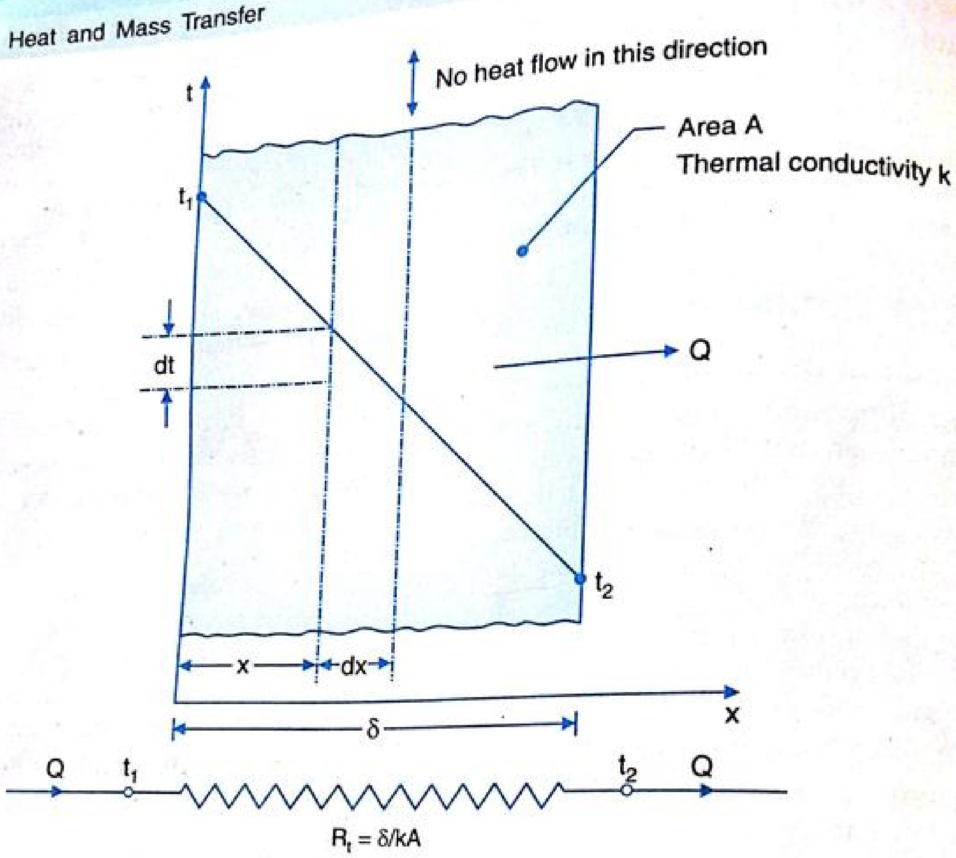


Fig. 3.1. Stready state conduction through a plane wall

dt and obviously the temperature gradient is *dt/dx*. Since *Q* is constant in a steady state, the terms of the Fourier rate equation

$$Q = -kA\,(dt/dx)$$

may be separated and integrated directly between the limits $t = t_1$ at $x = 0$ and $t = t_2$ at $x = \delta$. Thus

$$Q\int_0^{\delta} dx = -kA\int_{t_1}^{t_2} dt$$

$$Q\,\delta = kA\,(t_1 - t_2)\ ; \quad Q = \frac{kA\,(t_1 - t_2)}{\delta}$$

which is identical to equation 3.5 derived above.

The expression for steady state temperature distribution can be set up by integrating the Fourier rate equation between the limits :

(*i*) $x = 0$ where the temperature is stated to be t_1

(*ii*) $x = x$ where the temperature is to be worked out

Thus,

$$Q\int_0^{x} dx = -kA\int_{t_1}^{t} dt$$

$$Q\,x = -kA\,(t - t_1)\ ;$$

$$Q = \frac{kA\,(t_1 - t)}{x} \qquad \text{...(3.6)}$$

Comparing the expressions 3.5 and 3.6,

$$\frac{kA\,(t_1 - t)}{x} = \frac{kA\,(t_1 - t_2)}{\delta}$$

or

$$t = t_1 + \left(\frac{t_2 - t_1}{\delta}\right)x$$

or

$$\frac{t - t_1}{t_2 - t_1} = \frac{x}{\delta}$$

which is same as equation 3.4.

In terms of heat flow rate, the expresion for temperature distribution can be recast as

$$t = t_1 - \frac{Q}{kA}x = t_1 - \frac{q}{k}x$$

where $q = Q/A$ expresses the heat flow rate per unit area and is called ***heat flux***

The expression for heat flow rate, equation 3.5, can be written as

$$Q = \frac{t_1 - t_2}{\delta/kA} = \frac{t_1 - t_2}{R_t} \quad ...(3.7)$$

where $R_t = \delta/kA$ is the thermal resistance to heat flow. Equivalent thermal circuit for heat flow through a plane wall has been included in Fig. 3.1.

Let us develop the condition when weight, not the space, required for insulation of a plane wall is the significant criterion.

For one-dimensional steady heat condition

$$Q = \frac{kA(t_1 - t_2)}{\delta} = \frac{t_1 - t_2}{\delta/kA}$$

Thermal resistance of the wall,

$$R_t = \frac{\delta}{kA} \quad ...(i)$$

Weight of the wall,

$$W = \rho A \delta \quad ...(ii)$$

Eliminating the wall thickness δ from expressions (*i*) and (*ii*)

$$R_t = \frac{W}{\rho k A^2}$$

or $$W = (\rho\ k)\ R_t\ A^2 \quad ...(3.8)$$

Apparently when the product (ρk) for a given resistance is smallest, the weight of the wall would also be so. That stipulates the condition that the lightest insulation for a specified thermal resistance is that insulation which has the smallest product of density times thermal condcutivity.

EXAMPLE 3.1

(a) Fig. 3.2 shows the temperature profiles through a homogeneous plane wall. Comment on the direction of conduction heat flow.

(b) The interior of an oven is maintained at a temperature of 850°C by means of a suitable control apparatus. The oven walls are 500 mm thick and are fabricated from a material of thermal conductivity 0.3 W/m-deg. For an outside wall temperature of 250°C, workout the resistance to heat flow and the heat flow per square metre of wall surface. Also calculate the temperature at a point 200 mm from the interior side.

Solution : For conduction heat flow through a homogeneous medium, the rate of heat transfer is

$$Q = -kA\frac{dt}{dx}$$

Regarding temperature profile (*a*), the temperature increases with increasing value of *x*. Obviously the temperature gradient dt/dx would be positive. With positive dt/dx, the heat conducted works out to be negative which implies that the direction of heat flow is from right to left. Adopting a similar

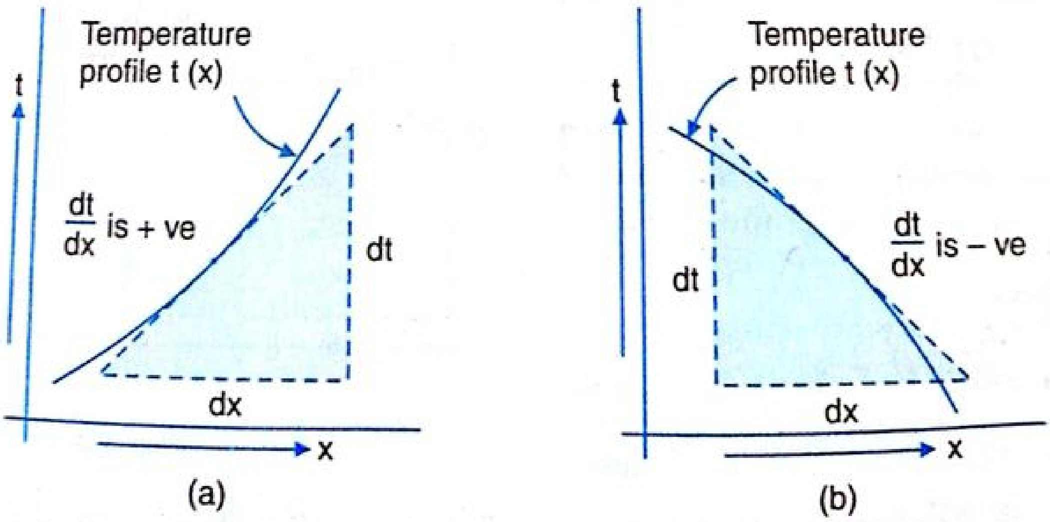


Fig. 3.2.

reasoning, the direction of heat flow would be from left to right conrresponding to temperature profile (*b*).

(*b*) For a plane wall of thickness δ, conductivity k and area A, the thermal resistance R_t is given by

$$R_t = \frac{\delta}{kA} = \frac{0.5}{0.3 \times 1} = \mathbf{1.667\ deg/W}$$

For one-dimensional steady state, the conduction heat flow through the wall is

$$Q = \frac{t_1 - t_2}{R_t} = \frac{850 - 250}{1.667} = \mathbf{359.92\ W}$$

The temperature at distance x from the interior side is

$$t = t_1 + \left(\frac{t_2 - t_1}{\delta}\right) x$$

$$= 850 + \left(\frac{250 - 850}{0.5}\right) \times 0.2$$

$$= \mathbf{610°C}$$

EXAMPLE 3.2

A plane slab of thickness δ = 60 cm is made of a material of thermal conductivity k = 17.5 W/m-deg. The left side of the slab absorbs a net amount of radiant energy from a radiant source at the rate q = 530 watt/m². If the right hand face of the slab is at a constant temperature t_2 = 38°C, set up an expression for temperature distribution within the slab as a function of relevant space co-ordinates. Therefrom work out the temperature at the mid-plane of the slab and the maximum temperature within the slab. It may be presumed that the temperature distribution is steady and there is no heat generation.

Solution : Under stipulations of steady state and no heat generation, the energy absorbed from the radiant source equals the rate at which it is conducted through the slab.

$$\text{Heat flux } q = \frac{k(t_1 - t_2)}{\delta}$$

$$530 = \frac{17.5(t_1 - 38)}{0.6}$$

$\therefore$ Temperature at the left side of slab,

$$t_1 = \frac{530 \times 0.6}{17.5} + 38 = \mathbf{56.17°C}$$

This also represents the maximum temperature within the slab.

From the expression for steady state temperature distribution,

$$t = t_1 + \frac{t_2 - t_1}{\delta} x$$

At the mid-plane x = 30 cm

$$\therefore \quad t = 56.17 + \frac{38 - 56.17}{0.6} \times 0.3$$

$$= \mathbf{47.08°C}$$

EXAMPLE 3.3

A homogeneous wall of area A and thickness δ has left and right hand surface temperatures of 0°C and 40°C respectively. Determine the temperature at the centre of the wall.

(*a*) *How much material must be added and to which side of the wall if the temperature at the centre is to be raised by 5°C ?*

(*b*) *How much material must be removed and from which side of the wall if the temperature at the centre line of the wall is to be lowered by 5°C?*

Express your answers in terms of δ. Presume that surface temperatures remain same before and after the alterations.

Solution : From the expression for steady state temperature distribution,

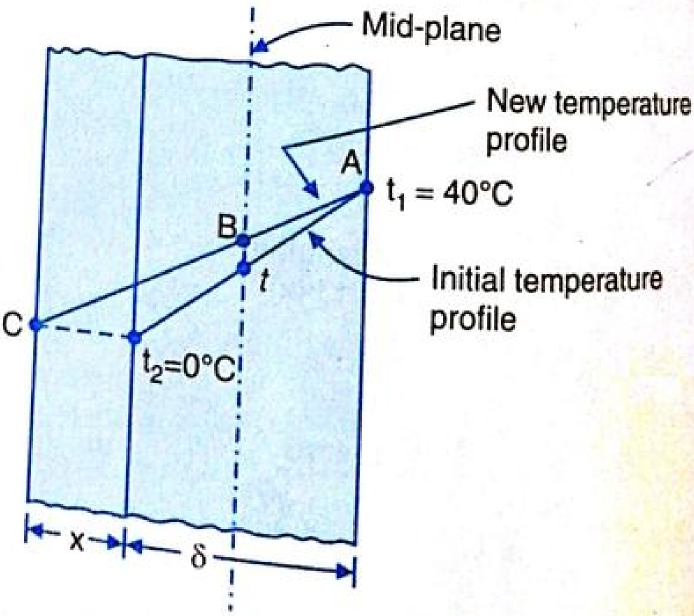


Fig. 3.3.

$$t = t_1 + \frac{t_2 - t_1}{\delta} x$$

At the mid-plane $x = \frac{\delta}{2}$

$$\therefore \quad t = 40 + \frac{0-40}{\delta} \times \frac{\delta}{2}$$
$$= \mathbf{20°C}$$

(*a*) Let x be the thickness of the material to be added to the left side of the wall to increase the temperature at the centre of the wall

$$t_B = (t + 5) = (20 + 5) = 25°C$$
$$t_C = t_2 = 0°C$$

After addition of material,

$$Q = Q_{AB}$$
$$= -kA\frac{t_A - t_B}{\delta/2}$$
$$= -kA\frac{40-25}{\delta/2} = -\frac{30\,kA}{\delta} \quad ...(i)$$

Also, $Q = Q_{BC} = -kA\frac{t_B - t_C}{\delta/2 + x}$

$$= -\frac{kA(25-0)}{\delta/2 + x} = \frac{-50\,kA}{\delta + 2x} \quad ...(ii)$$

Equating the expressions (*i*) and (*ii*) for heat flow through the wall,

$$-\frac{30\,kA}{\delta} = \frac{-50\,kA}{\delta + 2x}$$

or $30\,\delta + 60\,x = 50\,\delta$

$$\therefore \quad x = \frac{(50-30)\,\delta}{60} = \frac{\delta}{3}$$

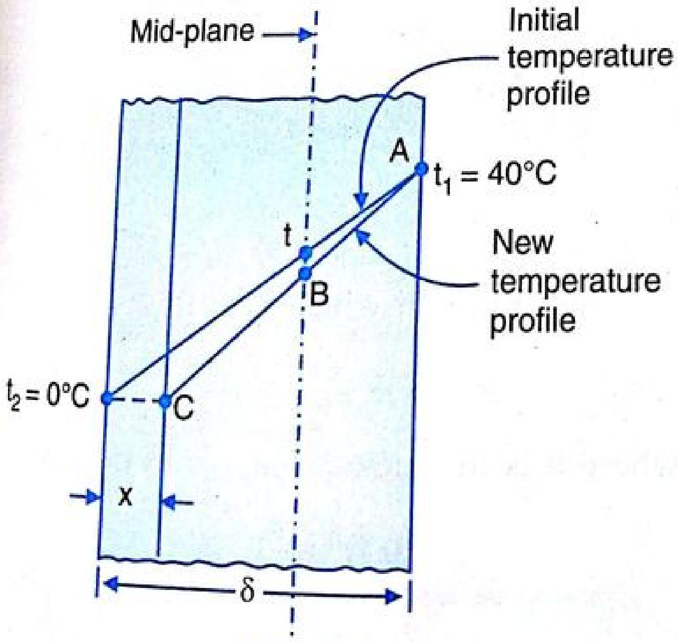


Fig. 3.4.

(*b*) Let x be the thickness of the material to be removed from the left side of the wall to lower the temperature at the mid-plane of the wall

$$t_B = (t - 5) = (20 - 5) = 15°C$$
$$t_C = t_2 = 0°C$$

After removal of material,

$$Q = Q_{AB} = -\frac{kA(t_A - t_B)}{\delta/2}$$
$$= -\frac{kA(40-15)}{\delta/2}$$
$$= -\frac{50\,kA}{\delta} \quad ...(i)$$

Also, $Q = Q_{BC} = -\frac{kA(t_B - t_C)}{\delta/2 - x}$

$$= -\frac{kA(15-0)}{\delta/2 - x} = -\frac{30\,kA}{\delta - 2x} \quad ...(ii)$$

Equating the expressions (*i*) and (*ii*) for heat flow through the wall,

$$-\frac{50\,kA}{\delta} = -\frac{30\,kA}{\delta - 2x}$$

or $\quad 50\,\delta - 100\,x = 30\,\delta$;

$$x = \frac{(50-30)\,\delta}{100} = \frac{\delta}{5}$$

EXAMPLE 3.4

A rod of 3 cm diameter and 20 cm length is maintained at 100°C at one end and 10°C at the other end. These temperature conditions are attained when there is heat flow rate of 6 watts. If cylindrical surface of the rod is completely insulated, determine the thermal condcutivity of the rod material.

Solution :

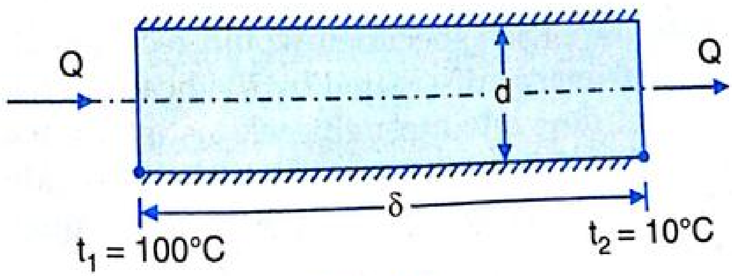


Fig. 3.5.

Invoking Fourier law of heat conduction, the heat flow through the rod is given by

$$Q = \frac{k A_c (t_1 - t_2)}{\delta}$$

$$6 = \frac{k\left(\frac{\pi}{4} \times 0.03^2\right)(100 - 10)}{0.2}$$

$$= 0.318\, k$$

$\therefore$ Thermal conductivity of the rod material,

$$k = \frac{6}{0.318} = \textbf{18.87 W/m-deg}$$

EXAMPLE 3.5

The accompanying sketch shows the schematic arrangement for measuring the thermal conductivity by the guarded hot plate method.

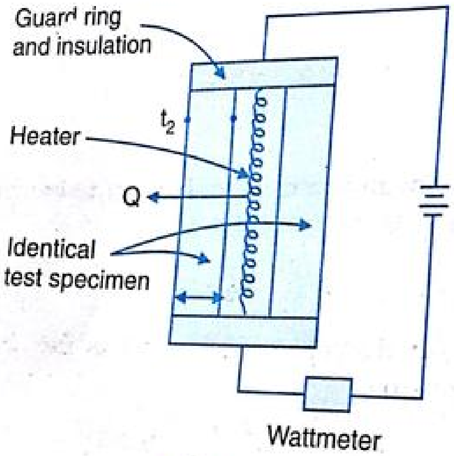


Fig. 3.6.

Two similar 1 cm thick specimens receive heat from a 6.5 cm × 6.5 cm guard heater. When the power dissipation as shown by the watt meter was 15 W, the thermocouples inserted at the hot and cold surfaces indicated temperatures as 325 K and 300 K respectively. Make calculatins for the thermal conductivity of the test specimen material.

Solution : Each specimen would receive half of total energy dissipated by the heater. That is, heat flow rate through each test-specimen would be 7.5 W. For one-dimensional steady state, the heat flow through the test specimen is

$$Q = \frac{kA(t_1 - t_2)}{\delta}$$

or
$$k = \frac{Q\delta}{A(t_1 - t_2)}$$

$$= \frac{7.5 \times 0.01}{(0.065 \times 0.065) \times (325 - 300)}$$

$$= \textbf{0.71 W/mK}$$

EXAMPLE 3.6

A metal piece of length 60 cm has a cross-section corresponding to a sector of a circle of radius 10 cm and included angle 60°C. Its ends are maintained at temperature of 125°C and 25°C, and the thermal conductivity of the material has a linear variation with temperature in degree Celsius.

$$k = (100 - 0.01\, t)\ \text{W/m-deg}$$

Find the heat flow rate through the metallic piece. Presume uni-directional heat conduction, i.e., neglect any variation of temperature in the θ and r-directions.

Solution : Average thermal conductivity,

$$k = 100 - 0.01\, t_m \text{ where } t_m = \frac{t_1 + t_2}{2}$$

$$= 100 - 0.01\left[\frac{125 + 25}{2}\right]$$

$$= 99.25\ \text{W/m-deg}$$

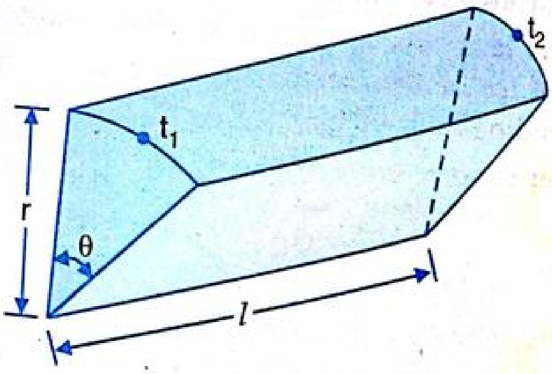


Fig. 3.7.

Area through which heat flow occurs is

$$A = \pi r^2 \times \frac{\theta}{2\pi} = \frac{r^2 \theta}{2}$$

where θ is the included angle in radians

$$A = \frac{(0.1)^2 \times \frac{\pi}{3}}{2} = 5.23 \times 10^{-3}\ \text{m}^2$$

Heat flow

$$Q = \frac{kA(t_1 - t_2)}{l}$$

$$= \frac{99.25 \times 5.23 \times 10^{-3} \times (125 - 25)}{0.6}$$

$$= \mathbf{86.51\ W}$$

EXAMPLE 3.7

Establish a relation for the time taken to form a layer of ice on the surface of a pond. How much time it will take for a layer of ice of thickness 20 cm to increase by 1 mm on the surface of a pond when the temperature of surroundings is – 20°C?

Thermal conductivity of ice = 7.53 kJ/m-hr-deg

Latent heat of ice = 335 kJ/kg

Density of ice at 0°C = 1000 kg/m³

Solution : An ice layer of thickness x has been formed on a pond where the temperature of air over the ice is – t°C land that of water below the ice is 0°C.

Consider that the thickness of ice layer increases by dx in time dt. If A is the surface area of the pond, ρ its mass density and L the latent heat of fusion of ice, then

mass of ice formed = $A\ dx\ \rho$

heat lost by water = $A\ dx\ \rho\ L$ ...(*i*)

This heat has to be conducted across a layer of ice of thickness dx upwards to the surroundings at – t°C.

heat conducted

$$= kA\frac{dt}{x}d\tau$$

$$= kA\frac{[0-(-t)]}{x}d\tau$$

$$= kA\frac{t}{x}d\tau \quad ...(ii)$$

Equating expressions (*i*) and (*ii*),

$$kA\frac{t}{x}d\tau = A\ dx\ \rho\ L$$

$$\frac{dx}{d\tau} = \frac{kt}{\rho Lx}$$

Here $dx/d\tau$ represents the rate of growth of thickness of ice

$$d\tau = \frac{\rho L}{kt}\ x\ dx$$

Total time taken by the layer of ice to increase in thickness by x

$$\int d\tau = \frac{\rho L}{kt}\int x\,dx$$

$$\tau = \frac{\rho L}{kt}\frac{x^2}{2} + \text{constant}$$

When $\tau = 0$, $x = 0$. That gives constant as zero

$$\therefore \quad \tau = \frac{\rho L}{kt}x^2 \quad ...(iii)$$

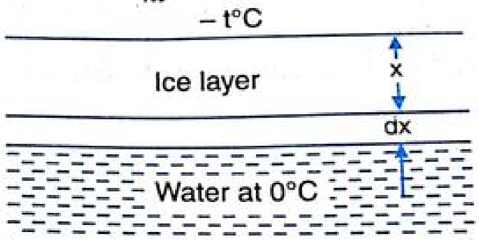


Fig. 3.8.

If the thickness of ice layer is to increase from x_1 to x_2

$$\tau = \frac{\rho L}{2kt}\left(x_2^2 - x_1^2\right) \quad ...(iv)$$

Substituting the given data in the above expression,

$$\tau = \frac{1000 \times 335}{2 \times 7.53 \times 20}\left[(0.201)^2 - (0.20)^2\right]$$

$$= 0.446 \text{ hours} = \mathbf{1605.6\ seconds}$$

EXAMPLES 3.8

Derive an expression for temperature distribution and conduction heat flow in a circular conical rod with diameter at any section given by D = ax where x is the distance measured from the apex of the cone and a is a certain numerical constant. It may be presumed that there is no internal heat generation, steady state conditions prevail and that the lateral surface is well insulated. Further proceed to obtain the numerical value for heat flow rate if the smaller end located at x_1 = 60 mm has a temperature 650 K and the larger end at x_2 = 300 mm has a temperature of 450 K. The parameter a equals 0.20 and the material of the conical rod has an average thermal conductivity k = 3.45 W/m-deg.

Solution : The well insulated lateral surface implies that the situation corresponds to one-dimensional conduction in the x-direction for which

$$Q = -kA\frac{dt}{dx}$$

Substituting for $A = (\pi/4)\,D^2 = (\pi/4)\,a^2x^2$ and separating the variables

$$-k\,dt = \frac{4Q}{\pi a^2}\frac{dx}{x^2}$$

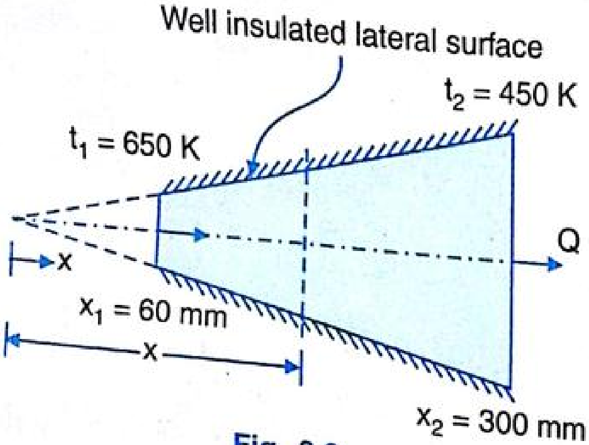


Fig. 3.9.

Since k and Q are constant, integration from x_1 to x within the conical rod yields

$$-k\int_{t_1}^{t} dt = \frac{4Q}{\pi a^2}\int_{x_1}^{x}\frac{dx}{x^2}$$

or $$-k\,(t - t_1) = \frac{4Q}{\pi a^2}\left(\frac{1}{x_1} - \frac{1}{x}\right)$$

and solving for t, one obtains

$$t = t_1 - \frac{4Q}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x}\right) \qquad ...(i)$$

Evaluation of the above expression at $x = x_2$ where $t = t_2$ gives,

$$t_2 = t_1 - \frac{4Q}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x_2}\right) \qquad ...(ii)$$

The heat flow rate then works out as

$$Q = \frac{\pi a^2 k\,(t_1 - t_2)}{4\left[\dfrac{1}{x_1} - \dfrac{1}{x_2}\right]} \qquad ...(iii)$$

When substitution for Q is made in expression (i), we get the temperature distribution as

$$t = t_1 + (t_1 - t_2)\left[\frac{\dfrac{1}{x} - \dfrac{1}{x_1}}{\dfrac{1}{x_1} - \dfrac{1}{x_2}}\right] \qquad ...(iv)$$

Substituting the given data in expression (iii) for the flow rate,

$$Q = \frac{\pi \times (0.20)^2 \times 3.45\,(650 - 450)}{4\left[\dfrac{1}{0.06} - \dfrac{1}{0.3}\right]}$$

$$= \mathbf{1.62\ W}$$

EXAMPLE 3.9

Heat flow occurs along the axis of a solid which has the shape of a truncated cone with circumferential surface insulated. The base is at 300°C and the area of the section at distance x measured from the base of the cone is given by

$$A = 1.2\,(1 - 1.5\,x)\ m^2 \text{ where } x \text{ is in metre}$$

If the plane at $x = 0.2$ m is maintained at 100°C, determine (a) heat flow, (b) temperature at $x = 0.1$ m, and (c) the temperature gradient at the two faces and at $x = 0.1$ m.

Take thermal conductivity of the solid material as 2.5 W/m-deg.

Solution : The insulatated circumferential surface implies that the situation corresponds to one-dimensional conduction in the x-direction for which

$$Q = -k\,A\frac{dt}{dx}$$

$$= -k\left[1.2(1 - 1.5x)\right]\frac{dt}{dx}$$

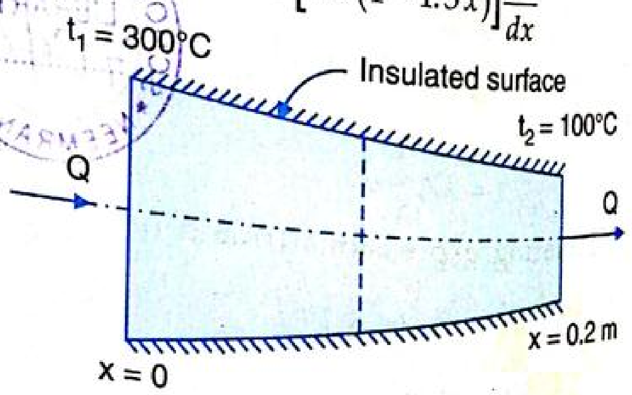


Fig. 3.10.

Separating the variables and upon integration

$$Q\int \frac{dx}{1-1.5x} = -1.2\,k \int dt$$

$$\text{or } Q\left[\frac{1}{-1.5}\log_e(1-1.5x)\right]_{x_1}^{x_2} = -1.2\,k\,(t_2 - t_1)$$

∴ Heat flow,

$$Q = \frac{1.2 \times 1.5\,k\,(t_1 - t_2)}{-\log_e\{1-1.5\,(x_2 - x_1)\}}$$

Heat flow,

$$Q = \frac{1.2 \times 1.5 \times 2.5\,(300-100)}{-\log_e[1-1.5\,(0.2-0.0)]} = 2523.85 \text{ W}$$

(b) Let t be the temperature at $x = 0.1$ m.

Then

$$2523.85 = \frac{1.2 \times 1.5 \times 2.5\,(300-t)}{-\log_e\{1-1.5\,(0.1-0.0)\}} = \frac{4.5\,(300-t)}{0.1625}$$

$$\therefore \quad t = 300 - \frac{2523.85 \times 0.1625}{4.5} = 208.86°\text{C}$$

(c) From identity, $Q = -k\,A\,dt/dx$, the temperature gradient at a section may be written as

$$\frac{dt}{dx} = -\frac{Q}{kA}$$

(i) At $x = 0$; $A = 1.2\,(1 - 1.5\,x) = 1.20 \text{ m}^2$

$$\frac{dt}{dx} = -\frac{2523.85}{2.5 \times 1.2} = -\,841.28°\text{C/m}$$

(ii) At $x = 0.1$ m ;

$$A = 1.2\,(1 - 1.5 \times 0.1) = 1.02 \text{ m}^2$$

$$\frac{dt}{dx} = \frac{2523.84}{2.5 \times 1.02} = -\,989.74°\text{C/m}$$

(iii) At $x = 0.2$ m ;

$$A = 1.2\,(1 - 1.5 \times 0.2) = 0.84 \text{ m}^2$$

$$\frac{dt}{dx} = -\frac{2523.84}{2.5 \times 0.84} = -\,1201.83°\text{C/m}$$

EXAMPLE 3.10

A conical cylinder of length l and radius r_1 and r_2 ($r_1 < r_2$) is fully insulated along the outer surface. The surface at r_1 is maintained at temperature t_1 and surface at r_2 is maintained at temperatrure t_2 ($t_1 > t_2$). Considering the heat flow along the axis of the cylinder, set up an expression for heat flow through the conical cylinder.

(b) *A tapered stainless steel rod, perfectly insulated on the curved surface, has end diameters of 6 cm and 12 cm respectively, and is 20 cm long. The thicker end is fixed to a hot wall and the thinner end is maintained at 25°C. The steady state rate of heat loss through the rod is estimated to be 50 W. Calculate the temperature at hot end of rod. Take thermal conductivity of the rod material as 15 W/m-deg.*

Solution : The perfectly insulated curved surface implies that the situation corresponds to one-dimensional conduction in the x-direction only for which

$$Q = -k\,A_x\,\frac{dt}{dx}$$

where A_x is the cross-sectional area at any axial position x from the thinner end of the rod.

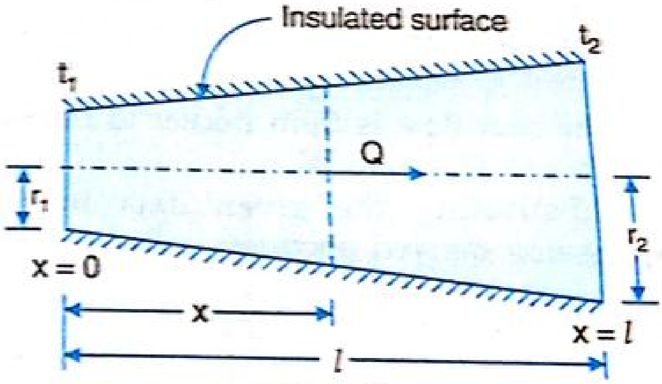


Fig. 3.11.

Radius at distance x :

$$r_x = r_2 - \frac{r_2 - r_1}{l}\,x$$

$$= r_2 - cx \quad \text{where} \quad c = \frac{r_2 - r_1}{l}$$

Then

$$A_x = \pi r_x^2 = \pi(r_2 - cx)^2, \quad \text{and}$$

$$Q = -k\,\pi\,(r_2 - cx)\,\frac{dt}{dx}$$

Separating the variables and upon integration, we get

$$\int_{t_1}^{t_2} -k\,\pi\,dt = \int_0^l \frac{Q\,dx}{(r_2 - cx)^2}$$

$$\text{or } k\,\pi\,(t_1 - t_2) = Q \int_0^l (r_2 - cx)^{-2}\, dx$$

$$= Q \left| \frac{(r_2 - cx)^{-1}}{(-1)\times(-c)} \right|_0^l$$

$$= \frac{Q}{c}\left[\frac{1}{r_2 - cl} - \frac{1}{r_2}\right]$$

Putting the value of $c = \dfrac{r_2 - r_1}{l}$ in the above expression,

$$k\,\pi\,(t_1 - t_2) = \frac{Q \times l}{r_2 - r_1}\left[\frac{1}{r_2 - \left(\frac{r_2 - r_1}{l}\right) l} - \frac{1}{r_2}\right]$$

$$= \frac{Q\,l}{r_2 - r_1}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q\,l}{r_1\, r_2}$$

$$\therefore \quad Q = \frac{k\,\pi\,(t_1 - t_2)\, r_1\, r_2}{l}$$

(*b*) Refer Fig. 3.11

$r_1 = 3$ cm; $r_2 = 6$ cm ; $t_1 = 25°C$

The heat flow is from thicker to thinner end and so $Q = -\ 50$ W

Substituting the given data in the expresssion derived above,

$$-\ 50 = \frac{15\,\pi\,(25 - t_2) \times 0.03 \times 0.06}{0.2}$$

$$= 0.4239\ (25 - t_2)$$

$\therefore$ Temperature t_2 at the thicker end of the rod is,

$$t_2 = 25 + \frac{50}{0.4239} = 142.95°C$$

EXAMPLE 3.11.

Heat is conducted through a uniformly tapered rod of square cross-section and length 50 cm. At the left end, the side of face is 3 cm and the temperature is 600°C. At the right end, the corresponding values are 8 cm and 150°C. Determine :

(*a*) *the rate of heat conduction, and*

(*b*) *the temperature at point 30 cm from the hot end.*

It may be presumed that thermal conductivity of the material of rod is 60 W/mK and heat is conducted only along the length of rod.

Solution : For one-dimensional heat conduction in the x-direction, we have

$$Q = -\,k\,A_x \frac{dt}{dx}$$

where A_x is the cross-sectional area at any axial position from the smaller end of the rod.

Side of face at distance x metre from the small end

$$= 0.03 + \frac{0.08 - 0.03}{0.5}\, x$$

$$= 0.1\,x + 0.03$$

Then: $A_x = (0.1x + 0.03) \times (0.1x + 0.03)$

$$= (0.1x + 0.03)^2$$

From the Fourier heat conduction equation

$$Q = -\,k\,(0.1x + 0.03)^2 \frac{dt}{dx}$$

Separating the variables and upon integration

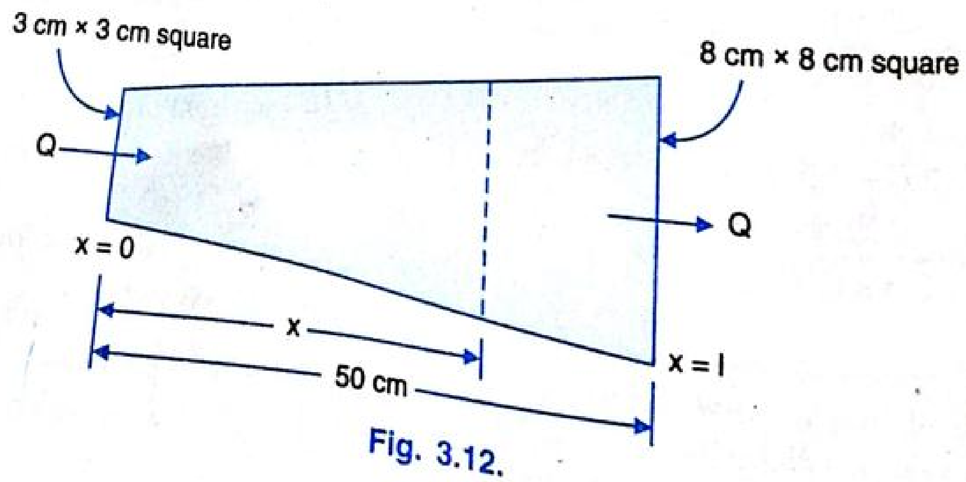


Fig. 3.12.

$$Q\int \frac{dx}{(0.1x+0.03)^2} = -k\int dt$$

The boundary conditions are :

(i) At $x = 0$ $\quad t = t_1 = 600°C$

(ii) At $x = l = 0.5$ m $\quad t = t_2 = 150°C$

$$\therefore Q\int_0^l \frac{dx}{(0.1x+0.03)^2} = -k\int_{t_1}^{t_2} dt$$

$$\text{or } \frac{Q}{0.1}\left[\frac{-1}{0.1x+0.03}\right]_0^l = k(t_1 - t_2)$$

$$\text{or } \frac{Q}{0.1}\left[\frac{1}{0.1l+0.03} + \frac{1}{0.03}\right] = k(t_1 - t_2)$$

$\therefore$ Heat flow,

$$Q = \frac{0.1\,k(t_1 - t_2)}{\frac{1}{0.03} - \frac{1}{0.1l+0.03}}$$

$$= \frac{0.1 \times 60(600-150)}{\frac{1}{0.03} - \frac{1}{0.1 \times 0.5 \times 0.03}}$$

$$= \frac{2700}{33.33 - 12.5} = \mathbf{129.62\ W}$$

(b) Let the boundary condition (ii) be generalized as

At $x = x \quad t = t_x$

$$\text{Then} \quad Q = \frac{0.1\,k(t_1 - t_x)}{\frac{1}{0.03} - \frac{1}{0.1x+0.03}}$$

When $x = 0.3$ m, then

$$129.62 = \frac{0.1 \times 60(600 - t_x)}{\frac{1}{0.03} - \frac{1}{0.1 \times 0.3 \times 0.03}}$$

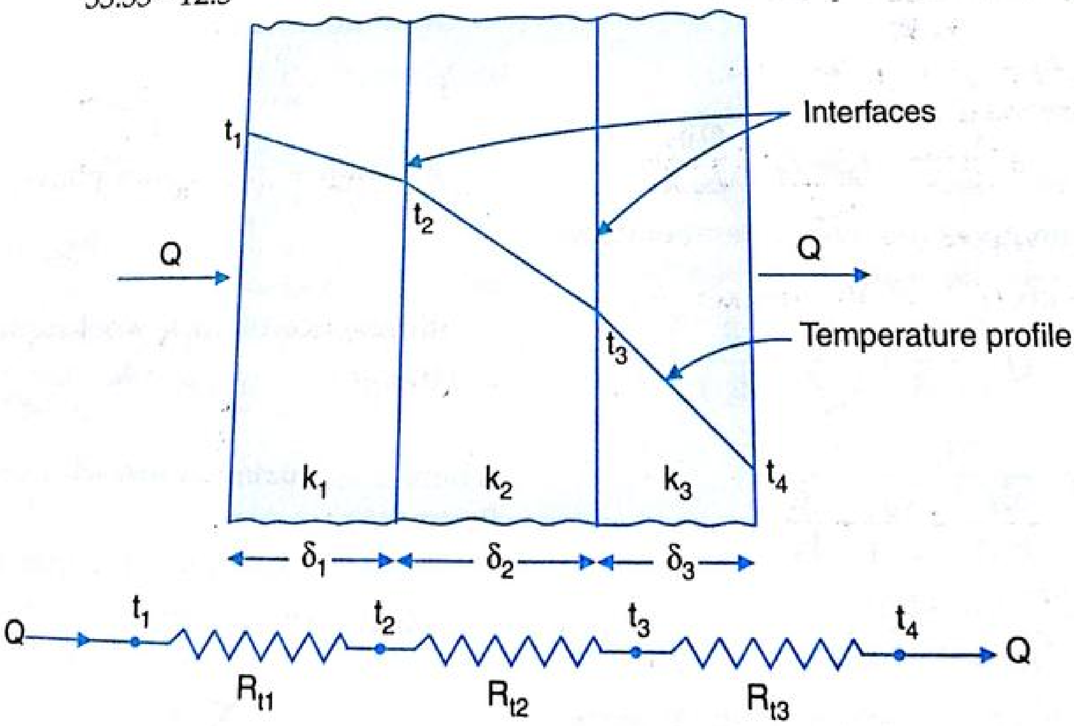
$$= \frac{6(600 - t_x)}{33.33 - 16.66}$$

$$\therefore \quad t_x = 600 - \frac{129.62(33.33 - 16.66)}{6}$$

$$= \mathbf{240°C}$$

3.2. CONDUCTION THROUGH A COMPOSITE WALL

A composite wall refers to a wall of a several heterogeneous layers, *e.g.*, walls of dwelling houses where bricks are given a layer of plaster on either side. Likewise walls of furnaces,



$$R_{t1} = \frac{\delta_1}{k_1 A}\ ;\ R_{t2} = \frac{\delta_2}{k_2 A}\ ;\ R_{t3} = \frac{\delta_3}{k_3 A}$$

Fig. 3.13. Steady state conduction through a composite wall

boilers and other heat exchange devices consist of several layers; a layer for mechanical strenght or for high temperature characteristics (fire brick), a layer of low thermal conductivity material to restrict the flow of heat (insulating brick) and another layer for structural requirements for good appearnce (ordinary brick).

Fig. 3.13 shows one such composite wall having three layers of different materials tightly fitted to one another. The layers have thicknesses δ_1, δ_2, δ_3 and their thermal conductivities correspond to the average temperature conditions. The surface temperature of the wall are t_1 and t_4 and the temperatures at the interfaces are t_2 and t_3.

Under steady state conditions, heat flow does not vary across the wall, *i.e.*, it is same for every layer. Thus

$$Q = \frac{k_1 A}{\delta_1} (t_1 - t_2)$$

$$= \frac{k_2 A}{\delta_2} (t_2 - t_3) = \frac{k_3 A}{\delta_3} (t_3 - t_4)$$

Rewriting the above expression in terms of temperature drop across each layer,

$$t_1 - t_2 = \frac{Q\delta_1}{k_1 A};$$

and $t_2 - t_3 = \dfrac{Q\delta_2}{k_2 A}$; $t_3 - t_4 = \dfrac{Q\delta_3}{k_3 A}$

Summation gives the overall temperature difference across the wall

$$t_1 - t_4 = Q\left(\frac{\delta_1}{k_1 A} + \frac{\delta_2}{k_2 A} + \frac{\delta_3}{k_3 A}\right)$$

Then $Q = \dfrac{(t_1 - t_4)}{\dfrac{\delta_1}{k_1 A} + \dfrac{\delta_2}{k_2 A} + \dfrac{\delta_3}{k_3 A}}$

$$= \frac{(t_1 - t_4)}{R_{t_1} + R_{t_2} + R_{t_3}} \qquad \ldots(3.9)$$

Distributing of temperature in a plane multilayer wall is represented by a polygonal line (Fig. 3.13).

When the above analysis is extended to a n-layer composite wall, one obtains :

$$Q = \frac{(t_1 - t_{n+1})}{\sum_1^n \delta/kA}$$

where $\sum_1^n \delta/kA$ is sum of the thermal resistances of different layers comprising the composite wall.

Anaysis of the composite wall assumes that there is a perfect contact between layers and no temperature drop occurs across the interface between materials.

EXAMPLE 3.12

An exterior wall of a house may be approximated by 10 cm layer of common brick (k = 0.75 W/m-deg) followed by 4 cm layer of gypsum plaster (k = 0.5 W/m-deg). What thickness of loosely packed rock wool insulation (k = 0.065 W/m-deg) should be added to reduce the heat loss or gain through the wall by 75% ?

Solution : Thermal resistance for a plane wall of thickness δ, area A and thermal condcutivity k is prescribed by the relation $R_t = \delta/kA$. Considering unit area of wall perpendicular to heat flow direction,

Resistance of brick work

$$= \frac{10 \times 10^{-2}}{0.75 \times 1} = 0.133 \text{ deg/W}$$

Resistance of gypsum plaster

$$= \frac{4 \times 10^{-2}}{0.5 \times 1} = 0.08 \text{ deg/W}$$

Resistance of rock wool insulation

$$= \frac{x \times 10^{-2}}{0.065} = 0.1538\, x \text{ deg/W}$$

where x is thickness of rock wool insulation in cm

Case I : *Rockwool insulation is not used :*

Heat loss or gain

$$= \frac{\Delta t}{\sum R_t}$$

$$= \frac{\Delta t}{0.133 + 0.08} = 4.695\, \Delta t$$

Case II : Rockwool insulation is used :

Heat loss or gain

$$= (1 - 0.75) \times 4.695\ \Delta t$$
$$= 1.174\ \Delta t$$

$$\therefore 1.174\ \Delta t = \frac{\Delta t}{0.133 + 0.08 + 0.1538\,x}$$

or $(0.133 + 0.08 + 0.1538\ x)$

$$= \frac{1}{1.174} = 0.8518$$

$\therefore$ Thickness of rock wool, x

$$= \frac{0.8518 - 0.133 - 0.08}{0.1538}$$

$$= \mathbf{4.153\ cm}$$

EXAMPLE 3.13

A storage chamber of interior dimensions

10 m × 8 m × 2.5 m

high has its inside maintained at a temperature of –20°C whilst the outside is at 25°C. The walls and ceiling of the chamber have three layers made of

60 mm thick board (k = 0.2 W/m-deg) on the inside

90 mm thick insulation (k = 0.04 W/m-deg) at mid

240 mm thick concrete (k = 1.8 W/m-deg) on the outside

Neglecting flow of heat through the floor, determine the rate at which heat can flow towards inside of the chamber.

Solution : Neglecting corners and edges, the area of heat flow is

$$A = 2\,(10 \times 2.5) + 2\,(8 \times 2.5) + (10 \times 8)$$

$$= 170\ m^2$$

Recalling that thermal resistance for a slab of thickness δ, conductivity k and area A equals δ/kA, we have

$$R_t = \frac{1}{A}\left[\frac{\delta_1}{k_1} + \frac{\delta_2}{k_2} + \frac{\delta_3}{k_3}\right]$$

$$= \frac{1}{170}\left[\frac{60 \times 10^{-3}}{0.2} + \frac{90 \times 10^{-3}}{0.04} + \frac{240 \times 10^{-3}}{1.8}\right]$$

$$= \frac{1}{170}\ (0.3 + 2.25 + 0.133)$$

$$= 0.01578\ deg/W$$

$\therefore$ Heat flow rate Q

$$= \frac{\Delta t}{R_t} = \frac{25 - (-20)}{0.01578} = \mathbf{2851.7\ W}$$

EXAMPLE 3.14

The walls of house in cold region comprise three layers

15 cm outer brick work (k = 0.75 W/m-deg)

1.25 cm inner wooden paneling (k = 0.2 W/m-deg)

7.5 cm intermediate layer of insulating material

The insulation layer is stated to offer resistance twice the thermal resistance of brick work. If the inside and outside temperatures of the composite wall are 20°C and –15°C respectively, determine the rate of heat loss per unit area of the wall and the thermal conductivity of the insulating material.

Solution : Thermal resistance for a plane wall of thickness δ, area A and thermal conductivity k is prescribed by the relation $R_t = \delta/kA$

Resistance of brick work,

$$= \frac{15 \times 10^{-2}}{0.75 \times 1} = 0.2\ deg/W$$

Resistance of wooden paneling,

$$R_{t_2} = \frac{1.25 \times 10^{-2}}{0.2 \times 1} = 0.0625\ deg/W$$

Resistance of insulating material,

$$R_{t_3} = 2 \times 0.2 = 0.4\ deg/W$$

Total resistance, R_t

$$= \sum R_t = 0.2 + 0.0625 + 0.4$$
$$= 0.6625\ deg/W$$

$\therefore$ Heat loss Q

$$= \frac{\Delta t}{\sum R_t} = \frac{20 - (-15)}{0.6625} = \mathbf{52.83\ W}$$

(*b*) Thermal conductivity of insulating material,

$$k = \frac{\delta}{A\,R_t} = \frac{7.5 \times 10^{-2}}{1 \times 0.4} = \mathbf{0.1875\ W/m\text{-}deg}$$

EXAMPLE 3.15

A furnace wall is a made up of steel plate 10 mm thick (k = 62.8 kJ/m-hr-deg) lined on inside with silica bricks 150 mm thick (k = 7.32 kJ/m-hr-deg) and on the outside with magnesia bricks 200 mm thick (k = 18.84 kJ/m-hr-deg). The inside and outside surfaces of the wall are at temperautre 650°C and 125°C respectively. Make calculations for the heat loss from unit area of the wall.

It is required that the heat loss be reduced to 10 MJ/hour by means of air gap between steel and magnesia bricks. Estimate the necessary width of air gap if thermal conductivity for air is 0.126 kJ/m-hr-deg

Solution : Thermal resistance for a plane wall of thickness δ, area A and thermal conductivity k is prescribed by the relation $R_t = \delta/kA$.

∴ Resistance of steel plate,

$$R_{t_1} = \frac{0.01}{62.8 \times 1}$$

$$= 0.000159 \text{ deg hr/kJ}$$

Resistance of silica bricks,

$$R_{t_2} = \frac{0.15}{7.32 \times 1}$$

$$= 0.02049 \text{ deg hr/kJ}$$

Resistance of magnesia bricks,

$$R_{t_3} = \frac{0.20}{18.84 \times 1}$$

$$= 0.01061 \text{ deg hr/kJ}$$

Total resistance of the composite wall,

$$\sum R_t = R_{t_1} + R_{t_2} + R_{t_3}$$

$$= 0.03126 \text{ deg hr/kJ}$$

Heat loss from the wall

$$= \frac{\Delta t}{\sum R_t}$$

$$= \frac{650 - 125}{0.03126} = 16795 \text{ kJ/hr}$$

To reduce the heat loss to 10 MJ/hr, the thermal resistance should be increased to :

$$\frac{650 - 125}{10 \times 10^3} = 0.0525 \text{ deg hr/kJ}$$

∴ Thermal resistance for the air gap,

$$\frac{\delta}{kA} = 0.0525 - 0.03126$$

$$= 0.02124 \text{ deg hr/kJ}$$

∴ Thickness of air gap, δ

$$= 0.02124 \times 0.126 \times 1$$

$$= 2.676 \times 10^{-3} \text{ m} = 2.676 \text{ mm}$$

EXAMPLE 3.16

A furnace wall comprises three layers: 13.5 cm thick inside layer of fire brick, 7.5 cm thick middle layer of insulating brick and 11.5 cm thick outside layer of red brick. The furnace operates at 870°C and it is anticipated that the outside of this composite wall can be maintained at 40°C by the circulation of air. Assuming close bonding of layers at their interfaces, find the rate of heat loss from the furnace and the wall interface temperature. The wall measures 5 m × 2 m and the data on thermal conductivities is :

Fire brick k_1 = 1.2 W/m-deg

Insulating brick k_2 = 0.14 W/m-deg

Red brick k_3 = 0.85 W/m-deg

Solution : The wall area (5 m × 2 m) = 10 m² is constant for all layers of the composite wall.

The thermal resistance R_t of a slab (thickness δ, conductivity k and area A) is given by

$$R_t = \frac{\delta}{kA}$$

∴ Resistance of fire brick,

$$R_{t_1} = \frac{0.135}{1.2 \times 10} = 0.01125 \text{ deg/W}$$

Resistance of insulating brick,

$$R_{t_2} = \frac{0.075}{0.14 \times 10} = 0.05357 \text{ deg/W}$$

Resistance of red brick,

$$R_{t_3} = \frac{0.115}{0.85 \times 10} = 0.01353 \text{ deg/W}$$

Total resistance of the composite wall,

$$\sum R_t = R_{t_1} + R_{t_2} + R_{t_3}$$

$$= 0.07835 \text{ deg/W}$$

The heat will flow from the inside of the furnace (temperature t_1 = 870°C) to outside of the composite wall (temperature t_4 = 40°C)

$$\text{Heat loss} = \frac{t_1 - t_4}{\sum R_t}$$

$$= \frac{870 - 40}{0.07835} = 10593.5 \text{ W}$$

(b) Since heat flowing through each layer is same, then for inside layer of fire brick

$$10593.5 = \frac{t_1 - t_2}{0.01125} = \frac{870 - t_2}{0.01125}$$

where t_2 is the temperature at the interface of fire brick and insulating brick.

$$t_2 = 870 - 10593.5 \times 0.01125$$
$$= 750.82°C$$

Similarlty for the mid layer of insulating brick,

$$10593.5 = \frac{t_2 - t_3}{0.05357} = \frac{750.82 - t_3}{0.05357}$$

where t_3 is the temperature at the interface of insulating brick and the red brick

$$t_3 = 750.82 - 10593.5 \times 0.05357$$
$$= 183.33°C$$

Check : The temperature t_3 could also be worked out by considering the heat flow through the outside layer of red brick.

$$10593.5 = \frac{t_3 - t_4}{0.01357} = \frac{t_3 - 40}{0.01357}$$

$$t_3 = 40 + 10593.5 \times 0.01357$$
$$= 183.75°C$$

This is same as calculated above.

EXAMPLE 3.17

A 30 cm thick wall of reactor is made up of an inner layer of fire brick (k_1 = 0.85 W/mK) covered with a layer of insulation (k_2 = 0.15 W/mK). The reactor operates at a temperature of 1600 K whilst the ambient temperature is 295 K. Calculate the thickness of fire brick and insulation which gives minimum heat loss. Also work out the heat loss presuming that the insulating material has a maximum temperature of 1475 K. If the calculated heat loss is unacceptable, would the addition of another layer of insulation be a satisfactory solution?

Solution : Under steady state conditions, the heat flux is constant throughout the wall and is same for each layer. Then for unit area of the wall,

$$Q = \underset{\text{composite wall}}{\frac{t_1 - t_2}{\frac{\delta_1}{k_1} + \frac{\delta_2}{k_2}}} = \underset{\text{fire brick}}{\frac{t_1 - t_i}{\frac{\delta_1}{k_1}}} = \underset{\text{insulation}}{\frac{t_i - t_2}{\frac{\delta_2}{k_2}}}$$

Since $\delta_2 = (0.3 - \delta_1)$, then from the first two equalities:

$$\frac{1600 - 295}{\frac{\delta_1}{0.85} + \frac{0.3 - \delta_1}{0.15}} = \frac{1600 - 1475}{\frac{\delta_1}{0.85}}$$

$$\frac{1305}{1.176\delta_1 + 2 - 6.667\delta_1} = \frac{106.25}{\delta_1}$$

$$1305\ \delta_1 - 124.958\delta_1 + 708.37\delta_1 = 212.50$$

$$\delta_1 = \frac{212.50}{1305 - 124.95 + 708.37}$$

$$= 0.112.5 \text{ m} = 11.25 \text{ cm}$$

Thus thickness of fire brick

$$\delta_1 = 11.25 \text{ cm}$$

and that of insulation is

$$= (30 - \delta_1)$$
$$= 30 - 11.25 = 18.75 \text{ cm}$$

Heat flux (heat flow per unit area) through the wall,

$$q = \frac{t_1 - t_i}{\frac{\delta_1}{k_1}}$$

$$= \frac{1600 - 1475}{\frac{0.1125}{0.85}} = 944.45 \text{ W/m}^2$$

The heat loss from the wall would get reduced if a further layer of insulating material is added. That would also reduce the temperature drop across the fire brick lining and that would cause an increase in the interfaces temperature t_i. Since this has already been set at the maximum permissible value, the addition of a further layer will not be a satisfactory solution.

EXAMPLE 3.18.

A 25 cm thick slab is made of a material having thermal conductivity 40 W/mK. Measurements

indicate that temperature variation within the slab can be prescribed by the relation.

$$t_x = 100 + 200\,x - 400\,x^2$$

where t is in °C and x is the distance measure from one face in meters. Make calculation for the temperature, temperature gradients and heat flow at the planes x = 0, x = 10 cm and x = 20 cm. What would be the heat generation rate per unit volume if the difference in heat flow at these sections is due to heat generation?

Solution : Refer Fig. 3.14 for the given slab.

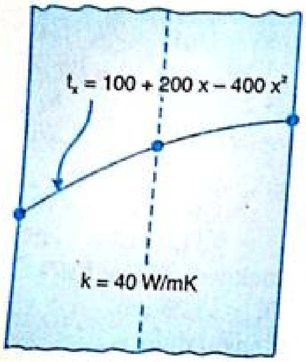


Fig. 3.14.

The temperature at the given planes can be determined from the given relation,

$$t_x = 100 + 200\,x - 400\,x^2$$

At $x = 0$:

$$t = 100°C$$

At $x = 0.125$:

$$t = 100 + 200 \times 0.125 - 400 \times (0.125)^2$$
$$= 100 + 25 - 6.25 = 118.75°C$$

At $x = 0.25$ m :

$$t = 100 + 200 \times 0.25 - 400 \times (0.25)^2$$
$$= 100 + 50 - 25 = 125°C$$

(*b*) Upon differentiating the given expression with respect to *x*, we get

$$\frac{dt}{dx} = 200 - 800\,x$$

Then the temperature gradients at the given planes are :

At $x = 0$;

$$\frac{dt}{dx} = 200°C/m$$

At $x = 0.125$ m :

$$\frac{dt}{dx} = 200 - 800 \times 0.125 = 100°C/m$$

At $x = 0.25$ m :

$$\frac{dt}{dx} = 200 - 800 \times 0.25 = 0$$

(*c*) From Fourier law $Q = -kA\dfrac{dt}{dx}$.

Considering unit area, the heat flows at different planes are:

At $x = 0$:

$$Q = -40 \times 1 \times 200 = -8000 \text{ W/m}^2$$

At $x = 0.125$ m :

$$Q = -40 \times 1 \times 100 = -4000 \text{ W/m}^2$$

At $x = 0.25$ m :

$$Q = -40 \times 1 \times 0.0 = 0$$

(*d*) Heat flow over 1 m^2 area and 0.25 m wall thickness is 9000 W

∴ Heat generation per unit volume

$$= \frac{9000}{0.25} = 3600 \text{ W}$$

EXAMPLE 3.19

Explain the concept of thermal contact resistance.

A furnace wall consists of an inside layer of silica brick 10 cm thick. (k = 6.28 kJ/m-hr°C) followed by a 20 cm layer of magnesite brick (k = 20.95 kJ/m-hr°C) on the outside. The inside surface of the silica brick wall is maintained at 750°C whilst the outside surface of magnesite is at 125°C. The conact thermal resistance between the two walls at the interface is 0.000716 hr-°C/kJ per unit wall area. What is the rate of heat loss per unit area of the wall? Also calculate the temperature drop at the interface.

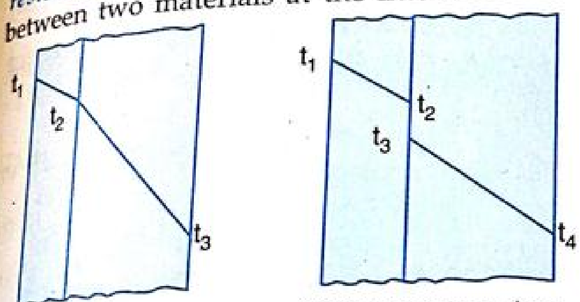
Solution : Calculation of heat flow through a multi-layer composite wall are made on the presumption that

(*i*) there is perfect contact between adjacent layers

(*ii*) the temperature is continuous at the interface although there is discontinuty in temperature gradient

(*iii*) there is no fall of temperature at the interface, *i.e.*, the temperature of the layers are same at the plane of contact

However in real systems, the contact surfaces touch only at discrete locations due to surface roughness, interspersed with void spaces which are usually filled with air. Obviously, there is not a single place of contact. This implies that the area for heat flow at the interface will be small compared to the geometric area of the face. Due to this apparent decrease in the heat flow area and also due to the presence of air voids, there occurs a large resistance to heat flow at the interface. This resistance is referred to as ***thermal contact resistance*** and it causes temperature drop between two materials at the interface.



(i) No temperature drop at interface (ii) Temperature drop at interface

Fig. 3.15.

Let t_2 and t_3 represent the temperature at the theoretical plane interface obtained by consideration of heat flow in materials on either side. Then for a given wall area A, the thermal contact resistance R_c is defined as

$$R_c = \frac{t_2 - t_3}{Q/A} \frac{\text{m}^2\text{-deg-hr}}{\text{kJ}}$$

$$= \frac{t_2 - t_3}{Q} \frac{\text{deg-hr}}{\text{kJ}}$$

The value of metallic contact resistance depends on the metals involved, the suface roughness, the contact pressure and temperature and the matter occupying void spaces. The values of contact thermal resistance are obtained through experiments. For solid to solid interference, the temperature drop is usually small and the effect of contact resistance is generally ignored.

Thermal resistance for a plane wall of thickness δ, area A and thermal conductivity k is prescribed by the relation $R_t = \delta/kA$

∴ Resistance of silica brick

$$R_{t_1} = \frac{0.1}{6.28 \times 1} = 0.01592 \frac{\text{hr-deg}}{\text{kJ}}$$

Resistance of magnesite brick,

$$R_{t_2} = \frac{0.2}{20.95 \times 1} = 0.009546 \frac{\text{hr-deg}}{\text{kJ}}$$

Contact thermal rsistance

$$R_c = 0.000716 \frac{\text{hr-deg}}{\text{kJ}}$$

Total resistance of the composite wall,

$$\Sigma R_t = (R_{t_1} + R_{t_2} + R_c)$$

$$= (0.01592 + 0.009546 + 0.000716)$$

$$= 0.02618 \frac{\text{hr-deg}}{\text{kJ}}$$

Heat loss form the wall

$$= \frac{\Delta t}{\Sigma R_t} = \frac{750 - 125}{0.02618}$$

$$= \mathbf{23873\ kJ/hr}$$

The same heat flows thorugh each layer of composite system. Accordingly for the silica brick

$$Q = \frac{t_1 - t_2}{R_{t_1}} \; ; \; 23873 = \frac{750 - t_2}{0.01592}$$

$$\therefore \quad t_2 = 750 - 23873 \times 0.01592$$

$$= 369.95°\text{C}$$

For the magnesite brick,

$$Q = \frac{t_1 - t_2}{R_{t_2}} \; ; \; 23873 = \frac{t_3 - 125}{0.009546}$$

$$\therefore \quad t_3 = 125 + 23873 \times 0.009546$$

$$= 352.89°\text{C}$$

Hence, temperature drop at the interface

$$= (t_2 - t_3) = 369.95 - 352.89$$

$$= \mathbf{17.06°C}$$

EXAMPLE 3.20

Determine the heat transfer rate across a composite slab which is made of different materials with top and bottom as shown in Fig. 3.16 The entire left-hand face is held at the temperature T_1 while the entire right hand face is at the temperature T_2. The conductivities of the two different materials are stated as k_a and k_b, and their areas as viewed

in the direction of slab thickness δ are A_a and A_b respectively. Steady state exists, there is no heat generation and the slab is so thin that any edge effects can be neglected. Interpret the result in terms of an electrical circuit.

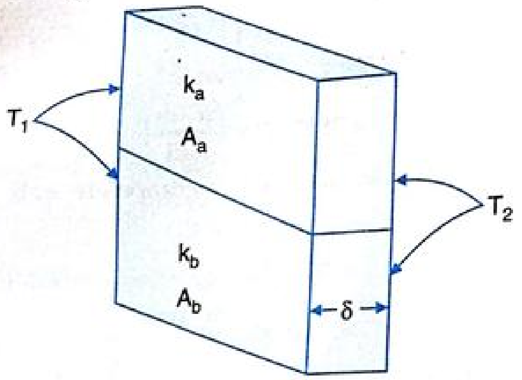


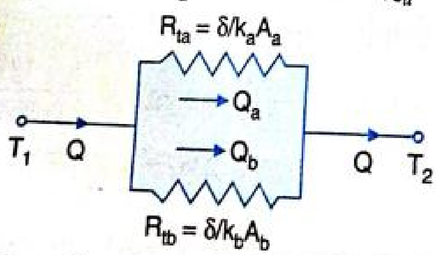
Fig. 3.16.

Solution : Applying Fourier law of heat conduction separately to materials a and b, we obtain

$$Q_a = \frac{k_a A_a (T_1 - T_2)}{\delta}$$

$$Q_b = \frac{k_b A_b (T_1 - T_2)}{\delta}$$

The desired quantity of heat transfer through the slab equals the sum of Q_a and Q_b



$$Q = Q_a + Q_b$$

$$= \frac{k_a A_a (T_1 - T_2)}{\delta} + \frac{k_b A_b (T_1 - T_2)}{\delta}$$

$$= \left\{ \frac{1}{\frac{\delta}{k_a A_a}} + \frac{1}{\frac{\delta}{k_b A_b}} \right\} (T_1 - T_2)$$

$$= \left(\frac{1}{R_{t_a}} + \frac{1}{R_{t_b}} \right) (T_1 - T_2) = \frac{1}{R_t} (T_1 - T_2)$$

Apparently, the two thermal resistances

$$R_{t_a} = \frac{\delta}{k_a A_a} \quad \text{and} \quad R_{t_b} = \frac{\delta}{k_b A_b}$$

appear in the same way as two electrical resistors in parallel. Accordingly the electrical circuit for the heat transfer through the given composite slab will be as indicated in Fig. 3.16.

EXAMPLE 3.21

Find the heat flow rate through the composite wall as shown in Fig. 3.17. Assume one dimensional flow and take

$$k_a = 150 \text{ W/m-deg} \; ; \; k_b = 30 \text{ W/m-deg}$$
$$k_c = 65 \text{ W/m-deg} \; ; \; k_d = 50 \text{ W/m-deg}$$

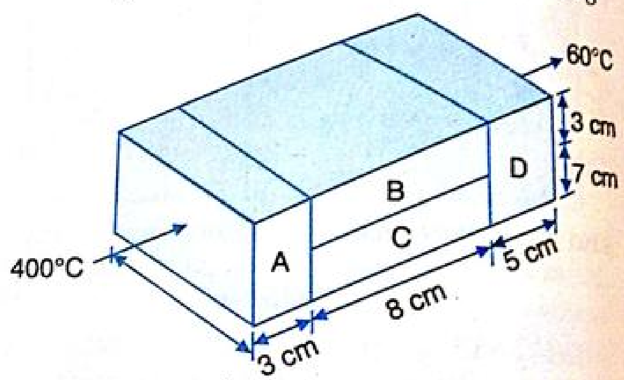


Fig. 3.17.

Solution : The equivalent thermal circuit for heat flow in the composite system has been shown in Fig. 3.18.

The various thermal resistances are given by

$$R_a = \frac{\delta_a}{k_a A_a} = \frac{0.03}{150 \times (0.1 \times 0.1)}$$
$$= 0.02 \text{ deg/W}$$

$$R_b = \frac{\delta_b}{k_b A_b} = \frac{0.08}{30 \times (0.1 \times 0.03)}$$
$$= 0.89 \text{ deg/W}$$

$$R_c = \frac{\delta_c}{k_c A_c} = \frac{0.08}{65 \times (0.1 \times 0.07)}$$
$$= 0.176 \text{ deg/W}$$

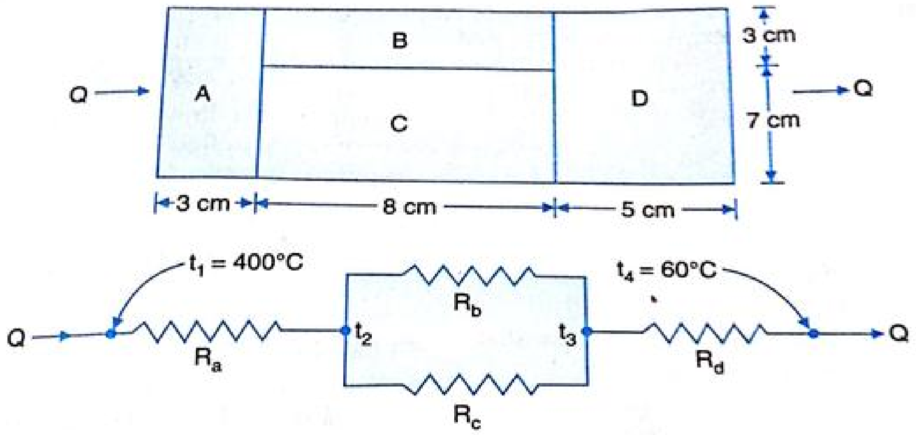


Fig. 3.18. Thermal circuit for heat flow in a composite system

$$R_d = \frac{\delta_d}{k_d A_d} = \frac{0.05}{50 \times (0.1 \times 0.1)}$$
$$= 0.01 \text{ deg/W}$$

The resistances R_b and R_c are in parallel and their equivalent resistance R_{eq} is

$$R_{eq} = \frac{R_b \times R_c}{R_b + R_c} = \frac{0.89 \times 0.176}{0.89 + 0.176}$$
$$= 0.1469 \text{ deg/W}$$

This equivalent resistance is now in series with resistance R_a and R_d. The total thermal resistance for the entire circuit then becomes

$$\Sigma R_t = R_a + R_{eq} + R_b$$
$$= 0.02 + 0.1469 + 0.1$$
$$= 0.2669 \text{ deg/W}$$

Hence heat transfer rate through the system is

$$Q = \frac{\Delta t}{\Sigma Rt} = \frac{400 - 60}{0.2669} = \mathbf{1273.88 \ W}$$

EXAMPLE 3.22.

For the configuration shown in Fig. 3.19 and conditions specified, determine the temperature t_2 and t_3.

Thermal conductivities conform to the following relation :

$$k_1 = k_4 = \frac{t_2}{2} = \frac{t_3}{3} = k$$

Solution : The equivalent thermal circuit for heat flow in the composite system is shown in Fig. 3.20.

The different thermal resistances are :

$$R_1 = \frac{\delta_1}{k_1 A_1} = \frac{x}{kA}$$

$$R_2 = \frac{\delta_2}{k_2 A_2} = \frac{2x}{2kA/4} = \frac{4x}{kA}$$

$$R_3 = \frac{\delta_3}{k_3 A_3} = \frac{2x}{3kA/4} = \frac{8x}{3k}$$

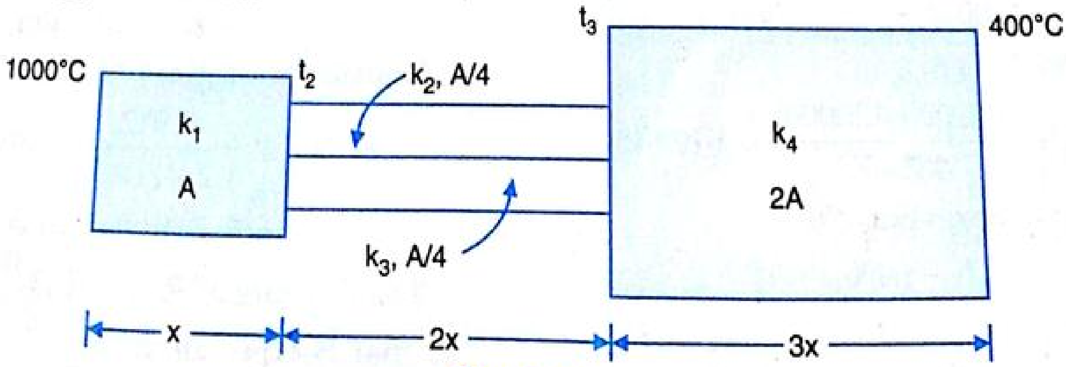


Fig. 3.19.

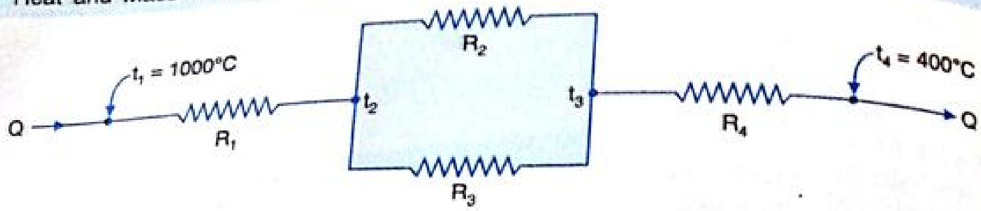


Fig. 3.20.

and $R_4 = \frac{\delta_4}{k_4 A_4} = \frac{3x}{k \times 2A} = \frac{3x}{2kA}$

The resistances R_2 and R_3 are in parallel and their equivalent resistance R_{23} is

$$R_{23} = \frac{R_2 \times R_3}{R_2 + R_3} = \frac{\frac{4x}{kA} \times \frac{8x}{3k}}{\frac{4x}{kA} + \frac{8x}{3k}} = \frac{8}{5\,kA}$$

Then under steady state condition,

$$Q = \frac{t_1 - t_2}{\frac{x}{kA}} = \frac{t_2 - t_3}{\frac{8x}{5kA}} = \frac{t_3 - t_4}{\frac{3x}{2kA}}$$

(i) (ii) (iii)

From identities (i) and (ii)

$8(t_1 - t_2) = 5(t_2 - t_3)$

or $8(1000 - t_2) = 5(t_2 - t_3)$

or $$t_2 = \frac{8000 + 5t_3}{13} \quad ...(a)$$

From identities (ii) and (iii),

$15(t_2 - t_3) = 16(t_3 - t_4)$

or $15(t_2 - t_3) = 16(t_3 - 400)$

or $31t_3 - 15t_2 = 6400 \quad ...(b)$

From expressions (a) and (b)

$$31t_3 - 15 \times \frac{8000 + 5t_3}{13} = 6400$$

or $403\,t_3 - 120000 - 75t_3 = 83200$

That gives :

$$t_3 = \frac{83200 + 120000}{403 - 75} = 619.51°C$$

Then from expression (b)

$$t_2 = \frac{31t_3 - 6400}{15}$$

$$= \frac{31 \times 619.51 - 6400}{15} = 853.65°C$$

EXAMPLE 3.23

A 30 cm thick wall of 5 m × 3 m size is made of red brick (k = 0.3 W/m-deg). It is covered on both sides by layers of plaster, 2 cm thick (k = 0.6 W/m-deg). The wall has a window size of 1 m × 2 m. The window door is made of 12 mm thick glass (k = 1.2 W/m-deg). If the inner and outer surface temperatures are 15 and 40°C, make calculations for the rate of heat flow through the wall.

Solution : Refer Fig 3.21 for the corfiguration of the wall with glass window and the equivalent electrical circuit for thermal resistance

Resistances of inner and outer plaster layers,

$$R_{p_1} = R_{p_2} = \frac{\delta}{kA}$$

$$= \frac{0.02}{0.6(5 \times 3 - 1 \times 2)}$$

$$= 2.564 \times 10^{-3} \text{ deg/W}$$

Resistance of brick work,

$$R_b = \frac{0.3}{0.3(5 \times 3 - 1 \times 2)}$$

$$= 76.92 \times 10^{-3} \text{ deg/W}$$

Resistance of glass window,

$$R_g = \frac{0.012}{1.2 \times (1 \times 2)}$$

$$= 5 \times 10^{-3} \text{ deg/W}$$

The resistances R_{p_1}, R_b and R_{p_2} are in series and that is equivalent to

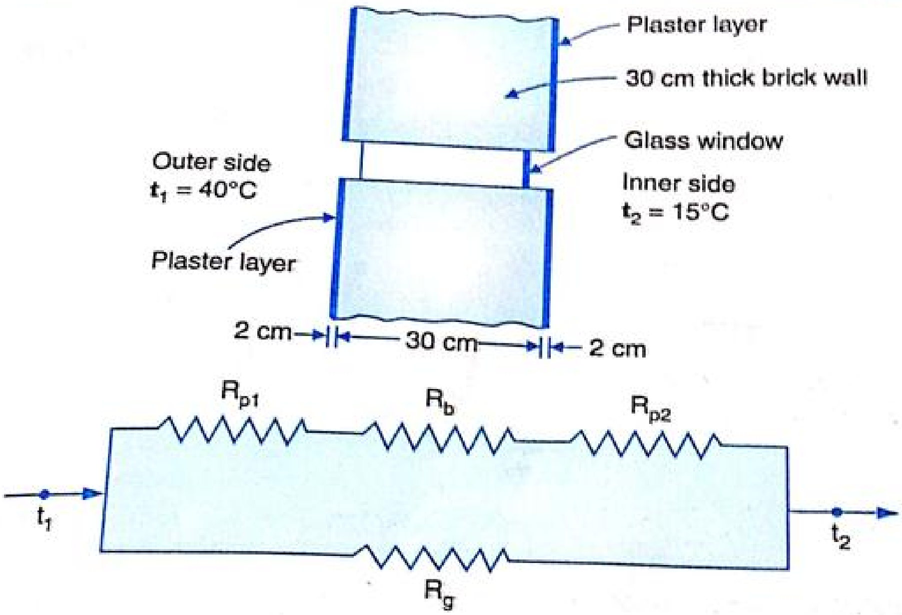


Fig. 3.21.

$$R_e = R_{p_1} + R_b + R_{p_2}$$
$$= 2.564 \times 10^{-3} + 76.92 \times 10^{-3} + 2.564 \times 10^{-3}$$
$$= 82.048 \times 10^{-3} \text{ deg/W}$$

This equivalent resistance R_e is in parallel with resistance R_g due to glass window.

∴ Resistance for the thermal circuit,

$$R_t = \frac{R_e \times R_g}{R_e + R_g}$$
$$= \frac{(82.048 \times 10^{-3}) \times (5 \times 10^{-3})}{(82.048 \times 10^{-3}) + (5 \times 10^{-3})}$$
$$= 4.713 \times 10^{-3} \text{ deg/W}$$

The heat flow through the wall is then given by

$$Q = \frac{t_1 - t_2}{R_t} = \frac{40 - 15}{4.713 \times 10^{-3}}$$
$$= 5.30 \times 10^3 \text{ W}$$
$$= \mathbf{5.30 \text{ kW}}$$

EXAMPLE 3.24

Two slabs, each 100 mm thick and made of materials with thermal conductivities of 16 W/m-deg and 1600 W/m-deg, are placed in contact which is not perfect. Due to roughness of surfaces, only 40% of area is in contact and air fills 0.02 mm thick gap in the remaining area. If the extreme surfaces of the arrangement are at temperatures of 250°C and 30°C, determine the heat flow through the composite system, the contact resistance and temperature drop in contact.

Take thermal conderctivity of air as 0.032 W/m-deg and assume that half of the contact (of the confact area) is due to either metal.

Solution : Refer Fig 3.22 for the composite system and its equivalent thermal resistance

The various thermal resistances to flow of heat are :

(i) $$R_{t_a} = \frac{\delta_a}{k_a A_a} = \frac{100 \times 10^{-3}}{16 \times 1} = 0.00625 \text{ deg/W}$$

(ii) $$R_{t_b} = \frac{\delta_b}{k_b A_b} = \frac{0.02 \times 10^{-3}}{16 \times 0.2} = 0.00000625 \text{ deg/W}$$

(iii) $$R_{t_c} = \frac{\delta_c}{k_c A_c} = \frac{0.02 \times 10^{-3}}{0.032 \times 0.6} = 0.001042 \text{ deg/W}$$

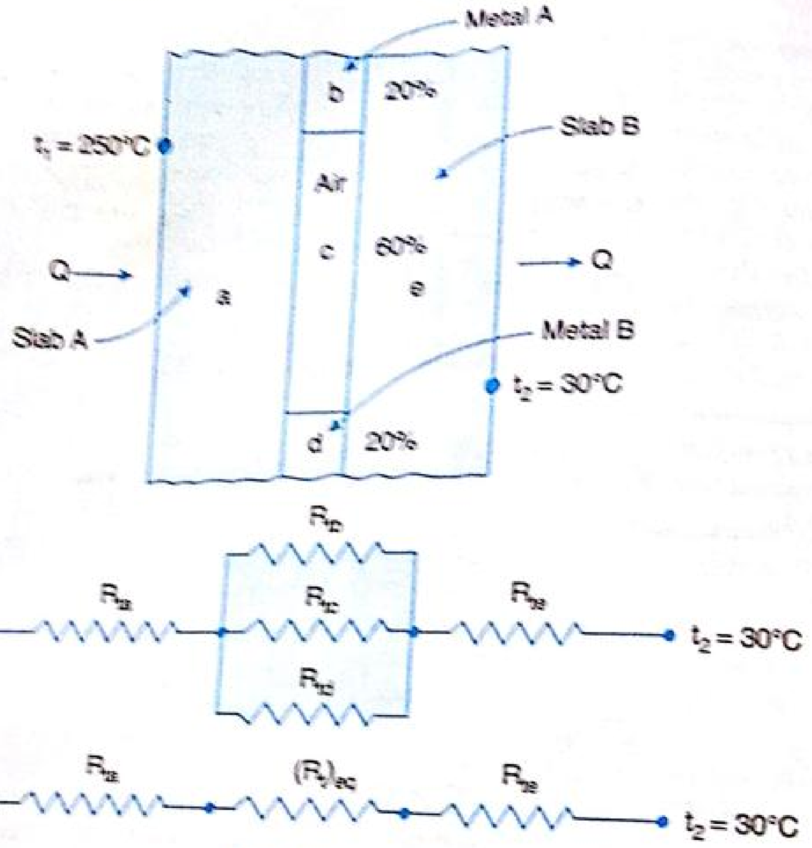


Fig. 3.22.

(iv) $R_{t_d} = \dfrac{\delta_d}{k_d A_d} = \dfrac{0.02 \times 10^{-3}}{200 \times 0.2}$

$= 0.0000005$ deg/W

(v) $R_{t_e} = \dfrac{\delta_e}{k_e A_e} = \dfrac{100 \times 10^{-3}}{200 \times 1}$

$= 0.0005$ deg/W

The resistances R_{t_b}, R_{t_c} and R_{t_d} are in parallel and their equivalent resistance $(R_t)_{eq}$ is

$$\frac{1}{(R_t)_{eq}} = \frac{1}{R_{t_b}} + \frac{1}{R_{t_c}} + \frac{1}{R_{t_d}}$$

$$= \frac{1}{0.00000625} + \frac{1}{0.001042} + \frac{1}{0.0000005}$$

$= 160000 + 959.7 + 2000000$

$= 2160959.7$

$\therefore \ (R_t)_{eq} = \dfrac{1}{2160959.7}$

$= 0.462 \times 10^{-6}$ deg/W

This equivalent resistance is now in series with resistance R_{t_a} and R_{t_e}. The total thermal resistance for the entire circuit then becomes

$\Sigma R_t = 0.00625 + 0.462 \times 10^{-6} + 0.0005$

$= 0.00675$ deg/W

Hence, heat transfer rate through the system is

$$Q = \frac{\Delta t}{\Sigma R_t} = \frac{250 - 30}{0.00675} = 32592 \text{ W}$$

(b) Contact resistance

$= 0.462 \times 10^{-6}$ deg/W

Temperature drop in contact

$= Q \times$ contact resistance

$= 32592 \times (0.462 \times 10^{-6})$

$= 0.01505°C$

EXAMPLE 3.25

Indicate a composite system having heat transfer with series and parallel barriers. A composite wall for thermal insulation has a rectangular section 2 m × 0.5 m and is made from timber 15 cm thick, cork board 30 cm thick and steel plate 5 cm thick. The temperatures at the outside faces of timber and steel are 25°C and 150°C respectively. How the heat transfer rate would be affected if aluminium rods of 4 cm diameter were inserted through each square metre of the composite wall. Neglect the effect of bolt heads and all lateral heat transfers.

The thermal conductivities are :

timber 0.12 W/mK, cork 0.035 W/mK

steel 45 W/mK and aluminium 205 W/mK

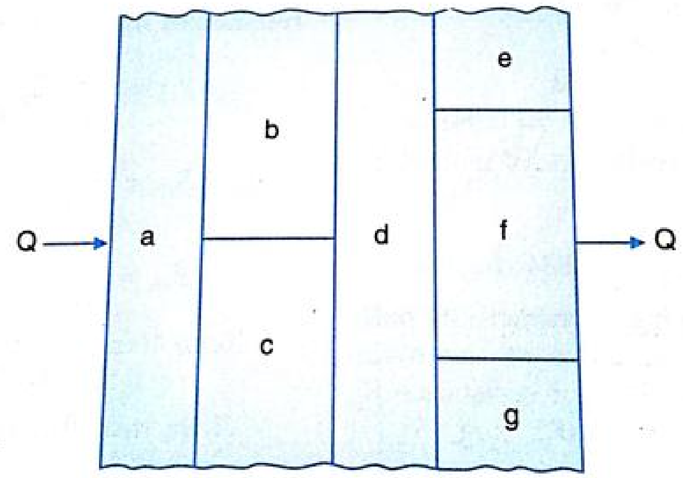
Solution : A thermal circuit for heat flow in a composite system has been illustrated in Fig. 3.23. When the heat flows in series first through one layer and then through another, overall thermal resistance is the sum of the component resistances. For heat flow through layers arranged in parallel, the overall thermal conductance (reciprocal of thermal resistance) equals the sum of the component thermal conductances.

(*b*) Total resistance for the composite wall is

$$\Sigma R_t = R_{timber} + R_{cork} + R_{steel}$$

$$= \frac{\delta_1}{k_1 A_1} + \frac{\delta_2}{k_2 A_2} + \frac{\delta_3}{k_3 A_3}$$

The wall area (2 m × 0.5 m) = 1 m² is constant for all layers of the composite wall



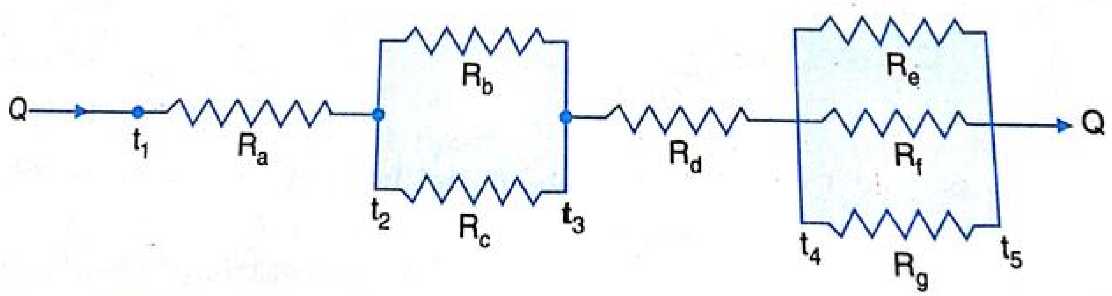


Fig. 3.23. Thermal circuit for a heat flow in a composite system

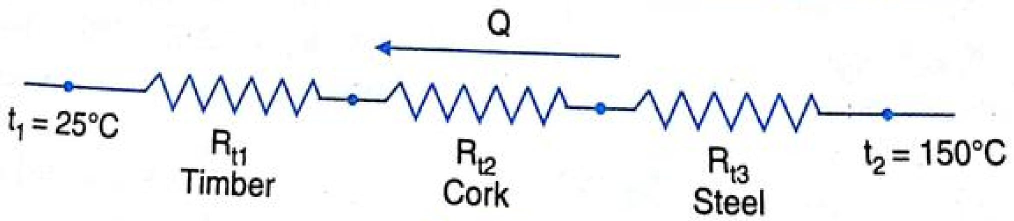


Fig. 3.24.

$$\therefore \quad \Sigma R_t = \frac{0.15}{0.12 \times 1} + \frac{0.30}{0.035 \times 1} + \frac{0.50}{45 \times 1}$$
$$= 1.25 + 8.571 + 0.001$$
$$= 9.822 \text{ deg/W}$$

Heat flow (without bolt)

$$= \frac{\Delta t}{\Sigma R_t} = \frac{150 - 25}{9.822} = 12.72 \text{ W}$$

When aluminium rods are inserted :

Area of the bolt

$$= \frac{\pi}{4}(0.04)^2 = 1.256 \times 10^{-3} \text{ m}^2$$

Thermal resistance due to bolt

$$= \frac{\text{length of the bolt}}{\text{thermal conductivity} \times \text{area}}$$
$$= \frac{(15 + 30 + 5) \times 10^{-2}}{205 \times (1.256 \times 10^{-3})}$$
$$= 1.942 \text{ deg/W}$$

Area of wall less bolt area

$$= 1 - 1.256 \times 10^{-3} = 0.99874 \text{ m}^2$$

Modified thermal resistance of insulating wall

$$= \frac{9.822 \times 1}{0.99874} = 9.834 \text{ deg/W}$$

The bolt provides a high conductivity path in parallel with the resistance of the main composite wall. The equivalent resistance R_e is given by

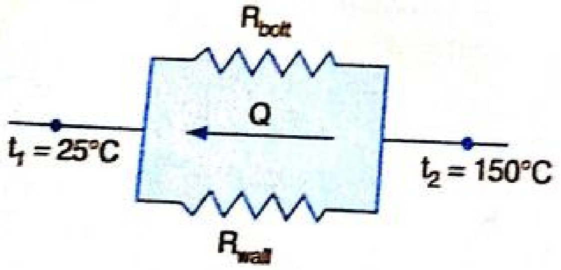


Fig. 3.25.

$$\frac{1}{R_e} = \frac{1}{R_{bolt}} + \frac{1}{R_{wall}}$$
$$R_e = \frac{R_{wall} \times R_{bolt}}{R_{wall} + R_{bolt}}$$
$$= \frac{9.834 \times 1.942}{9.834 + 1.942} \; 1.621 \text{ deg/W}$$

$\therefore$ Heat flux (with bolt)

$$= \frac{150 - 25}{1.621} = 77.11 \text{ W}$$

EXAMPLE 3.26.

The insulation boards for air conditioning purposes comprise three layers. A 12 cm thick layer of grass (k = 0.022 W/mK) is sandwiched between 3 cm thick layer of plywood (k = 0.15 W/mK) on each side. The bonding is achieved with glue which does not offer any resistance to heat flow. If the side surfaces of the board are maintained at 40°C and 20°C temperatures, determine the heat flux. How would the heat flux be affected if instead of glue, the three pieces are fastened by four steel bolts (k = 40 W/mK) of 1.2 cm diameter at the corners ?

Solution : $\delta_1 = 3$ cm, $\delta_2 = 12$ cm,

and $\delta_3 = 3$ cm

Considering unit area, the thermal resistances for the respective layers are :

$$R_{t1} = \frac{\delta_1}{k_1 A} = \frac{0.03}{0.15 \times 1} = 0.2 \text{ deg/W}$$
$$R_{t2} = \frac{\delta_2}{k_2 A} = \frac{0.12}{0.022 \times 1} = 5.45 \text{ deg/W}$$
$$R_{t3} = \frac{\delta_3}{k_3 A} = \frac{0.03}{0.15 \times 1} = 0.2 \text{ deg/W}$$

Total thermal resistance

$$R_t = 0.2 + 5.45 + 0.2 = 5.85 \text{ K/W}$$

Then, heat flux (heat flow per m² area)

$$q = \frac{t_1 - t_2}{R_t} = \frac{40 - 20}{5.85}$$
$$= 3.42 \text{ W/m}^2$$

(*b*) The arrangement and the thermal circuit for the system when the layers are joined by steel bolts is shown below in Fig. 3.27.

The resistance R_{t1}, R_{t2} and R_{t3} as calculated above are :

$$R_1 = 0.2 \text{ deg/W} ; \quad R_2 = 5.45 \text{ deg/W}$$
and $R_3 = 0.2$ deg/W

These resistances are in series with total resistance

$$R_t = 0.2 + 5.45 + 0.2$$
$$= 5.85 \text{ deg/W}$$

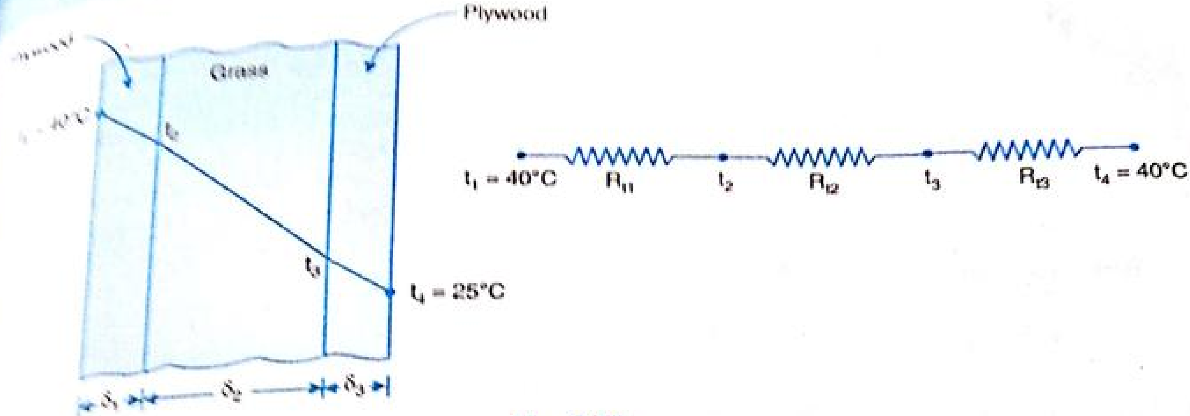


Fig. 3.26.

Resistance of a bolt

$$= \frac{\delta_1 + \delta_2 + \delta_3}{kA}$$

$$= \frac{0.03 + 0.12 + 0.03}{40 \times \frac{\pi}{4} \times (0.012)^2} = 39.81 \text{ deg/W}$$

There will be four such resistances in parallel with equivalent resistance

$$\frac{1}{R_b} = \frac{1}{39.81} + \frac{1}{39.81} + \frac{1}{39.81} + \frac{1}{39.81}$$

$$= \frac{4}{39.81}$$

Now, R_b and R_t are in parallel and therefore equivalent thermal resistance for the circuit becomes :

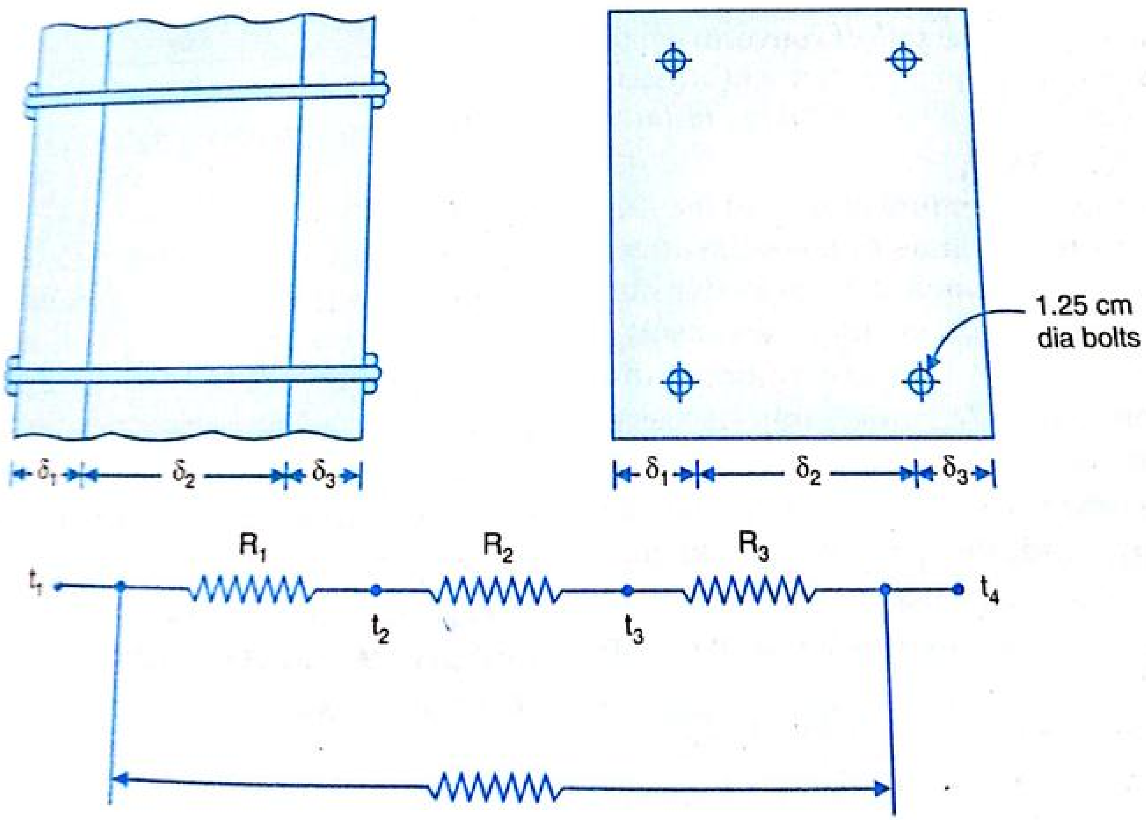


Fig. 3.27.

$$\frac{1}{R_{eq}} = \frac{1}{R_t} + \frac{1}{R_b}$$
$$= \frac{1}{5.85} + \frac{4}{39.81}$$
$$= 0.1709 + 0.1005 = 0.2714$$

$$\therefore \quad R_{eq} = \frac{1}{0.2714} = 3.685 \text{ deg/W}$$

Then, heat flux (heat flow per m^2 area)

$$q = \frac{t_1 - t_2}{R_{eq}} = \frac{40-20}{3.685} = 5.43 \text{ W/m}^2$$

3.3. HEAT FLOW BETWEEN SURFACE AND SURROUNDINGS : COOLING AND HEATING OF FLUIDS

When a moving fluid comes into contact with a stationary surface, a thin boundary layer develops adjacent to the wall and in this layer there is no relative velocity with respect to surface. In a heat exchange process, this layer is called ***stagnant film*** and heat flow in the layer is covered both by conduction and convection processes. Since thermal conductivity of fluids is low, the heat flow from the moving fluid of the wall is mainly due to convection. The rate of convective heat transfer between a solid boundary and adjacent fluid is given by the *Newton-Rikhman law* :

$$Q = hA\ (t_s - t_f) \quad \text{...(3.11)}$$

where t_f is the temperature of the hot moving fluid, t_s is the temperature of the wall surface and A is the area exposed to heat transfer. The convective coefficient (film coefficient) h depends upon the thermodynamic and transport properties (*e.g.*, density, viscosity, specific heat and thermal conductivity) of the fluid, the geomeny of the surface, the nature of fluid flow and the prevailing thermal conditions. The resistance of the stagnant boundary layer is included in the convection coefficient.

The dimensions of h are W/m^2-deg.

The factor $1/hA$ represents the thermal resistance of the film heat transfer process.

The heat transfer through a wall separating two moving fluids involves: (*i*) flow of heat from the fluid of high temperature to the wall, (*ii*) heat conduction through the wall and (*iii*) transport of heat from the wall to the cold fluid (Fig. 3.28)

Under steady state conditions, the heat flow can be expressed by the equations :

$$Q = h_a A\ (t_a - t_1)$$
$$= \frac{kA}{\delta}\ (t_1 - t_2)$$
$$= h_b A\ (t_2 - t_b) \quad \text{...(3.12)}$$

where h_a and h_b represent the convective film coefficients, k is the thermal conductivity of the solid wall having thickness δ. These expressions can be presented in the form :

$$t_a - t_1 = \frac{Q}{h_a A}$$

$$t_1 - t_2 = \frac{Q\delta}{kA}\ ; \quad t_2 - t_b = \frac{Q}{h_b A}$$

Summation of these equalities yields

$$(t_a - t_b) = Q\left[\frac{1}{h_a A} + \frac{\delta}{kA} + \frac{1}{h_b A}\right]$$

$$\therefore \quad Q = \frac{(t_a - t_b)}{\dfrac{1}{h_a A} + \dfrac{\delta}{kA} + \dfrac{1}{h_b A}} \quad \text{...(3.13)}$$

The denominator $(1/h_a A + \delta/kA + 1/h_b A)$ is the sum of thermal resistance of different sections through which heat has to flow.

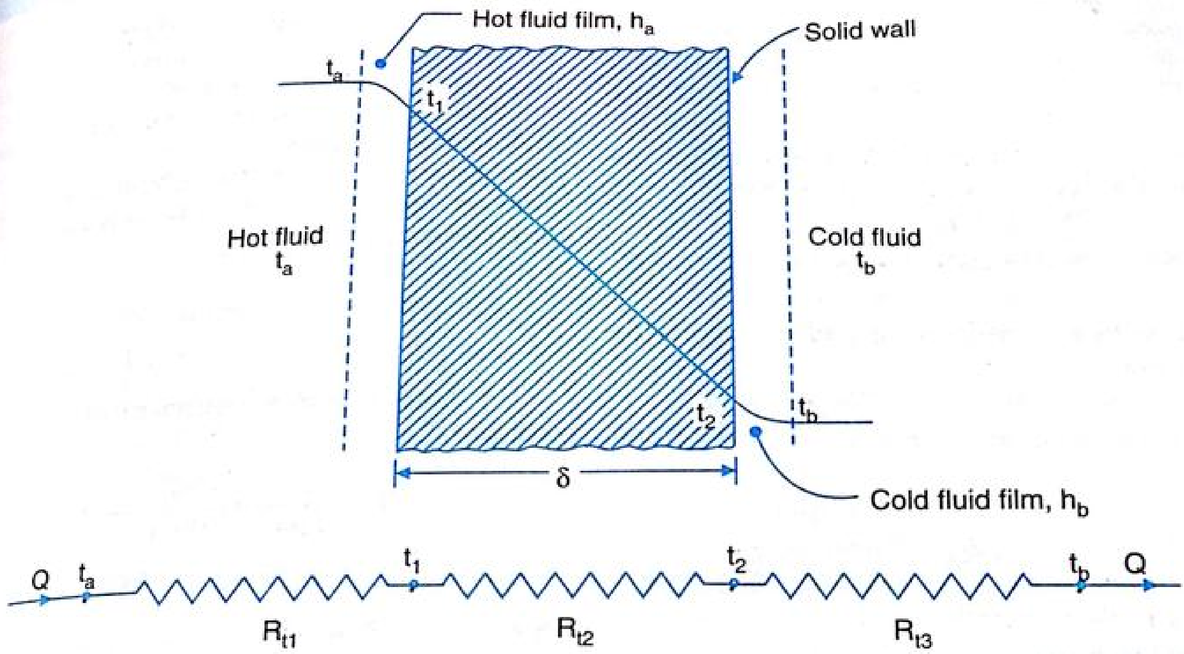
Quite often, the heat flow through a composite section is written in the form

$$Q = UA\ (t_a - t_b) \quad \text{...(3.14)}$$

where U is the *overall heat transfer coefficient*. It represents the intensity of heat transfer from one fluid to another through a wall separating them. Numerically it equals the quantity of heat passing through unit area of wall surface in unit time at a temperature difference of unit degree. The coefficient U has dimensions of W/m^2-deg.

Comparing equations 3.13 and 3.14.

$$\frac{1}{U} = \frac{1}{h_a} + \frac{\delta}{k} + \frac{1}{h_b} \quad \text{...(3.15)}$$



$$R_{t1} = \frac{1}{h_a A} \;;\; R_{t2} = \frac{\delta}{kA} \;;\; R_{t3} = \frac{1}{h_b A}$$

Fig. 3.28. Heat conduction through a wall separating two fluids

Apparently the overall heat transfer coefficient is reciprocal of unit thermal resistance to heat flow.

The overall heat transfer coefficient depends upon the geometry of the separating wall, its thermal properties and the convective coefficient at the two surfaces. The overall heat transfer coefficient is particularly useful in the case of composite walls, such as in the design of structural walls for boilers, refrigerators, airconditioned buildings, and in the design of heat exchangers.

EXAMPLE 3.27

An electric heater of exposed surface area 0.09 m^2 and output 600 watt is designed to operate fully submerged in water. Calculate the surface temperature of the heater when the water is at 37°C and the surface coefficient of heat transfer is 285.3 W/m^2-deg. How this value will be affected if the heater is mistakenly operated at 37°C in air with a surface co-efficient of 8.5 W/m^2-deg?

Solution : When the heater operates in water as per design,

$$Q = h\, A\, \Delta t$$

$$600 = 283.5 \times 0.09(T_s - 37)$$

∴ Surface temperature of the heater,

$$T_s = \frac{600}{283.5 \times 0.09} + 37 = 60.5°C$$

(*ii*) When the heater operates in air,

$$600 = 8.5 \times 0.09\ (T_s - 37)$$

∴ Surface temperature of the heater

$$T_s = \frac{600}{8.5 \times 0.09} + 37 = 821°C$$

This temperature is quite high and may result in mechanical failure (melting) of the material of the heating element.

EXAMPLE 3.28

A container with outside surface area 0.36 m and outside temperature of 0°C contain ice at 0°C. The container is placed in ambient ai

at 24°C and the surface coefficient of heat transfer between the container surface and the surrounding air is estimated to be 6.25 W/m²-deg. Calculate the rate at which ice would be changed into liquid water. Take latent heat of fusion of Ice as 340 J/g.

Solution : The convective heat flow from the air to the ice container is given by

$$Q = h A \Delta t$$
$$= 6.25 \times 0.36 \times (24 - 0)$$
$$= 54 \text{ W} = 54 \text{ J/s}$$

This heat is utilized in melting ice in the container

$$\therefore \quad mL = 54\ ; \quad m \times 340 = 54$$

Hence mass of ice melted,

$$m = \frac{54}{340} = 0.1588 \text{ g/s}$$
$$= \frac{0.1588}{1000} \times 3600 = \mathbf{0.572 \text{ kg/h}}$$

EXAMPLE 3.29

A lake surface is covered by a 8 cm thick layer of ice (k = 8 kJ/m-hr-deg) when the ambient air temperature is – 12.5°C. A thermo-couple embedded on the upper surface of the layer indicates a temperature of – 5°C. Assuming steady state conduction in ice and no liquid subcooling at the bottom surface of the ice layer, find the heat transfer coefficient at the upper surface. Also work out the heat loss per square kilometer of area.

Solution : Since water at the bottom surface of ice layer is to remain in liquid state (no sub-cooling), the minimum temperature required at that surface is 0°C.

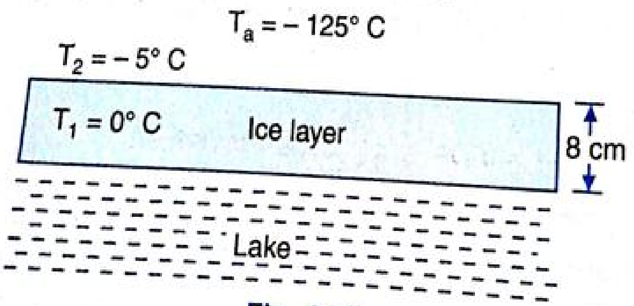


Fig. 3.29.

Invoking Fourier's law, the conduction heat transfer rate across the ice layer becomes

$$Q = \frac{k A \Delta t}{\delta}$$

$$= \frac{8 \times (1000 \times 1000) \times \{0 - (-5)\}}{0.08}$$
$$= 5 \times 10^8 \text{ kJ/hr}$$

Since steady state conditions prevail, the conduction heat flow across the ice slab equals the convective heat transfer between the top surface of ice slab and surrounding atmosphere. Accordingly

$$Q = h A \Delta t$$
$$5 \times 10^8 = h \times (1000 \times 1000) \times [-5 - (-12.5)]$$

$\therefore$ Heat transfer coefficient at the upper surface,

$$h = \frac{5 \times 10^8}{(1000 \times 1000) \times 7.5}$$
$$= \mathbf{66.67 \text{ m}^2\text{-hr-deg}}$$

EXAMPLE 3.30

The oven of an electric stove, of total outside surface area 2.9 m² dissipates electric energy at the rate of 600 watt. The surrounding room air is at 20°C and the surface coefficient of heat transfer between the room air and the surface of the oven is estimated to be 11.35 W/m²-deg. Determine the average steady-state temperature of the outside surface of the stove. What would be the inside surface temperature if wall thickness of stove is 3.8 cm and thermal conductivity of the stove material is 0.069 W/m-deg ?

Solution : The electric energy is dissipated as convective heat flow from the outside surface of stove to the ambient air. Therefore

$$Q = h A (t_o - t_a)$$
$$600 = 11.35 \times 2.9 \times (t_o - 20)$$

$\therefore$ Outside surface temperature of the stove,

$$t_o = \frac{600}{11.35 \times 2.9} + 20 = 38.22° \text{ C}$$

The electric energy is first conducted across the wall of the oven. Invoking Fourier's law of heat conduction

$$Q = \frac{kA(t_i - t_o)}{\delta}$$

$$600 = \frac{0.069 \times 2.9 \times (t_i - 38.22)}{0.038}$$
$$= 5.26\ (t_i - 38.22)$$

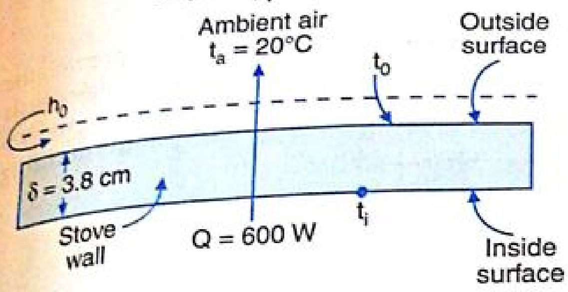


Fig. 3.30.

$\therefore$ Inside surface temperature of the stove

$$t_i = \frac{600}{5.26} + 38.22 = \mathbf{152.29°C}$$

EXAMPLE 3.31

Hot gases at 980°C flow past the upper surface of the blade of a gas turbine and the lower surface is cooled by air bled off the compressor. The convective heat transfer coefficient at the upper and lower surfaces are estimated to be 2830 and 1415 W/m²-deg respectively. The blade material has a thermal conductivity of 11.6. W/m-deg. If metallurgical considerations limit the blade temperature at 870°C, workout the temperature of the cooling air. Consider the blade as a flat plate 0.115 cm thick and presume that steady state conditions have been reached.

Solution : Heat flow rate per unit area, from the hot gases to the upper surface of the blade is

$$Q = h_u A\ (t_g - t_u)$$
$$= 2830 \times 1 \times (980 - 870)$$
$$= 311300\ \text{W/m}^2$$

Since steady state conditions prevail, this heat would be conducted across the blade. Then from Fourier's law of heat conduction

$$Q = \frac{kA(t_u - t_l)}{\delta}$$

$$311300 = \frac{11.6 \times 1 \times (870 - t_l)}{0.00115}$$
$$= 10087(870 - t_l)$$

$\therefore$ Temperature at the lower surface of the blade,

$$t_l = 870 - \frac{311300}{10087} = 839°C$$

The heat conducted across the blade would be finally transferred to the coolant by convection, then

$$311300 = 1415 \times 1 \times (839 - t_c)$$

$\therefore$ Temperature of the coolant air

$$t_c = 839 - \frac{311300}{1415} = 619°C$$

EXAMPLE 3.32

A kitchen oven has its maximum operating temperature set at 290°C, where as the temperature in the kitchen may vary from 15°C to 30°C due to seasonal variations. Workout the necessary thickness of fibre glass (k = 0.035 W/m-deg) insulation to ensure that the outside surface temperature of oven does not exceed 40°C. The average heat transfer coefficient between the outside oven surface and the kitchen air is 10 W/m²-deg. Neglect thermal resistance of metal wall and presume that steady conditions prevail.

Solution : The thermal resistance to heat flow is essentially provided by the fibre glass

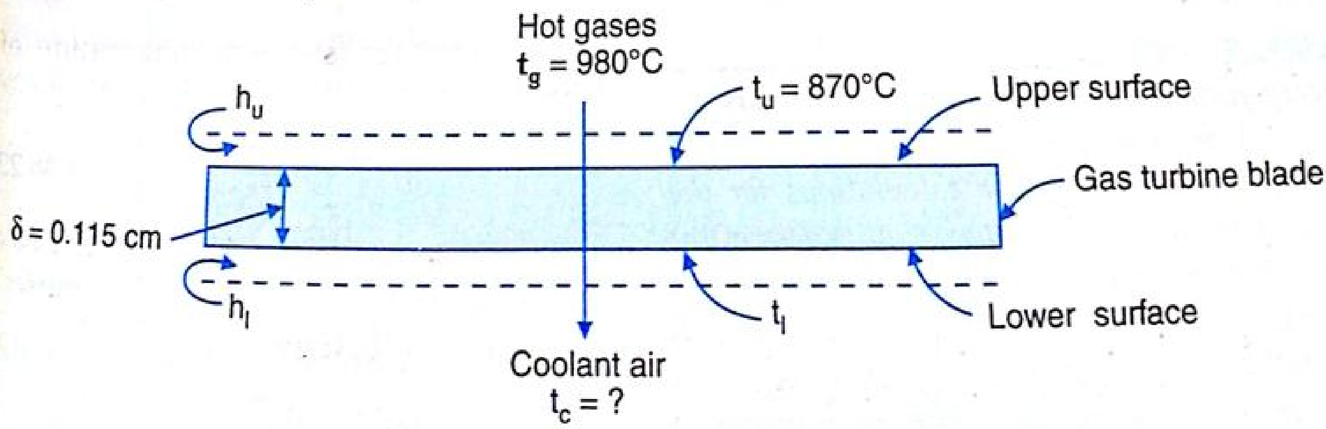


Fig. 3.31.

insulation and the convective heat transfer coefficient between the outside oven surface and the kitchen air. Therefore for unti area of the wall,

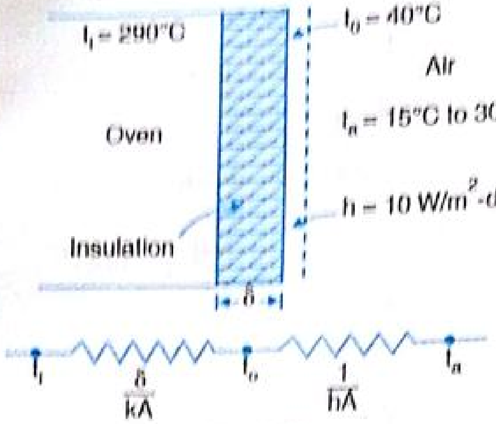


Fig. 3.32.

$$\frac{Q}{A} = \frac{t_i - t_a}{\frac{\delta}{k} + \frac{1}{h}}$$

Further for steady state conditions, heat flowing through each section is same.

$$\therefore \quad \frac{t_i - t_a}{\frac{\delta}{k} + \frac{1}{h}} = \frac{t_o - t_a}{\frac{1}{h}}$$

$$\text{or } (t_i - t_a)\frac{1}{h} = (t_o - t_a)\frac{\delta}{k} + (t_o - t_a)\frac{1}{h}$$

$$\text{or} \quad \delta = \frac{t_i - t_o}{t_o - t_a} \times \frac{k}{h}$$

The thickness of fibre glass insulation will be large for $t_a = 30°C$.

$$\therefore \quad \delta = \frac{290 - 40}{40 - 30} \times \frac{0.035}{10}$$

$$= 0.0875 \text{ m} \quad \text{or} \quad 8.75 \text{ cm}$$

EXAMPLE 3.33

The temperatures at the inside and outside surfaces of the brick work of a furnace have been noted to be 650°C and 225°C. Make calculations for the percentage decrease in heat loss if thickness of the brick work is increased by 100 percent. The ambient temperature is 30°C and assume that thermal conductivity of brickwork and convective coefficient remain the same before and after the increase in thickness.

Solution : Under steady state conditions and considering unit area perpendicular to the direction of heat flow

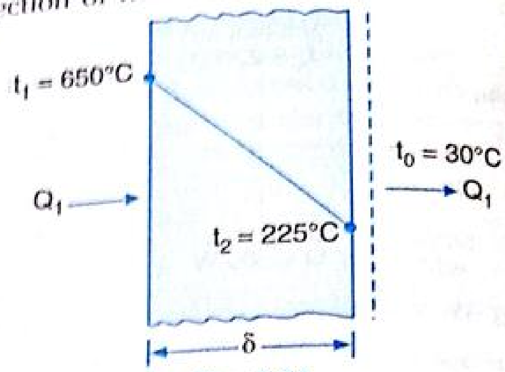


Fig. 3.33.

$$Q_1 = \frac{t_1 - t_0}{\frac{\delta}{k} + \frac{1}{h_0}}$$

Further the steady state conditions, the same quantity of heat flows through each section

$$\therefore \quad \frac{t_1 - t_0}{\frac{\delta}{k} + \frac{1}{h_0}} = \frac{t_1 - t_2}{\frac{\delta}{k}}$$

$$\text{or} \quad \frac{650 - 30}{C + \frac{1}{h_0}} = \frac{650 - 225}{C} \quad \text{where } C = \delta/k$$

$$\text{or} \quad \frac{620}{C + \frac{1}{h_0}} = \frac{425}{C}$$

Simplification gives:

$$\frac{1}{h_o} = 0.458\ C$$

$$\text{Also } Q_1 = \frac{650 - 225}{C} = \frac{425}{C}$$

(*b*) Considering the arrangement when the thickness of the brickwork has been increased by 100 percent

$$Q_2 = \frac{t_1 - t_o}{\frac{2\delta}{k} + \frac{1}{h_0}}$$

$$= \frac{650 - 30}{2C + 0.458C} \text{ as } \frac{1}{h_o} = 0.458\ C$$

$$= \frac{620}{2.458\ C} = 252.2\ C$$

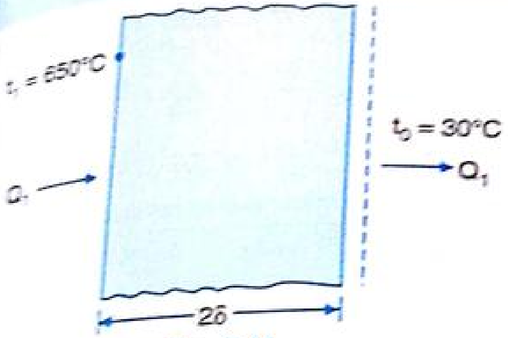


Fig. 3.34.

∴ Decrease in heat flow

$$= \frac{Q_1 - Q_2}{Q_1}$$

$$= \frac{425 - 252.2}{425}$$

$$= 0.4066 \text{ or } 40.66\%$$

EXAMPLE 3.34

A glazed window, made of 8 mm thick glass of thermal conductivity 1.5 W/mK, has its outside surface maintained at 5°C so that frosting is reduced. The surroundings are at – 10°C with convective coefficient 55 W/m²K. The desired condition is attained by providing a uniform heat flux at the inner surface of the window which is fitted into a room where the air temperature is 25°C with a convection coefficient of 12.5 W/m²K. Make calculations for the heating required per m² area.

Solution : Refer Fig. 3.35 for the window fixture with specified data and thermal circuit for the resistance involved.

Let t_1 be the temperature at the heater. Under steady state conditions heat conducted through the glass barrier equals the heat convected through the outde film. That is

$$\frac{kA(t_1 - t_2)}{\delta} = h_o A(t_2 - t_o)$$

Considering unit area,

$$\frac{1.5 \times 1(t_1 - 5)}{0.008} = 55 \times 1[5 - (-10)] = 825 \text{ W}$$

That gives :

$$t_1 = \frac{825 \times 0.008}{1.5} + 5 = 9.4°\text{C}$$

(*b*) From energy balance

heat flow (*Q*) + heat received by convection fram room (Q_2)

= heat conducted through the glass barrier (Q_1)

or heat flux *Q*

$$= Q_1 - h_i A(t_i - t_1)$$
$$= 825 - 12.5 \times 1\ (25 - 9.4)$$
$$= 630 \text{ W}$$

Thus, the heat required per m² area is 630 W.

EXAMPLE 3.35.

A square plate heater of 0.8 kW rating and measuring 15 cm × 15 cm is placed between two slabs A and B and the following data refers to these slabs :

Slab A is 1.8 cm thick with k = 55 W/m-deg

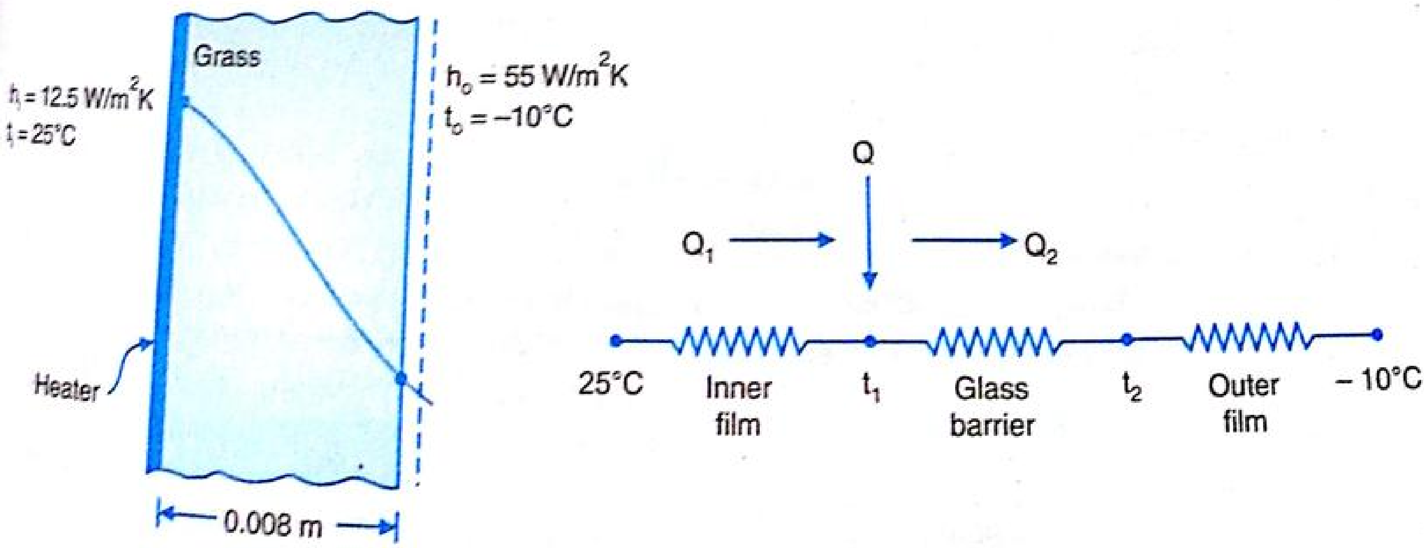


Fig. 3.35.

Slab B is 1 cm thick with k = 0.2 W/m-deg

The outside heat transfer coefficients on the sides of plate A and B are 200 W/m²-deg and 45 W/m²-deg respectively. If the surrounding environment is at 27°C temperature, make calculations for the maximum temperature of the system and outside surface temperature of both slabs.

Solution : Refer Fig. 3.36 for the arrangement and thermal resistance network for the system.

The individual resistances are evaluated as :

$$R_{t1} = \frac{\delta_1}{k_1 A_1} = \frac{1.8\times 10^{-2}}{55\times(0.15\times 0.15^{-2})^2}$$

$$= 0.145 \text{ deg/W}$$

$$R_{t2} = \frac{1}{h_1 A_1} = \frac{1}{200\times(0.15\times 0.15^{-2})}$$

$$= 0.222 \text{ deg/W}$$

These resistances are in series and accordingly for slab A(left branch of the circuit)

$$R_{t1} + R_{t2} = 0.0145 + 0.222 = 0.2365 \text{ deg/W}$$

$$R_{t3} = \frac{\delta_2}{k_2 A} = \frac{1\times 10^{-2}}{0.2\times(0.15\times 0.10^{-2})^2}$$

$$= 2.222 \text{ deg/W}$$

$$R_{t4} = \frac{1}{h_2 A_2} = \frac{1}{45\times(15\times 0.15^{-2})^2}$$

$$= 0.987 \text{ deg/W}$$

These resistances are in series and accordingly for slab B (right branch of the circuit)

$$R_{t3} + R_{t4} = 2.222 + 0.987 = 3.209 \text{ deg/W}$$

(*a*) Rating of heater,

$$Q = Q_A + Q_B$$

$$= \frac{T_{max} - T_a}{R_{t1} + R_{t2}} + \frac{T_{max} - T_a}{R_{t3} + R_{t4}}$$

$$= (T_{max} - T_a)\left[\frac{1}{0.2365} + \frac{1}{3.209}\right]$$

$$= 4.5396\ (T_{max} - T_a)$$

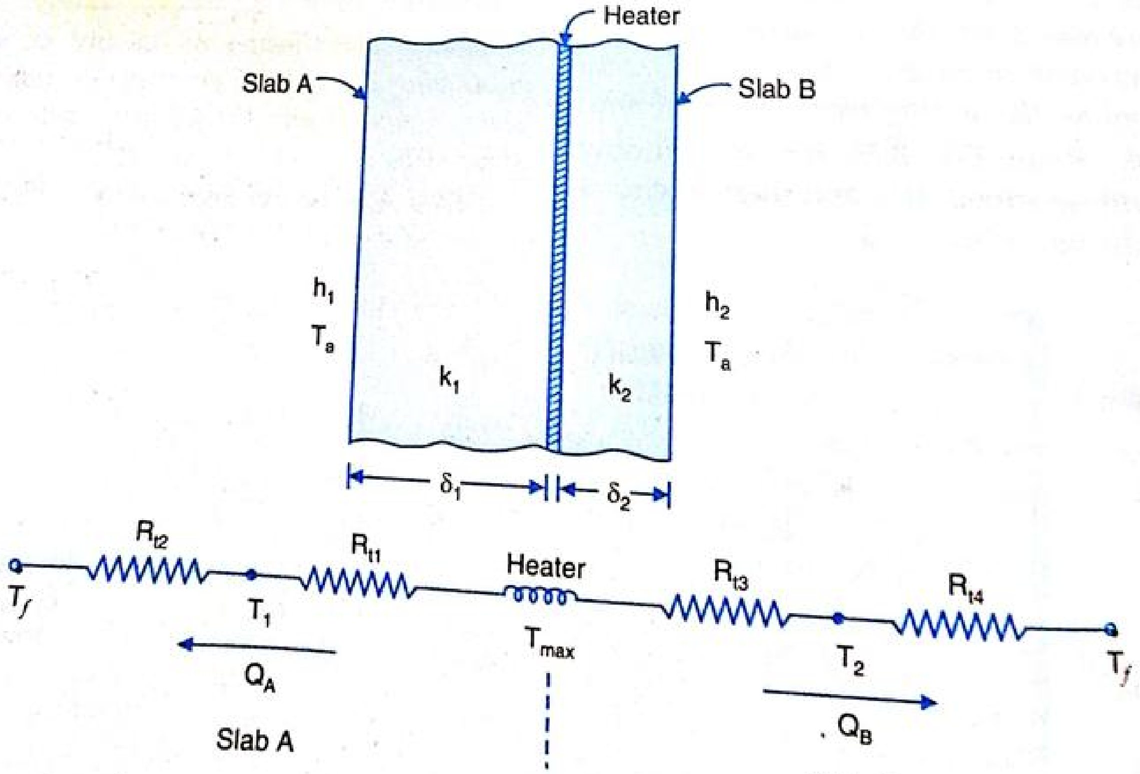


Fig. 3.36.

$\therefore$ Maximum temperature in the system,

$$T_{max} = \frac{Q}{4.5396} + T_a$$

$$= \frac{0.8 \times 10^3}{4.5396} + 27 = \mathbf{203.24°C}$$

(*b*) Considering left side branch of the circuit (slab *A*)

$$Q_A = \frac{T - T_a}{R_{t1} + R_{t2}}$$

$$= \frac{203.24 - 27}{0.2365} = 745.2 \text{ W}$$

If T_1 is the temperature at exposed surface of slab *A*, then

$$Q_A = \frac{T_1 - T_a}{R_{t2}} ;$$

$$T_1 = Q_A R_{t2} + T_a$$

$$= 745.2 \times 0.222 + 27 = \mathbf{192.43°C}$$

Considering right side branch of the circuit (slab *B*)

$$Q_B = \frac{T_2 - T_a}{R_{t4}} ;$$

$$T_2 = Q_B R_{t4} + T_a$$

$$= 54.8 \times 0.987 + 27 = \mathbf{81.09°C}$$

EXAMPLE 3.36

A 30 cm thick furnace wall (k = 1.5 W/m-deg) has inner surface temperature of 1250°C. If the heat transfer coefficient on the outer surface is prescribed by the relation

$$h_o = 8 + 0.09\ \Delta t$$

where Δt is the temperature difference between the outer wall surface and surroundings, calculate the rate of heat loss per unit area of the wall. Take surrounding temperature as 30°C.

(*b*) *Subsequently to keep the heat loss within maximum limit of 450 W/m², the outside surface of the wall is provided a layer of heat resistance brick (k = 0.7 W/m-deg) followed by 30 cm thick layer of silica brick (k = 0.15 W/m-deg). If the surrounding temperature remains unchanged at 30°C, what should be the thickness of heat resistance brick.*

Solution : Considering unit area of the furnace wall,

$$Q_1 = \frac{t_1 - t_o}{\delta_1 / k_1 + 1/h_o}$$

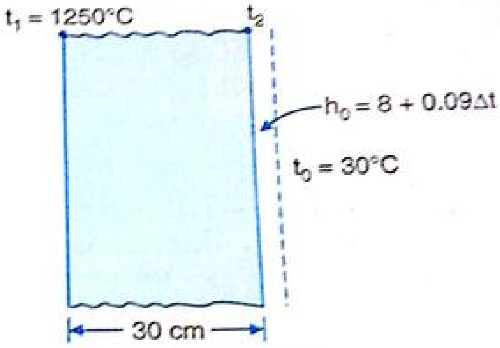


Fig. 3.37.

Further for steady state conditions, the same quantity of heat flows through each section.

$$\therefore \frac{t_1 - t_o}{\frac{\delta_1}{k_1} + \frac{1}{h_o}} = \frac{t_2 - t_0}{\frac{1}{h_o}}$$

$$\text{or } (t_2 - t_0) = \frac{1}{h_o}\left[\frac{t_1 - t_o}{\frac{\delta_1}{k_1} + \frac{1}{h_o}}\right]$$

$$= C\left[\frac{1250 - 30}{\frac{0.3}{1.5} + C}\right]$$

where $C = 1/h_o$

Substituting this value of $(t_2 - t_0)$ in the given expression for h_0, we get

$$h_0 = 8 + 0.09 \times C\left[\frac{1220}{0.2 + C}\right]$$

$$\text{or} \quad \frac{1}{C} = 8 + 0.09C \times \frac{1220}{0.2 + C}$$

Upon simplification, we get

$$117.8\ C^2 + 0.6C - 0.2 = 0$$

Solution of this quadratic equation gives:

$$C = \frac{-0.6 + \sqrt{(0.6)^2 - 4 \times 117.8(-0.2)}}{2 \times 117.8}$$

$$C = \frac{-0.6 + 9.72}{2 \times 117.8} = \frac{9.06}{2 \times 117.8} = \frac{1}{26}$$

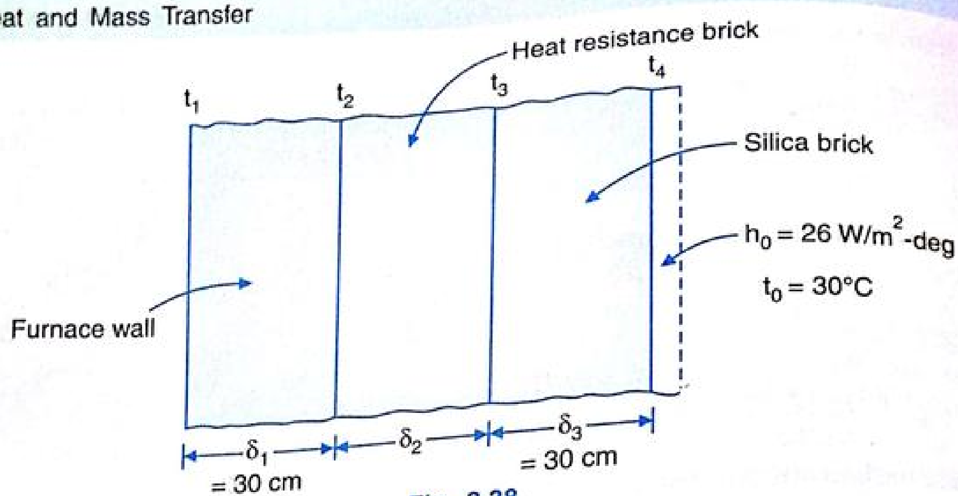


Fig. 3.38.

That gives :

$$\frac{1}{h_o} = \frac{1}{26} \quad \text{or} \quad h_o = 26 \text{ W/m}^2\text{-deg}$$

$$Q_1 = \frac{t_1 - t_o}{\frac{\delta}{k_1} + \frac{1}{h_o}}$$

$$= \frac{1250 - 30}{\frac{0.3}{1.5} + \frac{1}{26}}$$

$$= \frac{1220}{0.2 + 0.3846} = \mathbf{5116.2\ W}$$

(*b*) After providing layers of heat resistance brick and silica brick, the heat flow rate is given by

$$Q_2 = \frac{t_1 - t_o}{\frac{\delta_1}{k_1} + \frac{\delta_2}{k_2} + \frac{\delta_3}{k_3} + \frac{1}{h_o}}$$

Substituting the relevant data,

$$450 = \frac{1250 - 30}{\frac{0.3}{1.5} + \frac{\delta_2}{0.7} + \frac{0.3}{0.15} + \frac{1}{26}}$$

$$= \frac{1220}{2.238 + \frac{\delta_2}{0.7}}$$

$$\text{or } 2.238 + \frac{\delta_2}{0.7} = \frac{1220}{450} = 2.71$$

$$\therefore \quad \delta_2 = (2.71 - 2.238) \times 0.7$$

$$= 0.33 \text{ m} = \mathbf{33\ cm}$$

EXAMPLE 3.37

Consider a plane composite wall that is made of two materials of thermal conductivities $k_a = 735$ kJ/m-hr-deg and $k_b = 165$ kJ/m-hr-deg and thickness $\delta_a = 5$ cm and $\delta_b = 2.5$ cm. Material A adjoins a hot fluid at 150°C for which $h_a = 42$ kJ/m²-hr-deg and the material B is in contact with a cold fluid at 30°C and $h_b = 85$ kJ/m²-hr-deg. Calculate (a) the rate of heat transfer through a wall which is 2 m high and 2.5 m wide (b) the overall coefficient of heat transfer.

Solution : The wall area (2 m × 2.5 m) = 5 m² is constant for all the layers. Total thermal resistance of the composite system is given by

$$\Sigma R_t = \frac{1}{A}\left[\frac{1}{h_a} + \frac{\delta_a}{k_a} + \frac{\delta_b}{k_b} + \frac{1}{h_b}\right]$$

$$= \frac{1}{5}\left[\frac{1}{42} + \frac{0.05}{735} + \frac{0.025}{165} + \frac{1}{85}\right]$$

$$= 0.007013 \ \frac{\text{hr-deg}}{\text{kJ}}$$

Rate of heat transfer

$$= \frac{\Delta t}{\sum R_t} = \frac{150 - 30}{0.007013}$$

$$= 17111 \text{ kJ/hr}$$

(*b*) Heat transfer from the hot to cold fluid can also be expressed as :

$$Q = U A \Delta t$$

where U is the overall coefficient of heat transfer

$\therefore$ $17111 = U \times 5 \times (150 - 30)$

or $U = \dfrac{17111}{5 \times 120} = \mathbf{28.518\ kJ/m^2\text{-}hr\text{-}deg}$

EXAMPLE 3.38

The interior of a refrigerator has inside dimensions 60 cm × 45 cm base area and 120 cm high. The composite wall is made of two 3 mm mild steel sheets (k = 145 kJ/m-hr-deg) with 6 cm of glass wool (k = 0.188 kJ/m-hr-deg) insulation sandwiched between them. The average values of convective heat transfer coefficients at the interior and exterior wall are 40.8 and 52.3 kJ/m²-hr-deg respectively.

(a) Calculate the individual resistance of this composite wall and the resistances at the surfaces, and the overall conductance.

(b) Draw the thermal circuit

(c) For the air temperature inside the refrigerator at 6.5°C and outside of 25°C, determine the rate at which heat must be removed from the refrigerator. Also, calculate the temperature on the outer surface of the metal sheet.

Solution : Neglecting corner effects, the area for heat flow is

$$A = 2(1.2 \times 0.6 + 0.6 \times 0.45 + 0.45 \times 1.2)$$
$$= 3.06\ m^2$$

This is constant for all layers of the composite wall and the various thermal resistances to flow of heat are offered by :

(*i*) outside air firm

$$= \frac{1}{h_o A} = \frac{1}{52.3 \times 3.06}$$
$$= 6.248 \times 10^{-3}\ \text{deg-hr/kJ}$$

(*ii*) mild sheet sheet

$$= \frac{\delta_1}{k_1 A} = \frac{0.003}{145 \times 3.06}$$
$$= 6.761 \times 10^{-6}\ \text{deg-hr/kJ}$$

(*iii*) glass wool insulation

$$= \frac{\delta_2}{k_2 A} = \frac{0.06}{0.188 \times 3.06}$$
$$= 0.1043\ \text{deg-hr/kJ}$$

(*iv*) mild steel sheet

$$= \frac{\delta_3}{k_3 A} = \frac{0.003}{145 \times 3.06}$$
$$= 6.761 \times 10^{-6}\ \text{deg-hr/kJ}$$

(*v*) inside air firm

$$= \frac{1}{h_i A} = \frac{1}{40.8 \times 3.06}$$
$$= 8.0097 \times 10^{-3}\ \text{deg-hr/kJ}$$

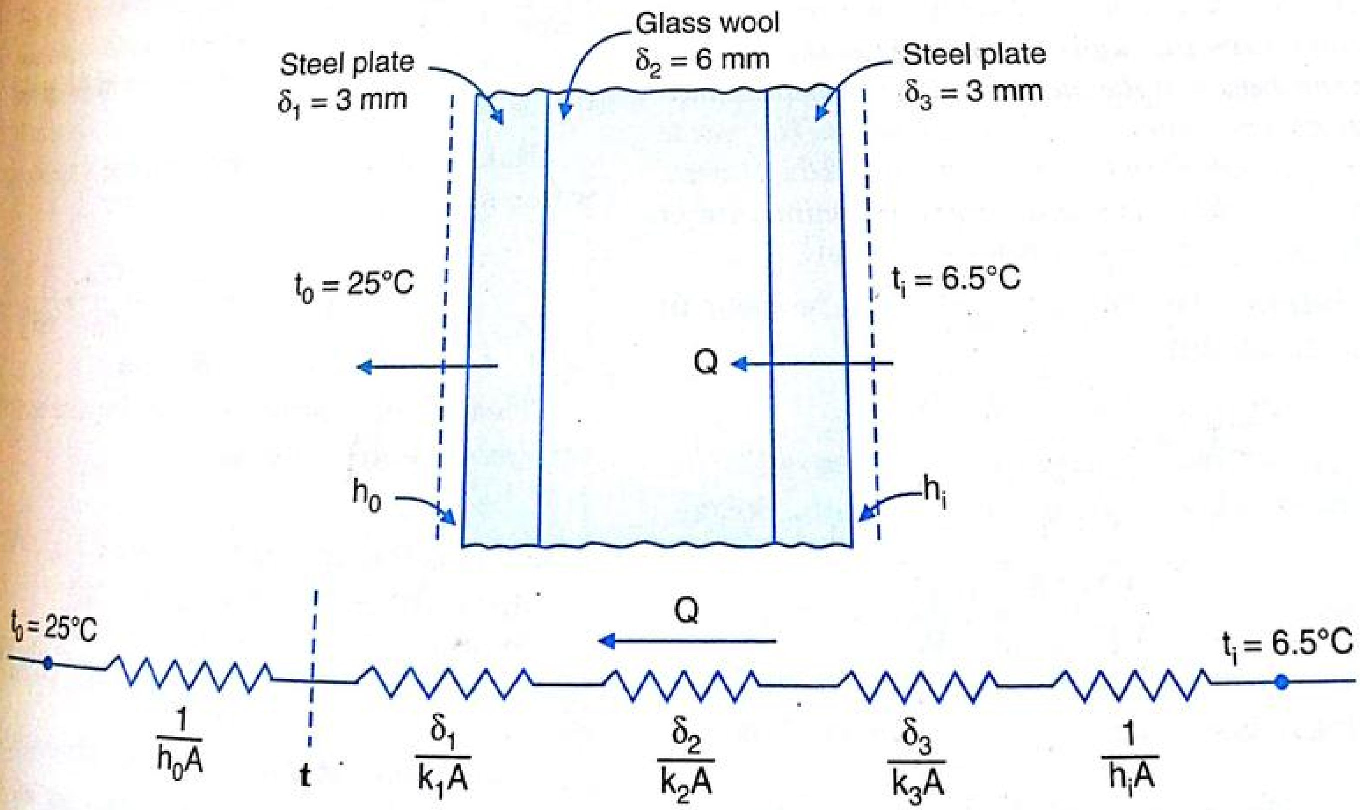


Fig. 3.39.

The total resistance equals the sum of individual resistances and the summation gives

$$\Sigma R_t = 0.11857 \text{ deg-hr/kJ}$$

Overall conductance

$$= \frac{1}{\sum R_t} = \frac{1}{0.11857}$$

$$= 8.434 \text{ kJ/hr-deg}$$

$$\text{Heat loss} = \frac{\Delta t}{\sum R_t} = \frac{25 - 6.5}{0.11857} = \mathbf{156 \text{ kJ/hr}}$$

Since heat flowing through each layer is same; then for the outside air film

$$156 = h_o A (t_o - t)$$

$$= 52.3 \times 3.06(25 - t)$$

where t is the temperature at the outer surface of metal sheet. Solution gives

$$t = 25 - \frac{156}{52.3 \times 3.06} = 24.02°\text{C}$$

EXAMPLE 3.39

The door of a domestic refrigerator has an area of 0.7 m² and is essentially made of a thin metal sheet with a 2.5 cm thick layer of insulation on the inside. The thermal conductivity of the insulation is 0.25 W/mK and the heat transfer coefficients to the surrounding air on each side of the door are both 10 W/m² K. Estimate the heat flow rate through the door and the surface temperature of the metal sheet which is presumed to have negligible thermal resistance. The inside of refrigerator (cold chamber) and the kitchen (where the refrigerator is placed) are to be maintained at 0°C and 20°C respectively.

Solution : Thermal resistance of the door of refrigerator is

$$\Sigma R_t = \frac{1}{A}\left[\frac{1}{h_o} + \frac{\delta}{k} + \frac{1}{h_i}\right]$$

$$= \frac{1}{0.7}\left[\frac{1}{10} + \frac{0.025}{0.25} + \frac{1}{10}\right]$$

$$= 0.4286 \text{ deg/W}$$

$$\text{Heat loss} = \frac{\Delta t}{\sum R_t} = \frac{20 - 0}{0.4286} = 46.7 \text{ W}$$

(*ii*) The heat flow through each layer is same. Accordingly for the outsider air film

$$46.7 = h_o A (t_o - t) = 10 \times 0.7 (20 - t)$$

where t is the surface temperature of metal sheet. Solution gives

$$t = 20 - \frac{46.7}{10 \times 0.7} = 13.33°\text{C}$$

EXAMPLE 3.40

A 3 mm thick metal plate, having thermal conductivity k = 98.6 W/m-deg, is exposed to vapour at 100°C on one side and cooling water at 30°C on the opposite side.

The heat transfer coefficients are :

$$h_i = 14200 \text{ W/m}^2\text{-deg on the vapour side}$$

$$h_o = 2325 \text{ W/m}^2\text{-deg on the water side}$$

Determine the rate of heat transfer, the overall heat transfer coefficient and the drop in temperature at each side of heat transfer.

Solution : Thermal resistance of composite system is given by

$$\Sigma R_t = \frac{1}{A}\left[\frac{1}{h_i} + \frac{\delta}{k} + \frac{1}{h_o}\right]$$

The wall area A is constant for all the layers. Considering unit area and inserting appropriate values :

$$\Sigma R_t = \frac{1}{1}\left[\frac{1}{14200} + \frac{0.003}{98.6} + \frac{1}{2325}\right]$$

$$= 7.04 \times 10^{-5} + 3.04 \times 10^{-5} + 43.01 \times 10^{-5}$$

$$= 53.09 \times 10^{-5} \text{ deg/W}$$

Rate of heat transfer from the vapour to water side,

$$Q = \frac{\Delta t}{\sum R_t} = \frac{100 - 30}{53.09 \times 10^{-5}}$$

$$= \mathbf{1.318 \times 10^5 \text{ W/m}^2}$$

Heat flow through the composite system can be written in the form

$$Q = U A \Delta t$$

where U is the overall heat transfer coefficient

$$1.318 \times 10^5 = U \times 1 \times (100 - 30)$$

or

$$U = \frac{1.318 \times 10^5}{70} = 1882 \text{ W/m}^2\text{deg}$$

The same heat flows through each layer of the system.

Temperature drop in any layer = heat flow × thermal resistance of that layer

∴ Temperature drop in the vapour film
$= 1.318 \times 10^{-5} \times 7.04 \times 10^{-5}$
$= 9.28°C$

Temperature drop in the metal
$= 1.318 \times 10^{-5} \times 3.04 \times 10^{-5}$
$= 4.00°C$

Temperature drop in the water film
$= 1.318 \times 10^{-5} \times 43.01 \times 10^{-5}$
$= 56.68°C$

Apparently the greatest temperature would occur in the water film.

EXAMPLE 3.41

A guest house has a multilayer composite wall constructed as shown in the figure given below. The temperature of air inside the room is 20°C and the surface coefficient of heat transfer between the room air and wall is 6.25 W/m²-deg. The outside temperature is – 12°C with an outside surface coefficient of heat transfer 17.25 W/m²-deg. The wall measures 2 m high and 4 m deep. The different wall thicknesses are as indicated in the figure and thermal conductivities of wall materials are :

$k_a = 0.16$ W/m-deg ; $k_b = 0.21$ W/m-deg
$k_c = 0.04$ W/m-deg ; $k_d = 0.17$ W/m-deg

Calculate the heat transfer rate across the wall in steady state.

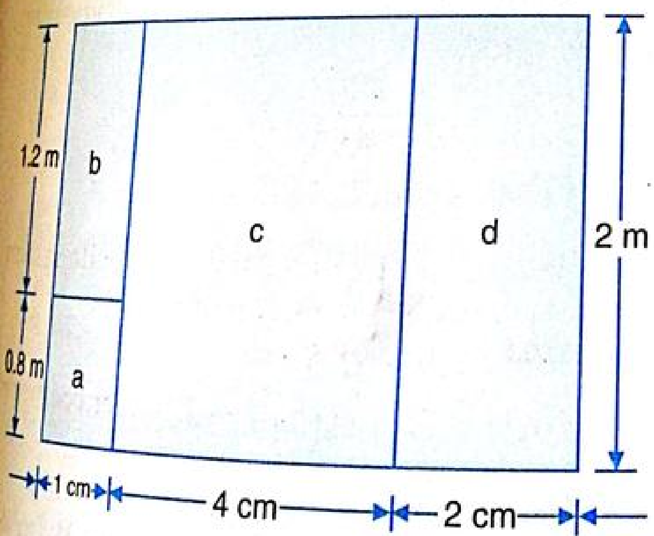


Fig. 3.40.

Solution : The heat transfer areas for the differed layers of the composite wall are :

$A_a = 0.8 \times 4 = 3.2$ m²
$A_b = 1.2 \times 4 = 4.8$ m²
$A_c = A_d = 2 \times 4 = 8$ m²

The area at the inside and outside surfaces are :

$$A_i = A_o = 2 \times 4 = 8 \text{ m}^2$$

The various thermal resistances to flow of heat are :

(*i*) Inside air film :

$$R_{t_i} = \frac{1}{h_i A_i} = \frac{1}{6.25 \times 8} = 0.02 \text{ deg/W}$$

(*ii*) Material a :

$$R_{t_a} = \frac{\delta_a}{k_a A_a} = \frac{0.01}{0.16 \times 3.2} = 0.019 \text{ deg/W}$$

(*iii*) Material b :

$$R_{t_b} = \frac{\delta_b}{k_b A_b} = \frac{0.01}{0.21 \times 4.8} = 0.01 \text{ deg/W}$$

(*iv*) Material c :

$$R_{t_c} = \frac{\delta_c}{k_c A_c} = \frac{0.04}{0.04 \times 8.0} = 0.125 \text{ deg/W}$$

(*v*) Material d :

$$R_{t_d} = \frac{\delta_d}{k_d A_d} = \frac{0.02}{0.17 \times 8.0} = 0.015 \text{ deg/W}$$

(*vi*) Outside air film :

$$R_{t_o} = \frac{1}{h_o A_o} = \frac{1}{17.25 \times 8} = 0.007 \text{ deg/W}$$

The analogous electrical circuit for the various thermal resistances is as shown in Fig. 3.41.

The resistance R_{t_a} and R_{t_b} are in parallel and their equivalent resistance is

$$= \frac{R_{t_a} \times R_{t_b}}{R_{t_a} + R_{t_b}}$$

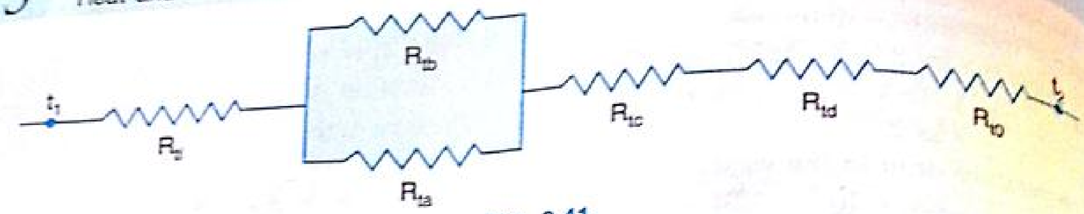


Fig. 3.41.

$$= \frac{0.019 \times 0.01}{0.019 + 0.01} = 0.0065 \text{ deg/W}$$

This equivalent resistance is now in series with the first and the last three resistances. The total thermal resistance for the entire circuit becomes

$$\Sigma R_t = 0.02 + 0.0065 + 0.125 + 0.015 + 0.007$$
$$= 0.1735 \text{ deg/W}$$

∴ Heat transfer rate through the wall of the guest house is

$$\frac{\Delta t}{\sum R_t} = \frac{20-(-12)}{0.1735} = 184.4 \text{ W}$$

EXAMPLE 3.42

A furnace wall is made of 7.5 cm of fire clay and 0.65 cm of mild steel plate. The inside surface of brick is exposed to hot gases at 650°C and the outside air temperature is 27°C. The convection and radiation heat transfer coefficients towards the air side are stated to be 60 and 8 W/m²-deg respectively. Make calculations for the heat loss per square metre area of the furnace wall. Also obtain the outside surface temperature of steel plate.

(b) How these values would change if 18 steel bolts each of 20 mm diameter are used for fixing the steel plate on the brick wall per square metre of wall area. Assume that the temperature of the bolts at sections flush with the outer faces of the wall are those of the wall surfaces. Further, a change in the wall resistance caused by the bolt holes is negligible. The thermal conductivities are:

Fire clay brick : 1.15 W/mK

Steel : 40 W/mK

Solution : The effective outside (air side) heat transfer coefficient,

$$h_0 = 60 + 8 = 68 \text{ W/m}^2\text{K}$$

Total resistance for the composite wall is,

$$\Sigma R_t = R_{brick} + R_{steel} + R_{con}$$
$$= \frac{\delta_1}{k_1 A_1} + \frac{\delta_2}{k_2 A_2} + \frac{1}{h_0 A}$$

The wall surface area is constant for all layers of the wall. Considering unit surface area,

$$\Sigma R_t = \frac{0.075}{1.15 \times 1} + \frac{0.0065}{40 \times 1} + \frac{1}{68 \times 1}$$
$$= 0.0652 + 0.0001625 + 0.01470$$
$$= 0.08 \text{ deg/W}$$

Heat flow (without bolts) per unit area of the wall,

$$\frac{\Delta t}{\sum R_t} = \frac{650 - 27}{0.080} = 7787 \text{ W}$$

The heat flow through each layer is same. Accordingly, for the outside air film :

$$7787 = h_o A (T_3 - T_a)$$
$$= 68 \times 1 \times (T_3 - 27)$$

∴ Outside surface temperature of steel plate,

$$T_3 = \frac{7787}{68 \times 1} + 27 = 141.5°\text{C}$$

(b) Minimum length of steel bolts

$$= \text{thickness of composite wall}$$
$$= 0.075 + 0.0065 = 0.0815 \text{ m}$$

Area of 18 bolts

$$= 18 \times \frac{\pi}{4}(0.02)^2 = 0.00565 \text{ m}^2$$

Thermal resistance of the bolts,

$$= \frac{\delta}{kA} = \frac{0.0815}{40 \times 0.00565}$$
$$= 0.3605 \text{ deg/W}$$

The thermal resistance of the bolts is in parallel with resistance due to brick and steel plate. The equivalent resistance of these resistors is given as

$$= \frac{R_{bolt} \times (R_{bolt} + R_{steel})}{R_{bolt} + (R_{brick} + R_{steel})}$$

$$= \frac{0.3606 \times (0.0652 + 0.0001625)}{0.3606 + (0.0652 + 0.0001625)}$$

$$= 0.05533 \text{ deg/W}$$

This equivalent resistance is now in series with R_{con} and the total resistance for the circuit becomes

$$\Sigma R_t = 0.05533 + 0.0147$$
$$= 0.07 \text{ deg/W}$$

Heat flux (with bolts) per unit area of the wall,

$$= \frac{\Delta t}{\Sigma R_t} = \frac{650 - 27}{0.07} = 8900 \text{ W}$$

∴ Percentage increase in the heat flow,

$$\frac{8900 - 7787}{7787} = 14.29\%$$

Further, corresponding to outside air film,

$$8900 = h_o A (T_3 - T_a)$$
$$= 68 \times 1 \times (T_3 - 27)$$

∴ Outside surface temperature of steel plate,

$$T_3 = \frac{8900}{68 \times 1} + 27 = 157.88°\text{C}$$

Apparently the steel surface temperature is increased by

(157.88 − 141.5)

= 16.38°C after putting the bolts.

3.4. CONDUCTION THROUGH A CYLINDRICAL WALL

Cylindrical metal tubes constitute an essential element of power stations, oil refineries and most process industries. The boilers have tubes in them, the condensers contain bank of tubes, the heat exchangers are tubulars and all these units are connected by tubes. Evidently then the radial heat transfer rate through a tube or any insulation which may surround it is quite important.

Consider heat conduction through a cylindrical tube of inner radius r_1, outer radius

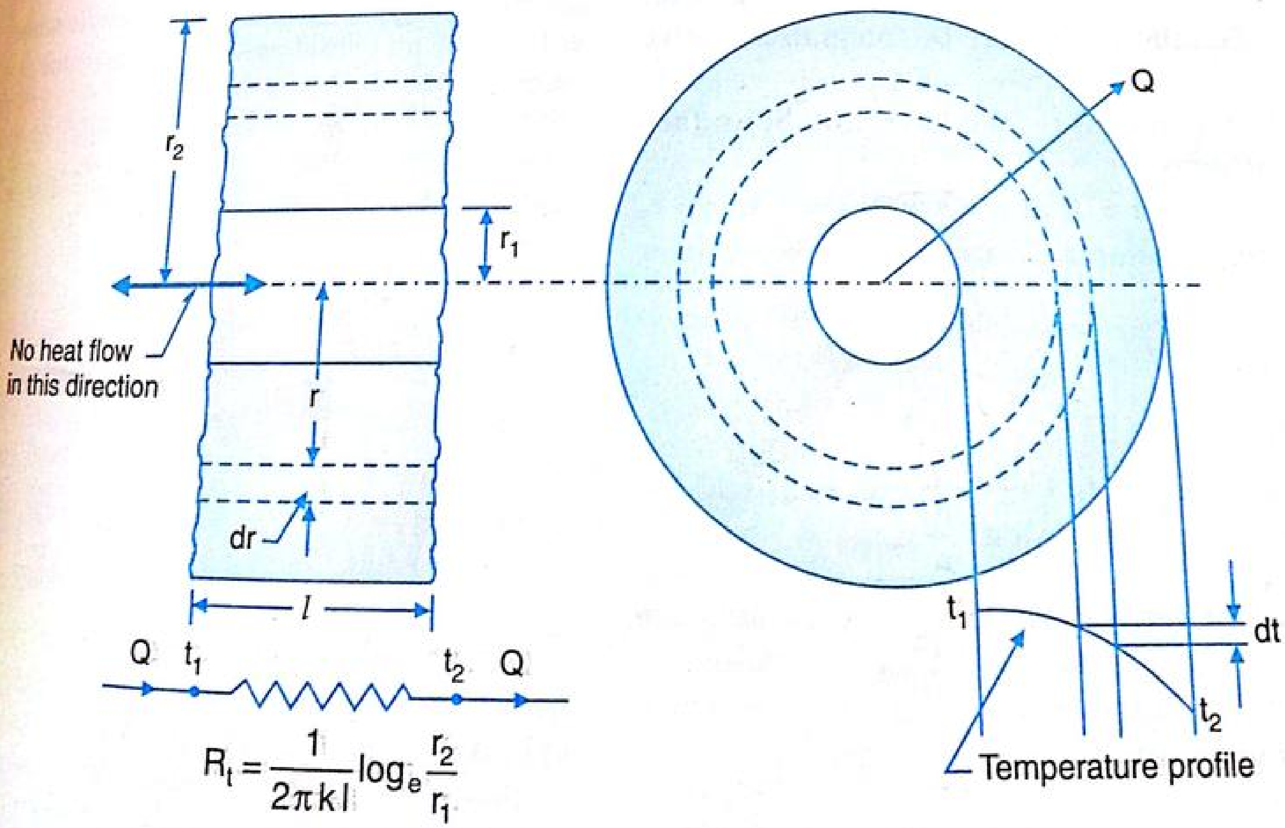


Fig. 3.42. Steady state conduction through a cylindrical wall

r_2 and lenth l. The inside and outside surfaces of the tube are at constant temperatures t_1 and t_2 and thermal conductivity k of the tube material is constant within the given temperature range. If both ends are perfectly insulated, the heat flow is limited to radial direction only. Further if temperature t_1 at the inner surface is greater than temperature t_2 at the outer surface, the heat flows radially outwards.

Recapitulate the general heat equation in cylindrical co-ordinates :

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial x^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

For steady state $(\partial t / \partial \tau = 0)$, uni-directional $\{t \neq f(\phi, x)\}$ heat flow in the radial direction and with no internal heat generation $(q_g = 0)$, the above equation reduces to

$$\frac{d^2 t}{dr^2} + \frac{1}{r}\frac{dt}{dr} = 0 \quad \text{or} \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dt}{dr}\right) = 0$$

Since, $\frac{1}{r} \neq 0$

$$\frac{d}{dr}\left(r\frac{dt}{dr}\right) = 0 \quad \text{or} \quad r\frac{dt}{dr} = \text{constant } C_1 \qquad ...(3.16)$$

Equation 3.16 may be integrated to give

$$t = C_1 \log_e r + C_2 \qquad ...(3.17)$$

Incorporating the relevant boundary conditions

$$t = t_1 \text{ at } r = r_1 \text{ and } t = t_2 \text{ at } r = r_2$$

the constants C_1 and C_2 take the values

$$C_1 = -\frac{t_1 - t_2}{\log_e \frac{r_2}{r_1}} ;$$

$$C_2 = t_1 + \frac{t_1 - t_2}{\log_e \frac{r_2}{r_1}} \log_e r_1 \qquad ...(3.18)$$

When the values of these constants are substituted in equation 3.17, one obtains the following expression for temperature distribution in a hollow cylinder

$$t = t_1 + \frac{t_1 - t_2}{\log_e \frac{r_2}{r_1}} \log_e r_1 - \frac{t_1 - t_2}{\log_e \frac{r_2}{r_1}} \log_e r$$

$$\text{or } (t - t_1)\log_e \frac{r_2}{r_1}$$

$$= (t_1 - t_2)\log_e r_1 - (t_1 - t_2)\log_e r$$

$$= (t_2 - t_1)\log_e r - (t_2 - t_1)\log_e r_1$$

$$= (t_2 - t_1)\log_e \frac{r}{r_1}$$

Therefore in dimensionless form we have:

$$\frac{t - t_1}{t_2 - t_1} = \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_2}{r_1}} \qquad ...(3.19)$$

Evidently the temperature distribution associated with radial conduction through a cylinder is logarithmic; not linear as for a plane wall. Further temperature at any point in the cylinder can be expressed as a function of radius only. Isotherms or lines of constant temperature are then concentric circles lying between the inner and outer cylinder boundaries.

The conduction heat transfer rate is determined by utilizing the temperature distribution in conjunction with the Fourier law:

$$Q = -kA\frac{dt}{dr}$$

$$= -kA\frac{d}{dr}\left[t_1 + \frac{(t_1 - t_2)}{\log_e \frac{r_2}{r_1}}\log_e r_1 - \frac{(t_1 - t_2)}{\log_e \frac{r_2}{r_1}}\log_e r\right]$$

$$= -k(2\pi r l)\left(\frac{-(t_1 - t_2)}{r \log_e \frac{r_2}{r_1}}\right)$$

$$= 2\pi k l \frac{(t_1 - t_2)}{\log_e \frac{r_2}{r_1}} = \frac{(t_1 - t_2)}{R_t} \qquad ...(3.20)$$

In the alternative approach to estimate heat flow, consider an infinitesimally thin cylindrical element at radius r. Let thickness of this elementary ring be dr and the change of temperature across it be dt. Then invoking Fourier law of heat conduction

$$Q = -kA\frac{dt}{dr} = -k(2\pi r l)\frac{dt}{dr}$$

Separating the variables and integrating within the prescribed boundary conditions, we obtain :

$$\frac{Q}{2\pi k l}\int_{r_1}^{r_2}\frac{dr}{r} = \int_{t_1}^{t_2} dt$$

$$\frac{Q}{2\pi k l}\log_e\frac{r_2}{r_1} = (t_1 - t_2)$$

or
$$Q = 2\pi k l\,\frac{(t_1 - t_2)}{\log_e\frac{r_2}{r_1}}$$

$$= \frac{(t_1 - t_2)}{R_t} \qquad \text{...(3.20a)}$$

Evidently for conduction in a hollow cylinder, the thermal resistance takes the form

$$R_t = \frac{\log_e\frac{r_2}{r_1}}{2\pi k l}$$

Special Remarks :

(*i*) Typical examples of heat conduction through cylindrical tubes are found in power plants, oil refineries and most process industries. The boilers have tubes in them, the condensers contain banks of tubes, the heat exchangers are tubulars and all these units are connected by tubes. In all these units, heat is transferred from the hot fluid to cold fluid through a pipe made of suitable material.

(*ii*) Surface area of a cylindrical surface changes with radius. Therefore the rate of heat conduction through a cylinder is usually expressed per unit length rather than per unit area as done for plane walls.

(*iii*) The heat flow rate through a plane wall is the same for all isothermal surfaces and therefore the temperature gradient remains constant for all isothermal surfaces. With a cylindrical surface, the heat flux falls off inversely with radius. This has to be so because the same amount of heat flows through a steadily increasing area.

At any radius r,

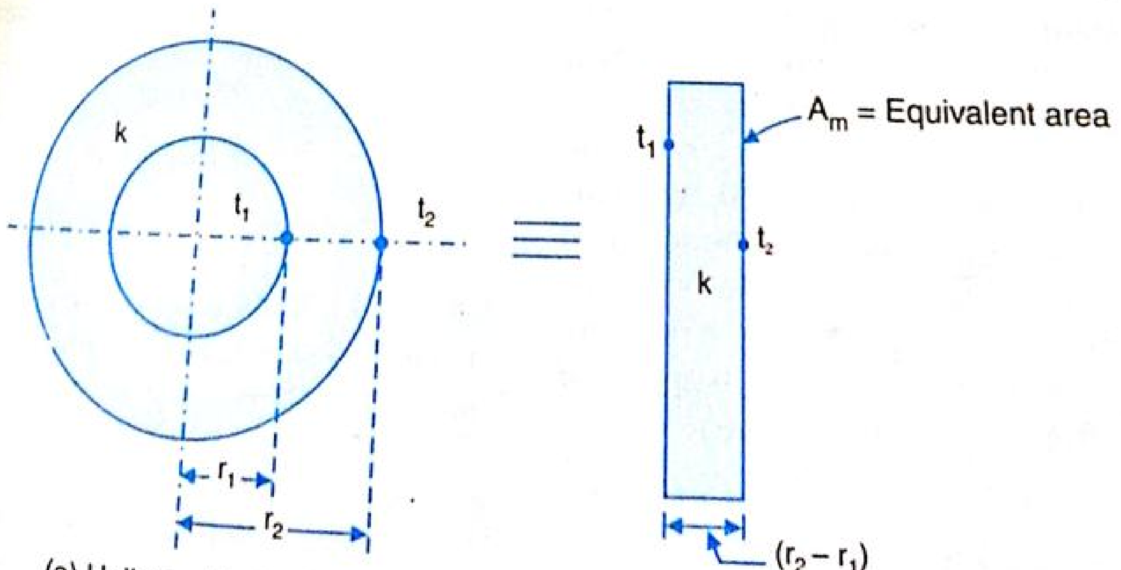
Heat flux, q

$$= -k\frac{dt}{dr}$$

$$= -k\frac{d}{dr}\left[t_1 + \frac{(t_1 - t_2)}{\log_e\frac{r_2}{r_1}}\log_e r_1 - \frac{(t_1 - t_2)}{\log_e\frac{r_2}{r_1}}\log_e r\right]$$

$$= \frac{k(t_1 - t_2)}{\log_e\frac{r_2}{r_1}}\times\frac{1}{r}$$

Thus the heat flux is proportional to $1/r$.



(a) Hollow cylinder

(b) Plane wall

Fig. 3.43. Concept of logarithmic mean area

(*iv*) Quite often it is considered advantageous to write the heat flow equation through a cylinder in the same form as that for heat flow through a plane wall. Then thickness δ will be equal to $(r_2 - r_1)$ and the area A will be an equivalent area A_m. Thus

$$Q = \frac{kA}{\delta}(t_1 - t_2) = kA_m \frac{(t_1 - t_2)}{r_2 - r_1} \quad ...(3.21)$$

Comparing equations 3.20 and 3.21,

$$A_m = \frac{2\pi(r_2 - r_1)l}{\log_e \frac{r_2}{r_1}} = \frac{A_2 - A_1}{\log_e \frac{A_2}{A_1}} \quad ...(3.22)$$

where A_1 and A_2 are the inner and outer surface areas of the cylindrical tube. The equivalent area A_m is called the *logarithmic mean area* of the tube. Further,

$$A_m = 2\pi r_m l = \frac{2\pi(r_2 - r_1)l}{\log_e \frac{r_2}{r_1}}$$

Obviously, *logarithmic mean radius* of the cylindrical tube is :

$$r_m = \frac{r_2 - r_1}{\log_e \frac{r_2}{r_1}} \quad ...(3.23)$$

When $r_2/r_1 = 1$, the arithmetic mean radius $(r_1 + r_2)/2$ deviates from the logarithmic mean radius by about 4% and when $r_2/r_1 = 1.5$, the deviation is only 1.3%. Apparently when thickness of a cylindrical tube is small in comparison with radius, *i.e.*, $r_2/r_1 < 2$, the heat conduction may by calculated (without appreciable error) by using arithmetic mean radius instead of logarithmic mean radius. Use of logarithmic mean area is primarily important in the problem of the lagging of pipes and in the insulation of electric cables.

(*v*) The temperature profile

$$\frac{t - t_1}{t_2 - t_1} = \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_2}{r_1}}$$

is nearly linear for values of (r_2/r_1) of the order of unity, but decidedly non-linear for large values of (r_2/r_1).

EXAMPLE 3.43

A cylindrical cement tube of radii 0.05 cm and 1.0 cm has a wire embedded into it along its axis. To maintain a steady temperature difference of 120°C between the inner and outer surfaces, a current of 5 ampere is made to flow in the wire. Make calculations for the amount of heat generated per metre length and the thermal conductivity of cement.

Take resistance of wire equal to 0.1 ohm per cm of length.

Solution : Resistance of wire,

$$R = 0.1\ \Omega \text{ per cm length}$$
$$= 10\ \Omega \text{ per m length}$$

Heat generated

$$= I^2 R = 5^2 \times 10$$
$$= 250 \text{ W/m length}$$

Under steady state conditions, the heat generated equals the heat transfer through the cylindrical element.

$$\therefore \quad Q = \frac{2\pi kl(t_1 - t_2)}{\log_e \frac{r_2}{r_1}}$$

$$250 = \frac{2\pi \times k \times 1 \times 120}{\log_e \frac{1.0}{0.05}}$$

$\therefore$ Thermal conductivity of cement,

$$k = \frac{250 \log_e \frac{1.0}{0.05}}{2\pi \times 120}$$
$$= 0.994 \text{ W/m-deg}$$

EXAMPLE 3.44

A stainless steel tube with inner diameter 12 mm, thickness 0.2 mm and length 50 cm is heated electrically. The entire 15 kW of heat energy generated in the tube is transferred through its outer surface. Work out the intensity of current flow and the temperature drop across the tube wall. For the tube material, take thermal

conductivity = 18.5 W/m-deg and specific resistance = 0.85 Ω-mm^2/m.

Solution : Specific resistance, ρ

$= 0.85\ \Omega\ (\text{cm}/10)^2\ /100\ \text{cm}$

$= 8.5 \times 10^{-3}\ \Omega\text{-cm}$

$r_1 = 6\ \text{mm}\ ;\ r_2 = 6 + 0.2 = 6.2\ \text{mm}$

Electrical resistance, R_e

$$= \frac{\rho l}{A} = \frac{8.5 \times 10^{-3} \times 50}{\pi\left(0.62^2 - 0.6^2\right)}$$

$= 5.548\ \Omega$

Power generated

$= I^2 R_e = 15\ \text{kW} = 15000\ \text{W}$

$\therefore$ Intensity of current flow

$$= \sqrt{\frac{15000}{5.548}} = 52\ \text{amps}$$

Under steady state conditions, the heat generated equals the heat transfer through the cylindrical tunbe. Thus

$$Q = \frac{2\pi k l}{\log_e \dfrac{r_2}{r_1}}\Delta t$$

$$15000 = \frac{2\pi \times 18.5 \times 0.5}{\log_e \dfrac{6.2}{6}}\Delta t$$

$= 1771.58\ \Delta t$

$\therefore$ Temperature drop across the tube wall,

$$\Delta t = \frac{15000}{1771.58} = 8.467°\text{C}$$

EXAMPLE 3.45

A hot fluid is being conveyed through a long pipe of 4 cm outer diameter and covered with 2 cm thick insulation. It is proposed to reduce the conduction heat loss to the surroundings to one-third of the present rate by further covering with same insulation. Calculate the additional thickness of insulation.

Solution : $r_1 = 2$ cm and $r_2 = 4$ cm

Heat loss with existing insulation, Q_1

$$= \frac{2\pi k l\,(t_1 - t_2)}{\log_e \dfrac{r_2}{r_1}}$$

Heat loss with additional insulation, Q_2

$$= \frac{2\pi k l\,(t_1 - t_2)}{\log_e \dfrac{r_2 + x}{r_1}}$$

where x is the additional thickness of insulation

According to the given condition: Q_2

$= Q_1/3$

$$\therefore \quad \frac{2\pi k l\,(t_1 - t_2)}{\log_e \dfrac{r_2 + x}{r_1}} = \frac{1}{3}\,\frac{2\pi k l\,(t_1 - t_2)}{\log_e \dfrac{r_2}{r_1}}$$

$$\text{or} \quad \frac{r_2 + x}{r_1} = \left(\frac{r_2}{r_1}\right)^3 = \left(\frac{4}{3}\right)^3 = 8$$

$\therefore \quad x = 8\,r_1 - r_2$

$= 8 \times 2 - 4 = 12$ cm

EXAMPLE 3.46

A steam pipe, 7.5 cm external diameter and 25 m long, conveys 1000 kg of steam per hour at a pressure of 20 bar. The steam enters the pipe with dryness fraction 0.98 and it is required that the steam at exit from the pipe must have a minimum dryness fraction of 0.96. The task is accomplished by suitably lagging the pipe; the thermal conductivity of lagging being 0.7 kJ/m-hr-deg. Neglecting thermal resistance of the steam pipe (no temperature drop across the steam pipe), determine the minimum thickness of lagging required to meet the necessary conditions. The temperature at the outside surface of lagging be taken as 25°C.

Also determine the temperature at the point half way between the inner and outer surfaces.

Solution : A reference to steam tables indicates that at 20 bar pressure

Saturation temperature of steam = 212.4°C

Latent heat of steam = 1890 kJ/kg

$\therefore$ Heat loss per kg of steam passing through the pipe,

$= (0.98 - 0.96) \times 1890 = 37.8$ kJ

Total heat loss Q

$= 37.8 \times 1000 = 37800$ kJ/hr

In terms of thermal resistance and temperature drop, the heat loss is given by :

$$Q = \Delta t / R_t$$

$$\therefore\ 37800 = \frac{212.4 - 25}{R_t};$$

$$R_t = 0.00496 \text{ deg-hr/kJ}$$

For a cylindrical wall, the thermal resistance is

$$R_t = \frac{\log_e \frac{r_2}{r_1}}{2\pi k l};$$

$$0.00496 = \frac{\log_e \frac{r_2}{r_1}}{2\pi \times 0.7 \times 25}$$

$$\log_e \frac{r_2}{r_1} = 0.545\ ;\ \frac{r_2}{r_1} = 1.725$$

r_1 = 3.75 cm (given), and hence

$r_2 = 3.75 \times 1.725 = 6.47$

∴ Thickness of lagging

$= 6.47 - 3.75 =$ **2.72 cm**

Radius at mid-plane of the pipe,

$$r = \frac{r_1 + r_2}{2} = \frac{3.75 + 6.47}{2} = 5.11 \text{ cm}$$

Thermal resistnace of the pipe upto its mid-plane

$$= \frac{1}{2\pi k l} \log_e \frac{r}{r_1}$$

$$= \frac{1}{2\pi \times 0.7 \times 25} \log_e \frac{5.11}{3.75}$$

$= 0.0028$ deg-hr/kJ

Since heat flowing through each section is same

$$37800 = \frac{212.4 - t}{0.0028}$$

∴ Temperature at the mid-plane of the steam pipe is

$t = 212.4 - 37800 \times 0.0028$

$=$ **105.84°C**

The temperature at point of the cylindrical pipe wall could also be obtained by using the relation

$$\frac{t - t_1}{t_2 - t_1} = \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_2}{r_1}}$$

$$\frac{t - 212.4}{25 - 212.4} = \frac{\log_e \frac{5.11}{3.75}}{\log_e \frac{6.47}{3.75}} = 0.5673$$

$\therefore\ t = 212.4 - (212.4 - 25) \times 0.5673$

$= 106.1°C$

EXAMPLE 3.47

The hot combustion gases at 150°C flow through a hollow cylindrical pipe of 10 cm inner diameter and 12 cm outer diameter. The pipe is located in a space at 30°C and the thermal conductivity of the pipe material is 200 W/mK. Neglecting surface heat transfer coefficients, calculate the heat loss through the pipe per unit length and the temperature at a point halfway between the inner and outer surface. What should be the surface area normal to the direction of heat flow so that the heat transfer through the pipe can be determined by considering material of the pipe as a plane wall of the same thickness ?

Solution : In terms of geometrical parameters, thermal resistance of a cylindrical pipe is

$$R_t = \frac{1}{2\pi k l} \log_e \frac{r_2}{r_1}$$

$$= \frac{1}{2\pi \times 200 \times 1} \log_e \frac{12}{10}$$

$= 1.4516 \times 10^{-4}$ deg/W

Heat loss Q

$$= \frac{\Delta t}{R_t} = \frac{150 - 30}{1.4516 \times 10^{-4}}$$

$=$ **826674 W**

(*ii*) Radius at halfway through the pipe wall,

$$r = \frac{r_1 + r_2}{2} = \frac{10 + 12}{2} = 11 \text{ cm}$$

Thermal resistnace of cylindrical pipe upto its mid-plane

$$= \frac{1}{2\pi k l} \log_e \frac{r}{r_1}$$

$$= \frac{1}{2\pi \times 200 \times 1} \log_e \frac{11}{10}$$

$$= 7.5884 \times 10^{-5} \text{ deg/W}$$

Since heat flow through each section is same ;

$$826674 = \frac{t_1 - t}{7.5884 \times 10^{-5}}$$

$\therefore$ Temperature at the mid plane,

$$t = 150 - 826674 \times 7.5884 \times 10^{-5}$$

$$= \mathbf{87.27°C}$$

Alternatively from the expression for temperature distribution

$$\frac{t - t_1}{t_2 - t_1} = \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_2}{r_1}}$$

$$t = t_1 - (t_1 - t_2) \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_2}{r_1}}$$

$$= 150 - (150 - 30) \frac{\log_e \frac{11}{10}}{\log_e \frac{12}{10}}$$

$$= \mathbf{87.18°C}$$

(*iii*) The equivalent logarithmic mean area is

$$A_m = \frac{A_2 - A_1}{\log_e \frac{A_2}{A_1}}$$

$$= \frac{2\pi (r_2 - r_1) l}{\log_e \frac{r_2}{r_1}}$$

$$= \frac{2\pi (0.12 - 0.10) \times 1}{\log_e \frac{12}{10}} = 0.689 \text{ m}^2$$

Check :

$$Q = \frac{k A_m (t_1 - t_2)}{r_2 - r_1}$$

$$= \frac{200 \times 0.689 \times (150 - 30)}{0.12 - 0.10}$$

$$= 826800 \text{ W}$$

This is approximately same as calculated above.

EXAMPLE 3.48

A piston barrel made of steel (k = 34.5 W/m-deg) has the shape of a hollow right circular cylinder of inside radius r_1 = 0.4 cm, outside radiust r_0 = 1 cm and its length is large compared to the outside radius. The barrel is mounted in a lathe undergoing a machining operation at its inside radius and that dissipates mechanical energy at the rate W = 3.95 × 10^4 watts per metre length of barrel. The outside of the barrel is held at constant temperature t_o = 15°C by a spray of cooling fluid. If steady state conditions prevail and if the temperature distribution in the barrel is primarily radial, set up an expression for the steady state temperature distribution and also work out the highest temperature of the piston barrel during the machining operation.

Solution : The temperature distribution is stated to be steady and radial with no generation. Obviously the dissipation of mechanical work occurs at the surface of the conduction region and not within. The situation then corresponds to conduction heat flow through the cylindrical walls of the hollow right circular cylinder.

The Fourier law of conduction for heat transfer rate along radial coordinates is,

$$Q = -kA\frac{dt}{dr} = -k\, 2\pi r l \frac{dt}{dr}$$

But $W = Q/l$ and therefore the above expression can be rewritten as

$$W = -k\, 2\pi r \frac{dt}{dr}$$

Since W and k are constants, we separate the variables and obtain

$$-\frac{W}{2\pi k}\frac{dr}{r} = dt$$

A generalised expression for steady state temperature distribution can be set up by integrating the above expression between the limits :

(i) general radial co-ordinate r where the temperature is t, and

(ii) the outside radius r_o where the temperature is stated to be t_o.

That gives :

$$\int_t^{t_o} dt = -\frac{W}{2\pi k}\int_r^{r_o}\frac{dr}{r}$$

or $$t_o - t = -\frac{W}{2\pi k}\log_e\frac{r_o}{r}$$

or $$t = t_o + \frac{W}{2\pi k}\log_e\frac{r_o}{r}$$

which is the required expression for steady state temperature distribution.

The maximum temperature of the barrel occurs at the inside radius r_i and therefore

$$t_{max} = t_o + \frac{W}{2\pi k}\log_e\frac{r_o}{r_i}$$

Substituting the given data in the above expression, we get :

$$t_{max} = 15 + \frac{3.95\times10^4}{2\pi\times34.5}\log_e\frac{1.0}{0.4}$$

$$= 15 + 167 = 182°\text{C}$$

EXAMPLE 3.49

The thickness of a gas turbine rotor of radius 50 cm varies linearly from 15 cm at the centre to 5 cm at the outer periphery where blades are attached. Thermocouples attached to the rotor at radial distances of 8 cm and 40 cm show temperature of 325°C and 650°C respectively. If thermal conductivity of the rotor material is 0.35 W/m-deg, workout the radial heat flow rate through the runner. What percentage change in heat flow rate would occur if the rotor has a uniform thickness of 5 cm ? Proceed from the basic principles.

Solution : Refer Fig. 3.44 which shows the gas turbine rotor section.

Thickness of rotor at distance r from the centre

$$x = x_c - \left(\frac{x_c - x_o}{R}\right)r = x_c - pr$$

Heat conducted at radius r

$$Q = -kA\frac{dt}{dr}$$

$$= -k\,2\pi r\,(x_c - pr)\frac{dt}{dr}$$

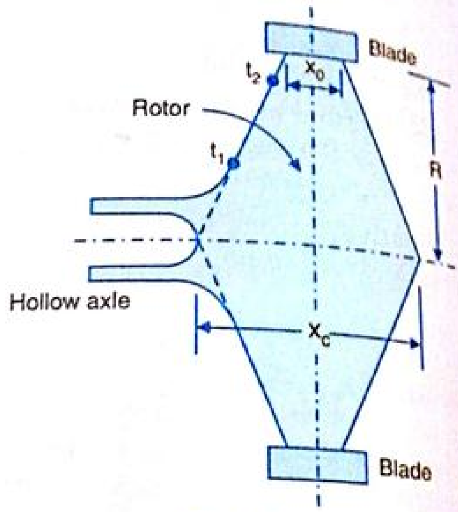


Fig. 3.44.

Separating the variables and integrating within the prescribed limits

$$-\int_{t_1}^{t_2} dt = \frac{Q}{2\pi k}\int_{r_1}^{r_2}\frac{dr}{r(x_c - pr)}$$

The right hand integral is of the form

$$\int\frac{dx}{x(a-bx)} = \frac{1}{a}\log_e\frac{x}{a-bx}$$

$$\therefore\ t_1 - t_2 = \frac{Q}{2\pi k x_c}\left|\frac{1}{x_c}\log_e\frac{r}{x_c - pr}\right|_{r_1}^{r_2}$$

$$= \frac{Q}{2\pi k x_c}\left[\log_e\left(\frac{r_2}{x_c - pr_2}\right) - \log_e\left(\frac{r_1}{x_c - pr_1}\right)\right]$$

$$= \frac{Q}{2\pi k x_c}\log_e\left[\frac{r_2}{r_1}\left(\frac{x_c - pr_1}{x_c - pr_2}\right)\right]$$

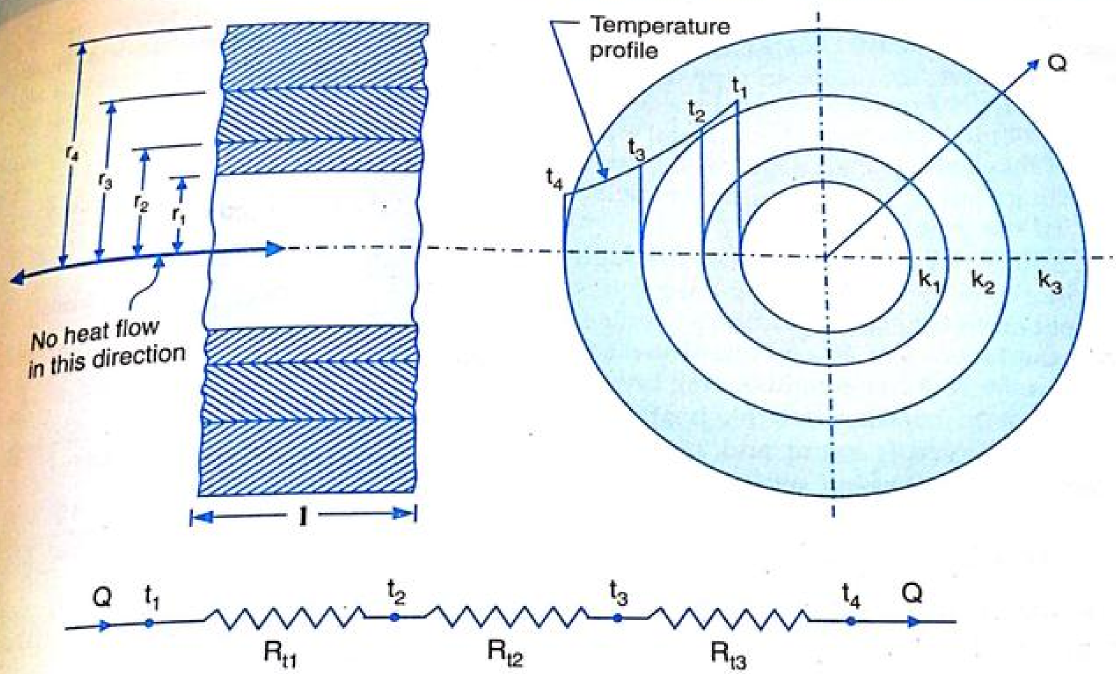


Fig. 3.45. Conduction through a composite cylindrical wall

$$\therefore \quad Q = 2\pi k x_c \,(t_1 - t_2) \div \log_e \left[\frac{r_2}{r_1}\left(\frac{x_c - pr_1}{x_c - pr_2}\right)\right] \quad ...(i)$$

$$\text{constant } p = \frac{x_c - x_0}{R} = \frac{15-5}{50} = 0.2$$

Substituting the given data in expression (i)

$$Q = 2\pi \times 0.35 \times 0.15\,(350 - 650) \div \log_e \left[\frac{40}{8}\left(\frac{15 - 0.2 \times 8}{15 - 0.2 \times 40}\right)\right]$$

$$= -\ 43.78 \text{ W}$$

The negative sign indicates that the heat flow is radially inward.

(b) For a rotor of constant thickness x_o

$$Q = -\,k_2 \pi r x_o \frac{dt}{dr}$$

$$-\int_{t_1}^{t_2} dt = \frac{Q}{2\pi k\, x_o}\int_{r_1}^{r_2} \frac{dr}{r}$$

$$\text{or} \quad (t_1 - t_2) = \frac{Q}{2\pi k\, x_o}\log_e \frac{r_2}{r_1}$$

$$\therefore \quad Q = \frac{2\pi k\, x_o\,(t_1 - t_2)}{\log_e \dfrac{r_2}{r_1}} \quad ...(ii)$$

The above expression is identical to the heat flow equation through a hollow cylinder of length x_o

$$Q = \frac{2\pi \times 0.35 \times 0.05\,(350 - 650)}{\log_e \dfrac{40}{8}}$$

$$= -\ 20.48 \text{ W}$$

Percentage reduction in heat flow rate

$$= \frac{43.78 - 20.48}{43.78} \times 100 = 53.2\%$$

3.5. CONDUCTION THROUGH A MULTI-LAYER CYLINDRICAL WALL

Multi-layer cylindrical walls are frequently employed to reduce heat losses from metallic

pipes meant for handling a hot fluid. The pipe is generally wrapped in one or more layers of heat insulation, *e.g.*, a steam pipe used for conveying high pressure steam in a steam power plant may have cylindrical metal wall, a layer of insulating material and then a layer of protecting plaster. The arrangement is called *lagging* of the pipe system.

Fig. 3.45 shows conduction of heat through a composite cylindrical wall having three layers of different materials. There is a perfect contact between the layers and so an equal interface temperature for any two neighbouring layers. For steady state conduction, the heat flow through each layer is same and it can be described by the following set of equations :

$$Q = 2\pi k_1 l \frac{(t_1 - t_2)}{\log_e \frac{r_2}{r_1}} = 2\pi k_2 l \frac{(t_2 - t_3)}{\log_e \frac{r_3}{r_2}} = 2\pi k_3 l \frac{(t_3 - t_4)}{\log_e \frac{r_4}{r_3}} \quad ...(3.24)$$

These equations help to determine the temperature difference for each layer of the composite cylinder, *i.e.*,

$$t_1 - t_2 = \frac{Q}{2\pi k_1 l} \log_e \frac{r_2}{r_1}$$

$$t_2 - t_3 = \frac{Q}{2\pi k_2 l} \log_e \frac{r_3}{r_2}$$

$$t_3 - t_4 = \frac{Q}{2\pi k_3 l} \log_e \frac{r_4}{r_3}$$

From summation of these equalities, one obtains

$$(t_1 - t_4) = Q\left[\frac{1}{2\pi k_1 l}\log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l}\log_e \frac{r_3}{r_2} + \frac{1}{2\pi k_3 l}\log_e \frac{r_4}{r_3}\right] \quad ...(3.25)$$

Thus the heat flow rate through a composite wall is

$$Q = \frac{(t_1 - t_4)}{\left[\frac{1}{2\pi k_1 l}\log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l}\log_e \frac{r_3}{r_2} + \frac{1}{2\pi k_3 l}\log_e \frac{r_4}{r_3}\right]} \quad ...(3.26)$$

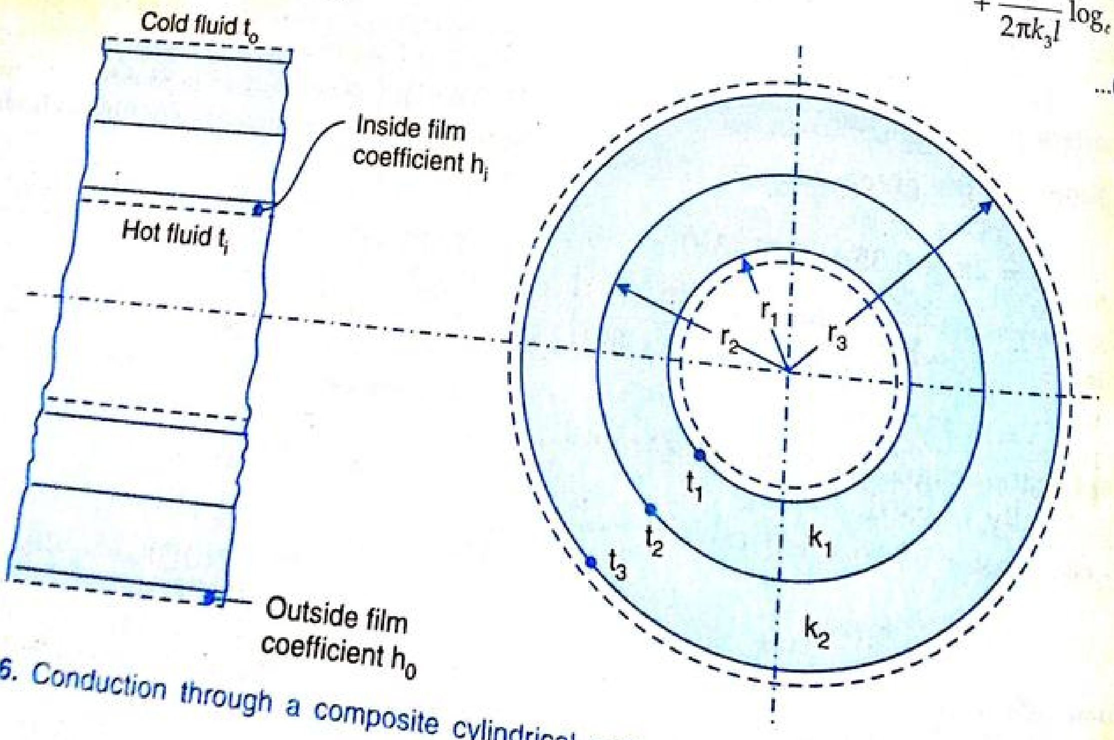


Fig. 3.46. Conduction through a composite cylindrical wall : heat transfer coefficients considered

The quantity in the denominator is the sum of the thermal resistance of the different layers comprising the composite cylinder.

When the above relation is extended to a hollow composite cylinder of n-concentric shells enveloping each other, the heat flow rate is given by

$$Q = \frac{t_1 - t_{n+1}}{\sum_{i=1}^{n} \frac{1}{2\pi k_i l} \log_e \frac{r_i + 1}{r_i}}$$

$$= \frac{t_1 - t_{n+1}}{\sum_{i=1}^{n} R_{t_i}} \qquad \text{...(3.27)}$$

If the internal and external heat transfer coefficients for the composite cylinder depicted in Fig. 3.38 are h_i and h_o respectively, then the total thermal resistance to heat flow would be :

$$R_t = \frac{1}{2\pi r_1 l h_i} + \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}$$

and

$$Q = \frac{(t_i - t_o)}{\left[\frac{1}{2\pi r_1 l h_i} + \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}\right]} \qquad \text{...(3.28)}$$

Introducing the concept of overall heat transfer coefficient U, the heat flow rate can be written as : $Q = UA\,(t_i - t_o)$. Since the flow area varies for a cylindrical tube, it becomes necessary to specify the area on which U is based. Thus depending upon whether the inner or outer area is specified, two different values are defined for U, *i.e.*,

$$Q = U_{in} A_{in} (t_i - t_o)$$

$$= U_{out} A_{out} (t_i - t_o) \qquad \text{...(3.29)}$$

Comparing equations 3.28 and 3.29, we note that

$$U_{in} \times 2\pi r_1 l\,(t_i - t_o)$$

$$= \frac{(t_i - t_o)}{\left[\frac{1}{2\pi r_1 l h_i} + \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}\right]}$$

$$\therefore\ U_{in} = \frac{1}{\left[\frac{1}{h_i} + \frac{r_1}{k_1} \log_e \frac{r_2}{r_1} + \frac{r_1}{k_2} \log_e \frac{r_3}{r_2} + \frac{r_1}{r_3 h_o}\right]}$$

Similarly,

$$U_{out} = \frac{1}{\left[\frac{r_3}{r_1 h_i} + \frac{r_3}{k_1} \log_e \frac{r_2}{r_1} + \frac{r_3}{k_2} \log_e \frac{r_3}{r_2} + \frac{1}{h_0}\right]}$$

EXAMPLE 3.50

A steam pipe 10 cm outside diameter is covered with two layers of insulation, each having a thickness of 2.5 cm. The average thermal conductivity of one material is 3 times that of other and the surface temperatures of the insulated steam pipe are fixed. Examine the position of better insulating layer relative to the steam pipe if heat dissipation from steam is to be minimum. What % age saving in heat dissipation results from that arrangement?

Solution : The geometrical dimensions of the lagged steam pipe are :

$$r_1 = 5 \text{ cm}\ ;\ r_2 = 7.5 \text{ cm}\ ;\ r_3 = 10 \text{ cm}$$

Let k and $3k$ be the thermal conductivities of the two insulating materials. Obviously the material with low thermal conductivity is a better insulator.

(*i*) Better insulator (material with low thermal conductivity) is placed inside, *i.e.*, next to the steam pipe.

Thermal resistance

$$= \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2}$$

$$= \frac{1}{2\pi kl}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi(3k)l}\log_e\frac{r_3}{r_2}$$

$$= \frac{1}{2\pi kl}\left[\log_e\frac{7.5}{5.0} + \frac{1}{3}\log_e\frac{10}{7.5}\right]$$

$$= \frac{0.2506}{\pi kl}$$

Heat dissipation Q_1

$$= \frac{\Delta T}{R_t} = \frac{\Delta T \times \pi kl}{0.2506} = 3.99\,\pi kl\Delta T$$

(ii) Better insulator is outside

Thermal resistance

$$= \frac{1}{2\pi k_2 l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi k_1 l}\log_e\frac{r_3}{r_2}$$

$$= \frac{1}{2\pi(3k)l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi kl}\log_e\frac{r_3}{r_2}$$

$$= \frac{1}{2\pi kl}\left[\frac{1}{3}\log_e\frac{r_2}{r_1} + \log_e\frac{r_3}{r_2}\right]$$

$$= \frac{1}{2\pi kl}\left[\frac{1}{3}\log_e\frac{7.5}{5.0} + \log_e\frac{10}{7.5}\right]$$

$$= \frac{0.2114}{\pi kl}$$

Heat dessipation Q_2

$$= \frac{\Delta T}{R_t} = \frac{\Delta T}{0.2114} \times \pi kl$$

$$= 4.7303\,\pi kl\,\Delta T$$

$$\therefore \quad \frac{Q_1}{Q_2} = \frac{3.99}{4.7303} = 0.843$$

Obviously the heat dissipation is small when the material with low thermal conductivity is placed next to the steam pipe

Saving in heat dissipation

$$= \frac{4.7303 - 3.99}{3.99} \times 100 = \mathbf{18.55\%}$$

EXAMPLE 3.51

An insulated steam pipe of 16 cm diameter is covered with 4 cm thick layer of insulation (k = 0.9 W/m-deg) and carries process steam. Determine the percentage change in the rate of heat loss if an extra 2 cm thick layer of lagging (k = 1.25 W/m-deg) is provided. Given that surrounding temperature remains constant and the heat transfer coefficient for both the configurations is 12 W/m²-deg.

Solution : Since the thickness and conductivity of the pipe material are not given, it will be assumed that resistance of the pipe material to heat flow is negligible.

$$r_1 = 8 \text{ cm} \;;\; r_2 = 8 + 4 = 12 \text{ cm} \;;$$
$$r_3 = 12 + 2 = 14 \text{ cm}$$

Case I : Resistance to heat flow,

$$\Sigma R_t = R_{t_1} + R_{t_2}$$

$$= \frac{1}{2\pi k_1 l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi r_2 l h_o}$$

$$= \frac{1}{2\pi l}\left[\frac{1}{0.9}\log_e\frac{12}{8} + \frac{1}{0.12 \times 12}\right]$$

$$= \frac{1}{2\pi l} \times 1.1449 \text{ deg/W}$$

Heat loss, Q_1

$$= \frac{\Delta t}{\Sigma R_t} = \frac{2\pi l\Delta t}{1.1449} = 1.747\,\pi l\Delta t$$

Case II : Resistance to heat flow,

$$\Sigma R_t = R_{t_1} + R_{t_2} + R_{t_3}$$

$$= \frac{1}{2\pi k_1 l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi k_2 l}\log_e\frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}$$

$$= \frac{1}{2\pi l}\left[\frac{1}{0.9}\log_e\frac{12}{8} + \frac{1}{1.25}\log_e\frac{14}{12} + \frac{1}{0.14 \times 12}\right]$$

$$= \frac{1}{2\pi l}[0.4505 + 0.1233 + 0.5952]$$

$$= \frac{1}{2\pi l} \times 1.169 \text{ deg/W}$$

Heat loss, Q_2

$$= \frac{\Delta t}{\Sigma Rt} = \frac{2\pi l\Delta t}{1.169} = 1.711\,\pi l\Delta t$$

$\therefore$ Decrease in heat loss

$$= \frac{Q_1 - Q_2}{Q_1} = \frac{1.747 - 1.711}{1.747}$$

$$= 0.0206 \quad \text{or} \quad \mathbf{2.06\ \%}$$

EXAMPLE 3.52

Saturated steam at 110°C flows inside a copper pipe (thermal conductivity 450 W/m K) having an internal diameter of 10 cm and an external diameter of 12 cm. The surface resistance on the steam side is 12000 W/m² K and that on the outside surface of pipe is 18 W/m² K. Determine the heat loss from the pipe if it is located in space at 25°C. How this heat loss would be affected if the pipe is lagged with 5 cm thick insulation of thermal conductivity 0.22 W/m K ?

Solution : When the pipe is not insulated :

$$r_1 = 5 \text{ cm} \quad \text{and} \quad r_2 = 6 \text{ cm}$$

and the effective resistance to flow of heat is

$$\Sigma R_t = R_{t_1} + R_{t_2} + R_{t_3}$$

$$= \frac{1}{2\pi r_1 l h_i} + \frac{1}{2\pi k_1 l}\log_e \frac{r_2}{r_1} + \frac{1}{2\pi r_2 l h_o}$$

Taking unit length of pipe

$$\Sigma R_t = \frac{1}{2\pi \times 0.05 \times 1 \times 12000} + \frac{1}{2\pi \times 450 \times 1}\log_e \frac{0.06}{0.05} + \frac{1}{2\pi \times 0.06 \times 1 \times 18}$$

$$= 2.65 \times 10^{-4} + 0.645 \times 10^{-4} + 0.147$$

$$\approx 0.147 \text{ deg/W}$$

$$\text{Heat loss} = \frac{\Delta t}{\Sigma R_t} = \frac{110 - 25}{0.147}$$

$$= \mathbf{578\ W\ per\ metre\ length\ of\ pipe}$$

(*b*) When the pipe is insulated

$$r_1 = 5 \text{ cm} \; ; \; r_2 = 6 \text{ cm}$$

and $\quad r_3 = 6 + 5 = 11$ cm

and the effective resistance to heat flow becomes

$$\Sigma R_t = R_{t_1} + R_{t_2} + R_{t_3} + R_{t_4}$$

$$= \frac{1}{2\pi r_1 l h_i} + \frac{1}{2\pi k_1 l}\log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l}\log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}$$

$$= \frac{1}{2\pi \times 0.05 \times 1 \times 12000} + \frac{1}{2\pi \times 450 \times 1}\log_e \frac{0.06}{0.05} + \frac{1}{2\pi \times 0.22 \times 1}\log_e \frac{0.11}{0.06} + \frac{1}{2\pi \times 0.11 \times 1 \times 18}$$

$$= 2.65 \times 10^{-4} + 0.645 \times 10^{-4} + 0.080 + 0.438$$

$$\approx 0.518 \text{ deg/W}.$$

$$\text{Heat loss} = \frac{\Delta t}{\Sigma R_t} = \frac{110 - 25}{0.518}$$

$$= \mathbf{164\ W\ per\ metre\ length}$$

Accordingly, addition of insulation reduces the heat loss from the steam by

$$= \frac{578 - 164}{578} \times 100$$

$$= 0.716 \quad \text{or} \quad \mathbf{71.6\%}$$

EXAMPLE 3.53

The shell of an experimental boiling water reactor is cylindrical having inside radius 1 m, length 1.25 m and 10 cm wall thickness. The shell is made from alloy steel and a concrete wall 50 cm thick surrounds it. The thermal conductivities of alloy steel and concrete are 22.5 W/m-deg and 1.12 W/m-deg respectively.

The reactor operates at a power level of 6 kW, of which 4% is lost in heat transfer through the shell. If the inside water is at 150°C, workout the interface temperatures (temperatures on the inside and outside of concrete covering). Neglect any resistance to heat flow between water and steel.

Solution : Heat loss through the reactor

$$= 4/100 \times 6000 = 240 \text{ W}$$

$r_1 = 1.0$ m ; $r_2 = 1.1$ m

and $r_3 = 1.6$ m

Thermal resistance due to a cylindrical

$$\text{wall} = \frac{1}{2\pi k l}\log_e\frac{r_o}{r_i}$$

$\therefore$ Resistance due to shell thickness

$$= \frac{1}{2\pi\times 22.5\times 1.25}\log_e\frac{1.1}{1.0}$$

$$= 5.396 \times 10^{-4}\ \text{deg/W}$$

Resistance due to concrete covering

$$= \frac{1}{2\pi\times 1.12\times 1.25}\log_e\frac{1.6}{1.1}$$

$$= 426 \times 10^{-4}\text{deg/W}$$

$$\therefore \quad \Sigma R_t = 5.396 \times 10^{-4} + 426 \times 10^{-4}$$

$$= 431.3964 \times 10^{-4}\text{deg/W}$$

In terms of thermal resistance and temperature drop, the heat transfer rate is given by;

$$Q = \Delta T/\Sigma R_t$$

$$\therefore \quad 240 = \frac{t_1 - t_2}{\Sigma R_t} = \frac{150 - t_3}{431.396\times 10^{-4}}$$

where t_3 is the temperature on the outside surface of concrete,

$$t_3 = 150 - 240 \times 431.396 \times 10^{-4}$$

$$= \mathbf{139.65°C}$$

Since the same heat passes through each layer of the reactor, then for the alloy steel shell:

$$240 = \frac{t_1 - t_2}{5.396\times 10^{-4}} = \frac{150 - t_2}{5.396\times 10^{-4}}$$

where t_2 is the temperature on the inside surface of concrete

$$t_2 = 150 - 240 \times 5.396 \times 10^{-4}$$

$$= \mathbf{149.87°C}$$

It may be noted that temperature drop across the steel shell is negligible.

Check. The temperature t_2 can also be worked out by considering the heat passing through the concrete covering

$$240 = \frac{t_2 - t_3}{426\times 10^{-4}} = \frac{t_2 - 139.65}{426\times 10^{-4}}$$

$$t_2 = 139.65 + 426 \times 10^{-4} \times 240$$

$$= \mathbf{149.87°C}$$

This is same as calculated above.

EXAMPLE 3.54

A steel pipe of 20 mm inner diameter and 2 mm thickness is covered with 20 mm thick of fibre glass insulation (k = 0.05 W/m-deg). If the inside and outside convective coefficients are 10 W/m²-deg and 5 W/m²-deg, calculate the overall heat transfer coefficient based on inside diameter of the pipe.

Solution : $r_1 = 10$ mm ; $r_2 = 10 + 2 = 12$ mm

and $r_3 = 12 + 20 = 32$ mm

The thermal resistances to flow of heat are offered by

(*i*) Inside fluid film, $R_{t_1} = \dfrac{1}{h_i A_i} = \dfrac{1}{h_i\, 2\pi r_1 l}$

(*ii*) Pipe material, $R_{t_2} = \dfrac{\log_e \frac{r_2}{r_1}}{2\pi k_1 l}$

(*iii*) Insulation, $R_{t_3} = \dfrac{\log_e \frac{r_3}{r_2}}{2\pi k_2 l}$

(*iv*) Outside air film, $R_{t_4} = \dfrac{1}{h_o A_o} = \dfrac{1}{h_o 2\pi r_3 l}$

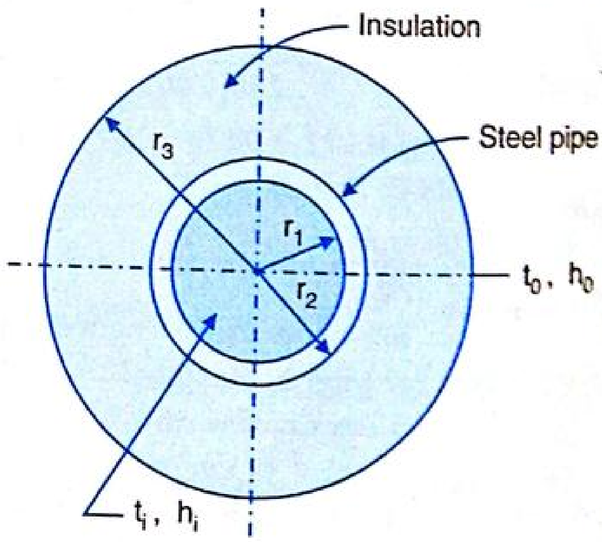


Fig. 3.47.

The heat transfer through the insulated pipe is than given by

$$Q = \frac{\Delta t}{\sum Rt}$$

$$= \frac{2\pi l (t_i - t_o)}{\frac{1}{h_i r_i} + \frac{1}{k_1}\log_e \frac{r_2}{r_1} + \frac{1}{k_2}\log_e \frac{r_3}{r_2} + \frac{1}{h_o r_3}}$$

The thermal conductivity of steel pipe is not given, and generally it is much higher than that of insulation. Accordingly thermal resistance due to pipe material can be neglected.

That gives :

$$Q = \frac{2\pi l (t_i - t_o)}{\frac{1}{h_i r_1} + \frac{1}{k_2}\log_e \frac{r_3}{r_2} + \frac{1}{h_o r_3}} \quad ...(i)$$

If U_i is the overall heat transfer coefficient based on inside area of the steel pipe, then heat flow rate can also be written as

$$Q = U_i A_i (t_i - t_o)$$

$$= U_i (2\pi r_1 l)(t_i - t_o) \quad ...(ii)$$

Comparing identities (*i*) and (*ii*), we note that

$$U_i (2\pi r_1 l)(t_i - t_o) = \frac{2\pi l (t_i - t_o)}{\frac{1}{h_i r_1} + \frac{1}{k_2}\log_e \frac{r_3}{r_2} + \frac{1}{h_o r_3}}$$

or $$\frac{1}{U_i r_1} = \frac{1}{h_i r_1} + \frac{1}{k_2}\log_e \frac{r_3}{r_2} + \frac{1}{h_o r_3}$$

or $$\frac{1}{U_i} = \frac{1}{h_i} + \frac{r_1}{k_2}\log_e \frac{r_3}{r_2} + \frac{r_1}{r_3} \times \frac{1}{h_o}$$

Upon substitution of given data,

$$\frac{1}{U_i} = \frac{1}{10} + \frac{10 \times 10^{-3}}{0.05}\log_e \frac{32}{12} + \frac{10}{32} \times \frac{1}{5}$$

$$= 0.1 + 0.196 + 0.0625 = 0.3585$$

$$\therefore \quad U_i = \frac{1}{0.3585} = \mathbf{2.789\ W/m^2\text{-}deg}$$

EXAMPLE 3.55

A steam main 75 mm inside diameter and 90 mm outside diameter is lagged with two successive layers of insulation. The layer in contact with the pipe is 38 mm asbestos and the asbestos layer is covered with 25 mm thick magnesia insulation. The surface coefficients for inside and outside surfaces are 227 W/m²K and 6.8 W/m²K respectively. If the steam temperature is 375°C and the ambient temperature is 35C, calculate the steady state loss of heat from steam for 60 m length of pipe. Also workout the overall coefficient of heat transfer based on the inside and outside surfaces of the lagged steam main. Comment upon the important generalisation of your result.

Thermal conductivity values are :

Pipe material : 45 W/mK

Asbestos : 0.14 W/mK

Magnesia Insulation : 0.07 W/mK

Solution : $r_1 = 37.5$ mm $= 0.0375$ m

$r_2 = 0.045$ m

$r_3 = 0.045 + 0.038 = 0.083$ m

$r_4 = 0.083 + 0.025 = 0.108$ m

The various thermal resistances to flow of heat are offered by:

(*i*) Inner steam film

$$= \frac{1}{h_i A_1} = \frac{1}{227 \times 2\pi \times 0.0375 \times 60}$$

$$= 3.117 \times 10^{-4}$$

(*ii*) Pipe material

$$= \frac{1}{2\pi k_1 l}\log_e \frac{r_2}{r_1}$$

$$= \frac{1}{2\pi \times 45 \times 60}\log_e \frac{0.045}{0.0375}$$

$$= 0.1075 \times 10^{-4}$$

(*iii*) Asbestos insulation

$$= \frac{1}{2\pi k_2 l}\log_e \frac{r_3}{r_2}$$

$$= \frac{1}{2\pi \times 0.14 \times 60}\log_e \frac{0.083}{0.045}$$

$$= 116.048 \times 10^{-4}$$

(*iv*) Magnesia insulation

$$= \frac{1}{2\pi k_3 l} \log_e \frac{r_4}{r_3}$$

$$= \frac{1}{2\pi \times 0.07 \times 60} \log_e \frac{0.108}{0.083}$$

$$= 99.822 \times 10^{-4}$$

(*v*) Outside air film

$$= \frac{1}{h_o A_o} = \frac{1}{6.8 \times 2\pi \times 0.108 \times 60}$$

$$= 36.137 \times 10^{-4}$$

The resistance of series path is equal to the sum of individual resistances and the summation gives

$$\Sigma R_t = 255.2315 \times 10^{-4} \text{deg/W}$$

Heat lost through conduction,

$$Q = \frac{\Delta t}{\Sigma R_t} = \frac{375 - 35}{255.2315 \times 10^{-4}}$$

$$= 13321 \text{ W}$$

(*b*) The heat loss can also be expressed as

$$Q = U_o A_o \Delta T = U_i A_i \Delta T$$

where U_o and U_i are the overall coefficients of heat transfer based on the outside area A_o and inside area A_i respectively.

$$U_o = \frac{13321}{(2\pi \times 0.108 \times 60)(375 - 35)}$$

$$= 0.962 \text{ W/m}^2\text{K}$$

$$U_i = \frac{13321}{(2\pi \times 0.0375 \times 60)(375 - 35)}$$

$$= 62.19 \text{ W/m}^2\text{K}$$

Important Generalisations:

(*a*) Resistance to heat flow rises mainly from the laggings and not from metal of the steam main.

(*b*) Total resistance is most strongly controlled by the film with lowest coefficient. Thus little gain would be obtained if the steam side coefficient were increased but a large gain would result from an increase in the air side coefficient.

(*c*) Resistance of the pipe material and inner steam film could have been neglected without introducing any appereciable error.

EXAMPLE 3.56

A steam pipe exposed to atmosphere is to be covered with two layers of insulation each 3 cm thick. The average thermal conductivity of the inferior insulation is 4 times that of the other. Show that for a given temperature difference between the steam and atmosphere, the location of better insulation should be next to the pipe surface.

Solution : Thermal resistance of two layers of insulation taken together is

$$R_t = \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2}$$

Let k and $4k$ be the thermal conductivities of the two insulating materials.

(*i*) When the better insulator (material with low thermal conductivity) is placed next to steam pipe, then

$$R_{t_1} = \frac{1}{2\pi kl} \log_e \frac{r_2}{r_1} + \frac{1}{8\pi kl} \log_e \frac{r_3}{r_2} \quad ...(i)$$

(*ii*) When the inferior insulator (material with greater thermal conductivity) is used closed to the pipe surface, then

$$R_{t_2} = \frac{1}{8\pi kl} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi kl} \log_e \frac{r_3}{r_2} \quad ...(ii)$$

Now, $(r_2 - r_1) = (r_3 - r_2)$, *i.e.*, r_1, r_2 and r_3 are in arithmetic progression.

Further $r_3 > r_2 > r_1$ and so $\frac{r_2}{r_1} > \frac{r_3}{r_2}$

This aspect can be visualised by considering any arithmetic progression (say 1, 2, 3 or 2, 4, 6).

In the expression for R_{t1}, bigger term $\log_e (r_2/r_1)$ has been reduced by small proportion by the factor $1/2\pi kl$, and the smaller term $\log_e (r_2/r_1)$ has been reduced in greater proportion by the factor $1/8\pi kl$. Converse is true for the identity for thermal resistance R_{t_2}. Evidently then

$$R_{t_1} > R_{t_2}$$

That is, the thermal resistance is more when the superior insulation is used closed to the pipe surface. This would be desirable to reduce heat loss from the steam pipe to the surrounding atmosphere.

EXAMPLE 3.57

A steel pipe having an external diameter of 8 cm carries saturated steam at 40 bar and is lagged with a layer of material 4 cm thick of thermal conductivity 0.05 W/mK. The ambient temperature is 19.5°C and the surface of the lagging has a heat transfer coefficient of 8.33 W/m² K. What thickness of lagging having a conductivity of 0.069 W/mK must be added to reduce the steam condensation rate by 50% if the surface coefficient remains unchanged? Neglect resistance due to pipe material and due to steam film on the inside of steam pipe.

Solution : With a single layer of insulation :

$r_1 = 4$ cm and $r_2 = 8$ cm

and the effective resistance to flow of heat is

$$\Sigma R_t = R_{t_1} + R_{t_2}$$

$$= \frac{1}{2\pi k l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi r_2 l h_o} \quad \ldots(i)$$

$$= \frac{1}{2\pi \times 0.05 \times 1}\log_2\frac{0.08}{0.04} + \frac{1}{2\pi \times 0.08 \times 1 \times 8.33}$$

$= 2.207 + 0.239$

$= 2.446$ deg/W per metre length

(*b*) With two layers of insulation :

$r_1 = 4$ cm ; $r_2 = 8$ cm and $r_3 = ?$

and the effective resistance to heat flow is

$$\Sigma R_t = R_{t_1} + R_{t_2} + R_{t_3}$$

$$= \frac{1}{2\pi k_1 l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi k_2 l}\log_e\frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}$$

$$= \frac{1}{2\pi \times 0.05 \times 1}\log_e\frac{0.08}{0.04} + \frac{1}{2\pi \times 0.069 \times 1}\log_e\frac{r_3}{0.08} + \frac{1}{2\pi r_3 \times 1 \times 8.33}$$

$$\Sigma R_t = 2.207 + 2.308\log_e\frac{r_3}{0.08} + \frac{0.019}{r_3}$$

When condensation rate is to be reduced to 50 percent, the resistance to heat flow must become two times.

The gives

$$2 \times 2.446 = 2.207 + 2.308\log_e\frac{r_3}{0.08} + \frac{0.019}{r_3}$$

$$\text{or } 2.308\log_e\frac{r_3}{0.08} + \frac{0.019}{r_3} = 2.685$$

By hit and trial :

$r_3 = 0.25$ m $= 25$ cm

∴ Required thickness of second insulation

$= r_3 - r_2 = 25 - 8 = 17$ cm

EXAMPLE 3.58

A 3 cm diameter pipe at 100°C is losing heat at the rate of 100 W per metre length of pipe to the surrounding air at 20°C. This is to be reduced to a minimum value by providing insulation.The following insulation materials are available :

Insulation A

Quantity = 3.15 × 10^{-3}m³ per metre length of pipe

Thermal conductivity = 5 W/m-deg

Insulation B

Quantity = 4 × 10^{-3}m³ per metre length of pipe

Thermal conductivity = 1 W/m-deg

Examine the position of better insulating layer relative to the pipe. What percentage saving in heat dissipation results from that arrangement ?

Solution : Thermal resistance due to pipe material works out as

$$R_t = \frac{\Delta t}{Q} = \frac{100 - 20}{100} = 0.8 \text{ deg/W}$$

For a pipe with two layers of insulation,

$$\Sigma R_t = R_t + R_{t_1} + R_{t_2}$$

$$= 0.8 + \frac{1}{2\pi k_1 l}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi k_2 l}\log_e\frac{r_3}{r_2}$$

1st arrangement : The insulation material A is placed inside, *i.e.*, next to the pipe

$$r_1 = 0.015 \text{ m}$$

$$\frac{\pi}{4}(r_2^2 - r_1^2)l = 3.15 \times 10^{-3}$$

$$\text{or } \frac{\pi}{4}(r_2^2 - 0.015^2) \times 1 = 3.15 \times 10^{-3};$$

$$r_2 = 0.065 \text{ m}$$

$$\text{Also, } \frac{\pi}{4}(r_3^2 - r_2^2)l = 4 \times 10^{-3}$$

$$\text{or } \frac{\pi}{4}(r_3^2 - 0.065^2) \times 1 = 4 \times 10^{-3};$$

$$r_3 = 0.0965 \text{ m}$$

$$\therefore \quad \Sigma R_t = 0.8 + \frac{1}{2\pi \times 5 \times 1}\log_e\frac{0.065}{0.015} + \frac{1}{2\pi \times 1 \times 1}\log_e\frac{0.0965}{0.065}$$

$$= 0.8 + 0.0467 + 0.0629$$

$$= 0.9096 \text{ deg/W}$$

Heat loss, Q

$$= \frac{\Delta t}{\Sigma R_t} = \frac{100 - 20}{0.9096} = 87.95 \text{ W}$$

2nd arrangement : The insulation material B is placed inside, *i.e.*, next to the pipe

$$r_1 = 0.015 \text{ m}$$

$$\frac{\pi}{4}(r_2^2 - r_1^2)l = 4 \times 10^{-3};$$

$$r_2 = 0.073 \text{ m}$$

$$\frac{\pi}{4}(r_3^2 - r_2^2)l = 3.15 \times 10^{-3};$$

$$r_3 = 0.0965 \text{ m}$$

$$\Sigma R_t = 0.8 + \frac{1}{2\pi \times 1 \times 1}\log_e\frac{0.073}{0.015} + \frac{1}{2\pi \times 5 \times 1}\log_e\frac{0.0965}{0.073}$$

$$= 0.8 + 0.252 + 0.0089$$

$$= 1.0609 \text{ deg/W}$$

Heat loss, Q

$$= \frac{100 - 20}{1.0609} = 75.4 \text{ W}$$

Obviously the heat loss is small when the insulation material B is placed next to pipe.

Saving in heat loss

$$= \frac{100 - 75.4}{100} \times 100 = \mathbf{24.6\%}$$

3.6. CONDUCTION THROUGH A SPHERE

Consider heat conduction through a hollow sphere of inner radius r_1, outer radius r_2 and made of a material of constant thermal conductivity. The inner and outer surfaces are maintained at constant but different temperatures t_1 and t_2 respectively. Geometrical symmetry indicates that the heat flow is limited to radial direction only. Further if temperature t_1 at the inner surface is greater than temperature t_2 at the outer surface, the heat flows radially outwards.

Recapitulate the general heat equation in spherical co-ordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial^2\phi} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial\tau} \quad ...(3.30)$$

For steady state $(\partial t/\partial\tau = 0)$, uni-directional heat flow in the radial direction $\{t \neq f(\theta, \phi)\}$ and with no internal heat generation $(q_g = 0)$, the above equation gets simplified to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dt}{dr}\right) = 0$$

$$\text{or} \quad \frac{d}{dr}\left(r^2\frac{dt}{dr}\right) = 0 \quad \text{as} \quad \frac{1}{r^2} \neq 0$$

$$\text{or} \quad r^2\frac{dt}{dr} = \text{constant } C_1 \quad ...(3.31)$$

Equation 3.31 may be integrated to give

$$t = -\frac{C_1}{r} + C_2 \quad ...(3.32)$$

Incorporating the relevant boundary conditions

$$t = t_1 \quad \text{at} \quad r = r_1$$

1st arrangement : The insulation material A is placed inside, *i.e.*, next to the pipe

$$r_1 = 0.015 \text{ m}$$

$$\frac{\pi}{4}(r_2^2 - r_1^2)l = 3.15 \times 10^{-3}$$

or $\frac{\pi}{4}(r_2^2 - 0.015^2) \times 1 = 3.15 \times 10^{-3}$;

$$r_2 = 0.065 \text{ m}$$

Also, $\frac{\pi}{4}(r_3^2 - r_2^2)l = 4 \times 10^{-3}$

or $\frac{\pi}{4}(r_3^2 - 0.065^2) \times 1 = 4 \times 10^{-3}$;

$$r_3 = 0.0965 \text{ m}$$

$$\therefore \quad \Sigma R_t = 0.8 + \frac{1}{2\pi \times 5 \times 1}\log_e \frac{0.065}{0.015} + \frac{1}{2\pi \times 1 \times 1}\log_e \frac{0.0965}{0.065}$$

$$= 0.8 + 0.0467 + 0.0629$$

$$= 0.9096 \text{ deg/W}$$

Heat loss, Q

$$= \frac{\Delta t}{\Sigma R_t} = \frac{100 - 20}{0.9096} = 87.95 \text{ W}$$

2nd arrangement : The insulation material B is placed inside, *i.e.*, next to the pipe

$$r_1 = 0.015 \text{ m}$$

$$\frac{\pi}{4}(r_2^2 - r_1^2)l = 4 \times 10^{-3};$$

$$r_2 = 0.073 \text{ m}$$

$$\frac{\pi}{4}(r_3^2 - r_2^2)l = 3.15 \times 10^{-3};$$

$$r_3 = 0.0965 \text{ m}$$

$$\Sigma R_t = 0.8 + \frac{1}{2\pi \times 1 \times 1}\log_e \frac{0.073}{0.015} + \frac{1}{2\pi \times 5 \times 1}\log_e \frac{0.0965}{0.073}$$

$$= 0.8 + 0.252 + 0.0089$$

$$= 1.0609 \text{ deg/W}$$

Heat loss, Q

$$= \frac{100 - 20}{1.0609} = 75.4 \text{ W}$$

Obviously the heat loss is small when the insulation material B is placed next to pipe.

Saving in heat loss

$$= \frac{100 - 75.4}{100} \times 100 = 24.6\%$$

3.6. CONDUCTION THROUGH A SPHERE

Consider heat conduction through a hollow sphere of inner radius r_1, outer radius r_2 and made of a material of constant thermal conductivity. The inner and outer surfaces are maintained at constant but different temperatures t_1 and t_2 respectively. Geometrical symmetry indicates that the heat flow is limited to radial direction only. Further if temperature t_1 at the inner surface is greater than temperature t_2 at the outer surface, the heat flows radially outwards.

Recapitulate the general heat equation in spherical co-ordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial t}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 t}{\partial^2 \phi} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial t}{\partial \theta}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau} \quad ...(3.30)$$

For steady state $(\partial t/\partial \tau = 0)$, uni-directional heat flow in the radial direction $[t \neq f(\theta, \phi)]$ and with no internal heat generation $(q_g = 0)$, the above equation gets simplified to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dt}{dr}\right) = 0$$

or $$\frac{d}{dr}\left(r^2 \frac{dt}{dr}\right) = 0 \quad \text{as} \quad \frac{1}{r^2} \neq 0$$

or $$r^2 \frac{dt}{dr} = \text{constant } C_1 \quad ...(3.31)$$

Equation 3.31 may be integrated to give

$$t = -\frac{C_1}{r} + C_2 \quad ...(3.32)$$

Incorporating the relevant boundary conditions

$$t = t_1 \text{ at } r = r_1$$

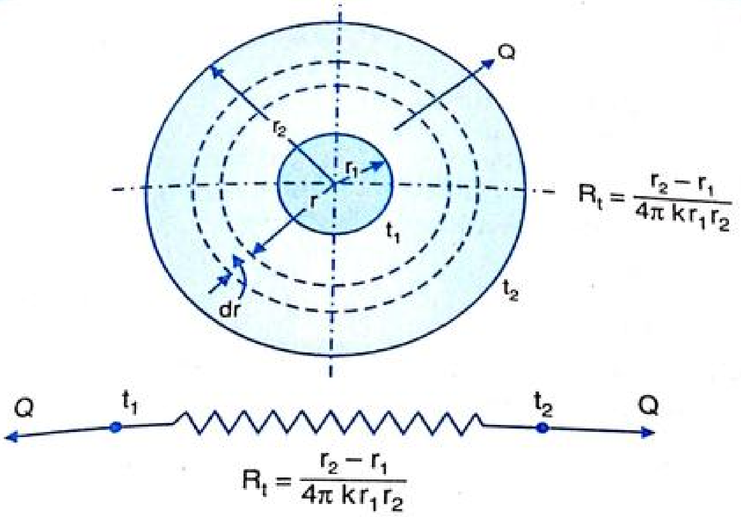


Fig. 3.48. Steady state conduction through a sphere

and $t = t_2$ at $r = r_2$

The integration constants C_1 and C_2 take the values:

$$C_1 = \frac{(t_1 - t_2) r_1 r_2}{(r_1 - r_2)}$$

and $$C_2 = t_1 + \frac{(t_1 - t_2) r_1 r_2}{r_1 (r_1 - r_2)} \qquad ...(3.33)$$

When the values of these constants are substituted in equation 3.32, one obtains the following expression for temperature distribution in a hollow sphere.

$$t = -\frac{(t_1 - t_2) r_1 r_2}{r(r_1 - r_2)} + t_1 + \frac{(t_1 - t_2) r_1 r_2}{r_1 (r_1 - r_2)}$$

$$= -\frac{(t_1 - t_2)}{r(1/r_2 - 1/r_1)} + t_1$$

$$+ \frac{(t_1 - t_2)}{r_1 (1/r_2 - 1/r_1)}$$

$$t = t_1 + \frac{(t_1 - t_2)}{(1/r_2 - 1/r_1)} \left[\frac{1}{r_1} - \frac{1}{r} \right]$$

$$= t_1 + \frac{(t_1 - t_2) r_1 r_2}{(r_1 - r_2)} \left[\frac{1}{r_1} - \frac{1}{r} \right]$$

or in non-dimensional form

$$\frac{t - t_1}{t_2 - t_1} = \frac{1/r - 1/r_1}{1/r_2 - 1/r_1}$$

$$= \frac{r_2}{r} \left(\frac{r - r_1}{r_2 - r_1} \right) \qquad ...(3.34)$$

Evidently the temperature distribution associated with radial conduction through a sphere is represented by a hyperbola.

The conduction heat transfer rate is determined by ultizing the temperature distribution in conjunction with the Fourier law :

$$Q = -kA \frac{dt}{dr}$$

$$= -kA \frac{d}{dr} \left[t_1 + \frac{(t_1 - t_2) r_1 r_2}{(r_1 - r_2)} \left\{ \frac{1}{r_1} - \frac{1}{r} \right\} \right]$$

$$= -k \times 4\pi r^2 \left[\frac{(t_1 - t_2) r_1 r_2}{(r_1 - r_2)} (-1) \times r^{-2} \right]$$

$$= \frac{4\pi k (t_1 - t_2) r_1 r_2}{(r_2 - r_1)}$$

$$= \frac{(t_1 - t_2)}{(r_2 - r_1)/4\pi k r_1 r_2} \qquad ...(3.35)$$

The quanity $(r_2 - r_1)/4\pi k r_1 r_2$ is the thermai resistance for heat conduction through a spherical wall.

In the alternative approach to estimate heat flow, consider an infinitesimal thin spherical element at radius r. Let thickness of this elementary ring be dr and the change of temperature across it be dt. Then invoking Fourier law of heat conduction.

$$Q = -kA\frac{dt}{dr} = -k(4\pi r^2)\frac{dt}{dr}$$

Separating the variables and integrating within the prescribed boundary conditions

$$\frac{Q}{4\pi k}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{t_1}^{t_2} dt$$

$$\frac{Q}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (t_1 - t_2)$$

or
$$Q = \frac{4\pi k(t_1 - t_2) r_1 r_2}{r_2 - r_1}$$

$$= \frac{(t_1 - t_2)}{(r_2 - r_1)/4\pi k r_1 r_2}$$

Quite often it is considered advantageous to write the heat flow equation through a sphere in the same form as that for heat flow through a plane wall. Then thickness δ will be equal to $(r_2 - r_1)$ and the areas A will be an equivalent area A_m. Thus

$$Q = \frac{kA(t_1 - t_2)}{\delta}$$

$$= \frac{kA_m(t_1 - t_2)}{(r_2 - r_1)} \quad ...(3.36)$$

Comparing equations 3.35 and 3.36,

$$A_m = 4\pi r_1 r_2 \quad ...(3.37)$$

Further, $A_m = 4\pi r_m^2 = 4\pi r_1 r_2$

Obviously *geometric mean radius* of the spherical shell is,

$$r_m = \sqrt{r_1 r_2} \quad ...(3.38)$$

Spherical composites are dealt with in the same way as composite walls and cylinders. The heat flow through a three-layer composite spherical shall will be:

$$Q = \frac{(t_1 - t_2)}{\dfrac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \dfrac{r_3 - r_2}{4\pi k_2 r_2 r_3} + \dfrac{r_4 - r_3}{4\pi k_3 r_3 r_4}} \quad ...(3.39)$$

Further, if the convective film coefficients at the inner and outer surfaces of the composite spherical shell are also considered, then:

$$Q = \frac{(t_1 - t_2)}{\dfrac{1}{4\pi r_1^2 h_i} + \dfrac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \dfrac{r_3 - r_2}{4\pi k_2 r_2 r_3} + \dfrac{r_4 - r_3}{4\pi k_3 r_3 r_4} + \dfrac{1}{4\pi r_4^2 h_o}} \quad ...(3.40)$$

EXAMPLE 3.59.

The following data pertains to a hollow cylinder and a hollow sphere made of the same material and having the same temperature drop over the wall thickness.

Inside radius = 0.1 m and outside surface area = 1 m²

If the outside radius for both the geometrics is same, calculate the ratio of heat flow in the cylinder to that in the sphere.

Solution : Let suffix 1 and 2 refer to inside and outside surface respectively.

For **sphere :**

$$A_2 = 4\pi\, r_2^2\ ; \quad 1 = 4\pi\, r_2^2$$

or
$$r_2 = \sqrt{\frac{1}{4\pi}} = 0.282 \text{ m}$$

For **cylinder :**

$$A_2 = 2\pi\, r_2\, l$$

The outside radius of the cylinder is same as that of sphere, *i.e.*, $r_2 = 0.282$ m

$$\therefore \quad l = \frac{A_2}{2\pi r_2} = \frac{1}{2\pi \times 0.282} = 0.565 \text{ m}$$

Now,

$$\frac{Q_{cyl}}{Q_{sph}} = \frac{2\pi k l \dfrac{(t_1 - t_2)}{\log_e(r_2/r_1)}}{\dfrac{4\pi k\, r_1 r_2 (t_1 - t_2)}{r_2 - r_1}}$$

$$\frac{Q_{cyl}}{Q_{sph}} = \frac{l(r_2 - r_1)}{2 r_1 r_2 \log_e (r_2/r_1)}$$

$$= \frac{0.565}{2 \times 0.1 \times 0.282 \log_e (0.282/0.1)}$$

$$\approx \mathbf{1.756}$$

EXAMPLE 3.60

A hollow sphere of inner radius 30 mm and outer radius 50 mm is electrically heated at the inner surface at a rate of 10^5 W/m^2. At the outer surface, it dissipates heat by convection into a fluid at 100°C and a heat transfer coefficient of 400 W/m^2 K. Determine the temperature at the inside and outside surfaces of the sphere.

It may be presumed that there is no energy generation and the thermal conductivity of the material is constant at 15 W/mK.

Solution : $r_1 = 0.03$ m and $r_2 = 0.05$ m

Rate of heat dissipation Q

$$= q \times 4\pi\, r_1^2$$

$$= 10^5 \times (4\pi \times 0.03^2) = 1130.44 \text{ W}$$

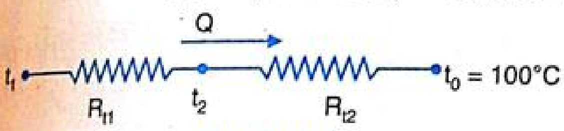


Fig. 3.49.

Heat dissipation $Q = \dfrac{t_1 - t_0}{R_{t1} + R_{t2}}$

where R_{t1}(resitance to conduction)

$$= \frac{r_2 - r_1}{4\pi k\, r_1 r_2}$$

$$= \frac{0.05 - 0.03}{4\pi \times 15 \times 0.05 \times 0.03}$$

$$= 0.0708 \text{ k/W}$$

and R_{t2}(resitance to convection)

$$= \frac{1}{4\pi r_2^2\, h} = \frac{1}{4\pi (0.05)^2 \times 400}$$

$$= 0.0796 \text{ k/W}$$

Then : $1130.4 = \dfrac{t_1 - 100}{0.0708 + 0.0796} = \dfrac{t_1 - 100}{0.1504}$

∴ Temperature at inner surface of the sphere,

$$t_1 = 1130.4 \times 0.1504 + 100 = 270°\text{C}$$

Under steady state conditions, heat flow through each section is same. Therefore

$$1130.4 = \frac{t_2 - t_0}{R_2} = \frac{t_2 - 100}{0.0796}$$

∴ Temperature at outer surface of the sphere,

$$t_2 = 1130.4 \times 0.0796 + 100$$

$$= \mathbf{189.98°C}$$

EXAMPLE 3.61

The thermal coductivity of a material is to be determined by fabricating the material into the shape of a hollow sphere, placing an electric heater at the centre and measuring the surface temperatures with thermocouples when steady state conditions have been attained. The sphere has internal radius 3 cm, external radius 8 cm and the corresponding temperatures are 95°C and 85°C when an electrical input to heater is 10 watts. Determine the experimental value of thermal conductivity and the temperature at a point halfway through the wall.

Solution : In terms of geometical parameters, thermal resistance of a sphere is

$$R_t = \frac{r_2 - r_1}{4\pi k r_1 r_2} = \frac{0.08 - 0.03}{4\pi k (0.08 \times 0.03)}$$

$$= \frac{1.6587}{k} \text{deg / W}$$

Heat flowing through the sphere is equal to

$$\frac{\Delta t}{R_t} = \frac{95 - 85}{1.6587 / k} = \frac{k}{0.16587} \text{W}$$

Under steady state conditions, this heat flow equals the power input to the heater.

$$\frac{k}{0.16587} = 10\ ;\ k = \mathbf{1.6587\ W/m\text{-}deg}$$

(*ii*) Radius at halfway through the wall,

$$r = \frac{r_1 + r_2}{2} = \frac{3 + 8}{2} = 5.5 \text{ cm}$$

Thermal resistance of the spherical wall upto its mid plane is equal to

$$\frac{0.055 - 0.03}{4\pi \times 1.6587(0.055 \times 0.03)} = 0.727 \text{ deg/W}$$

Since heat flowing through each section is same,

$$10 = \frac{95 - t}{0.727}$$

∴ Temperature at a point halfway through the wall is,

$$t = 95 - 10 \times 0.727 = 87.73°C$$

Alternatively from the expression for temperature distribution for a spherical wall

$$\frac{t - t_1}{t_2 - t_1} = \frac{(1/r - 1/r_1)}{(1/r_2 - 1/r_1)} = \frac{r_2}{r}\left(\frac{r - r_1}{r_2 - r_1}\right)$$

$$\frac{t - 95}{85 - 95} = \frac{8}{5.5}\left(\frac{5.5 - 3}{8 - 3}\right) = 0.7272$$

$$t = 95 - (95 - 85) \times 0.7272$$

$$= 87.728°C$$

EXAMPLE 3.62

A spherical vessel of 0.5 m outside diameter is insulated with 0.2 m thickness of insulation of thermal conductivity 0.04 W/m-deg. The surface temperature of the vessel is – 195°C and outside air is at 10°C. Determine :

(a) heat flow,

(b) heat flow per m² based on inside and outside area, and

(c) temperature gradients at the inner and outside surface.

Solution : The heat flow through a spherical tube with radii r_1 and r_2 is given by

$$Q = \frac{4\pi k r_1 r_2 (t_1 - t_2)}{r_2 - r_1}$$

$$= \frac{4\pi \times 0.04 \times 0.25 \times 0.45(-195 - 10)}{0.45 - 0.25}$$

$$= -57.93 \text{ W}$$

Alternatively : Thermal resistance of a spherical tube with radii r_1 and r_2 is given by

$$R_t = \frac{r_2 - r_1}{4\pi k r_1 r_2}$$

$$= \frac{0.45 - 0.25}{4\pi \times 0.04 \times 0.25 \times 0.45}$$

$$= 3.538°C/W$$

Heat flow, Q

$$= \frac{\Delta t}{R_t} = \frac{-195 - 10}{3.538} = -57.94 \text{ W}$$

The negative sign indicates that heat flow is in negative radial direction

(b) Heat flow based on inside area

$$= \frac{Q}{4\pi r^2} = \frac{-57.93}{4\pi(0.25)^2}$$

$$= -73.79 \text{ W/m}^2$$

Heat flow based on outside area

$$= \frac{Q}{4\pi r_2^2} = \frac{-57.93}{4\pi(0.45)^2}$$

$$= -22.78 \text{ W/m}^2$$

(c) The temperature gradients are worked out from the relation

$$Q = -kA\frac{dt}{dr}; \frac{dt}{dr} = \frac{-Q}{kA}$$

$$\therefore \text{ Inside } \frac{dt}{dr} = \frac{-(-57.93)}{0.04 \times 4\pi(0.25)^2}$$

$$= 1844.9°C/m$$

$$\therefore \text{ Outside } \frac{dt}{dr} = \frac{-(-57.93)}{0.04 \times 4\pi(0.45)^2}$$

$$= 569.41°C/m$$

Evidently the temperature gradient decreases along the radius. Further, the gradient is positive and heat flow is in the opposite direction.

EXAMPLE 3.63

Two insulation materials A and B, in powder form, with thermal conductivities of 0.005 W/m-deg and 0.03 W/m-deg were purchased for use over a sphere of 40 cm diameter. Material A was to form the first layer 4 cm thick and material B was to be the next layer 5 cm thick. Due to oversight during installation, whole of material B was applied first and subsequently there was a layer formed by material A. Investigate how the conduction heat fransfer would be affected.

Solution : Case I:

$r_1 = 0.2$ m ; $r_2 = 0.24$ m ; $r_3 = 0.29$ m

Thermal resistance to heat flow,

$$R_{t_1} = \frac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_2 r_2 r_3}$$

$$= \frac{0.24 - 0.2}{4\pi \times 0.005 \times 0.2 \times 0.24} + \frac{0.29 - 0.24}{4\pi \times 0.03 \times 0.24 \times 0.29}$$

$$= 13.27 + 1.906 = \mathbf{15.176\ °C/W}$$

Case II : When the materials get interchanged, there would be change in radii also.

Volume of material A

$$= \frac{4\pi}{3}\left(0.24^3 - 0.2^3\right)$$

$$= 0.024396 \text{ m}^3$$

Volume of material B

$$= \frac{4\pi}{3}\left(0.29^3 - 0.24^3\right)$$

$$= 0.044255 \text{ m}^3$$

The new radii are then worked out as

$$0.044255 = \frac{4}{3}\pi\left(r_2^2 - 0.2^2\right) ;$$

$$r_2 = 0.2648 \text{ m}$$

$$0.024396 = \frac{4}{3}\pi\left(r_3^2 - 0.2648^2\right) ;$$

$$r_3 = 0.29 \text{ m}$$

Thermal resistance R_{t_2}

$$= \frac{r_2 - r_1}{4\pi k_2 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_1 r_2 r_3}$$

$$= \frac{0.2648 - 0.2}{4\pi \times 0.03 \times 0.2 \times 0.2648} + \frac{0.29 - 0.2648}{4\pi \times 0.005 \times 0.2648 \times 0.29}$$

$$= 3.247 + 5.225 = \mathbf{8.472\ °C/W}$$

Heat transfer is inversely proportional to thermal resistance. As such the heat flow will increase by

$$\frac{15.176 - 8.472}{15.176} \times 100 = \mathbf{44.17\%}.$$

EXAMPLE 3.64

A hemispherical oven of 60 cm internal radius is lagged with 10 cm thick fire brick covering surrounded by a magnesia layer of thickness 5 cm. An electric heater is placed at the centre and under steady state conditions, the inner surface of the oven is maintained at 825 °C. The system is in a room for which the ambient temperature is 25 °C and the outside unit convective coefficient is 8.75 W/m²-deg. Compute the heat loss through the oven and the wattage required of the heater filament to be placed inside to affect the same heat transfer. Also determine the temperature at a point halfway through the fire-brick covering. Thermal conductivities of the insulating material are :

Fire brick = 0.315 W/m-deg ;

Magnesia = 0.0525 W/m-deg.

Neglect any thermal resistance due to the oven meterial.

Solution : $r_1 = 60$ cm $= 0.6$ m ;

$r_2 = 60 + 10 = 70$ cm $= 0.7$ m ;

$r_3 = 70 + 5 = 75$ cm $= 0.75$ m

Thermal resistance for a spherical body is given by :

$$R_1 = \frac{r_o - r_i}{4\pi k r_o r_i}$$

and for a hemisphere, it equals half of this value.

∴ Resistance of fire brick

$$= \frac{1}{2} \times \frac{0.70 - 0.60}{4\pi \times 0.315 \times (0.7 \times 0.6)}$$

$$= 0.03009 \text{ deg/W}$$

Resistance of magnesia

$$= \frac{1}{2} \times \frac{0.75 - 0.70}{4\pi \times 0.0525 \times (0.75 \times 0.70)}$$

$$= 0.07221 \text{ deg/W}$$

Resistance of outside air film

$$= \frac{1}{2} \times \frac{1}{h_o A_o}$$

$$= \frac{1}{2} \times \frac{1}{8.75 \times 4\pi (0.75)^2}$$

$= 0.03235$ deg/W

Total resistance, ΣR_t

$= 0.13465$ deg/W

Heat loss from the oven

$$= \frac{\Delta T}{\Sigma R_t} = \frac{825 - 25}{0.13465} = 5941.3 \text{ W}$$

Therefore the required filament wattage is 5.94 kW

(*b*) Radius at mid-plane of the fire brick lining

$$r = \frac{r_1 + r_2}{2} = \frac{60 + 70}{2} = 65 \text{ cm}$$

Thermal resistance of fire brick upto its mid plane

$$= \frac{1}{2} \times \frac{0.65 - 0.60}{4\pi \times 0.315 \times (0.65 \times 0.60)}$$

$= 0.0162$ deg/W

Since heat flowing through each section is same,

$$5941.3 = \frac{825 - t}{0.0162}$$

∴ Temperature at the mid plane of the fire brick lining is :

$t = 825 - 5941.3 \times 0.0162$

$= 728.75$ °C

EXAMPLE 3.65

A 6.5 m diameter vertical kiln has a hemi-spherical dome at the top; the dome is fabricated from a 25 cm thick layer of chrome brick which has a thermal conductivity of 1.16 W/m-deg. The kiln dome has inside temperature of 875°C, and 20°C atmospheric air results into 11.4 W/m²-deg heat transfer coefficient between the dome and air. Estimate the outside surface temperature of the dome and the heat loss from the kiln. Compare this heat with that would result from a flat dome fabricated from the same material and with kiln operating under identical temperature conditions.

Solution : Conduction heat loss through from a spherical body is given by

$$= 4\pi k (t_1 - t_2) \times \frac{r_1 r_2}{r_2 - r_1}$$

and for a hemi-sphere it equals half of this value.

∴ Conduction heat loss through the hemi-spherical dome,

$$= \left[2\pi \times 1.16 (875 - t_2) \times \frac{3.25 \times 3.5}{3.5 - 3.25}\right]$$

$$= 331.46 \ (875 - t_2) \qquad \text{...}(i)$$

Convective heat flow from outside surface of dome to the surrounding air,

$= h A \Delta t$

$$= 11.4 \times \left(\frac{1}{2} \times 4\pi \times 3.5^2\right) \times (t_2 - 20)$$

$$= 887 \ (t_2 - 20) \qquad \text{...}(ii)$$

Under steady state conditions,

$331.46 \ (875 - t_2) = 887 \ (t_2 - 20)$

$875 - t_2 = 2.65 t_2 - 52.92$

∴ Temperature at the outside surface of the dome,

$$t_2 = \frac{875 + 52.92}{3.65} = 254°\text{C}$$

The heat loss from the dome may now be obtained from either of the expression (*i*) and (*ii*).

$Q = 331.46 \ (875 - 254)$

$= 205837$ W

(*b*) For a dome with flat top :

$$\frac{kA (t_1 - t_2)}{\delta} = h A \ (t_2 - t_a)$$

The area for conduction and convection heat flow will be same

$$\therefore \frac{1.16 (875 - t_2)}{0.25} = 11.4 \ (t_2 - 20)$$

or $875 - t_2 = 2.45 \, t_2 - 49.14$

or $$t_2 = \frac{875 + 49.14}{3.45} = 268°\text{C}$$

∴ $Q = 11.4 \times (\pi \times 3.25^2) \times (268 - 20)$

$= 93768$ W

Reduction in heat loss

$$= \frac{205836 - 93768}{2058.36}$$

$= 0.544$ or **54.4 %**

EXAMPLE 3.66

A cylindrical shell with flat ends contains a fluid at 100°C and is lagged with 10 cm thick layer of insulating material of thermal conductivity 0.05 W/m-deg. The shell is placed in a room where the air is at 20°C and the convective coefficient at the outer surface of lagging is 10 W/m²-deg.If the volume enclosed is 10 m³ and the ratio of cylinder length to radius is 8, workout the rate of heat dissipation from the fluid. Neglect corner effects and any thermal resistance due to shell material.

(b) It is now required to hold the same eqantity (10 m³) of fluid in a spherical container lagged with the same insulating material. For the same heat flux, calculate the size of the spherical shell and thickness of lagging.

Solution : If l and r denote the length and radius of the cylindrical shell, then

$$\pi r^2 l = 10 \text{ or } \pi r^2 (8r) = 10$$

$\therefore$ $r = 0.736$ m and $l = 5.888$ m

Thus for the cylindrical shell with lagging, we have,

$r_1 = 0.736$ m,

$r_2 = 0.836$ m and $l = 5.888$ m

Thermal resistance of the cylindrical shell and surface

= resistance due to lagging

+ resistance due to outside film

$$= \frac{1}{2\pi k l} \log_e \frac{r_2}{r_1} + \frac{1}{h_o(2\pi r_2 l)}$$

$$= \frac{1}{2\pi \times 0.05 \times 5.88} \log_e \frac{0.836}{0.736} + \frac{1}{10 \times 2\pi \times 0.836 \times 5.888}$$

$= 0.0689 + 0.003235$

$= 0.07213$ deg/W

Neglecting corner effects, thermal resistance of the two ends is,

$$= \frac{\delta}{kA} + \frac{1}{hA}$$

For the two ends, A

$= 2 \times \pi(0.736)^2 = 3.402$ m²

$\therefore$ Thermal resistance,

$$= \frac{0.1}{0.05 \times 3.402} + \frac{1}{10 \times 3.402}$$

$= 0.5879 + 0.02939$

$= 0.6173$ deg/W

The two resistances are in parallel and the equivalent resistance R_e would be given by :

$$\frac{1}{R_e} = \frac{1}{0.07213} + \frac{1}{0.6173}$$

$$R_e = \frac{0.07213 \times 0.6173}{(0.07213 + 0.6173)}$$

$= 0.06458$ deg/W

$\therefore$ Heat loss form the fluid

$$= \frac{\Delta T}{R_e} = \frac{100 - 20}{0.06458} = \mathbf{1238.77\ W}$$

(*b*) For a spherical vessel to hold 10 m³

$$\frac{4}{3}\pi r_1^3 = 10 \text{ ; } r_1 = 1.336 \text{ m}$$

Thermal resistance of the spherical container

$$= \frac{r_2 - r_1}{4\pi k r_1 r_2} + \frac{1}{h_o\left(4\pi r_2^2\right)}$$

and this must equal the thermal resistance of the cylindrical container if the heat dissipation is to be the same

$\therefore$ 0.06458 =

$$\frac{r_2 - 1.336}{4\pi \times 0.05 \times 1.336 r_2} + \frac{1}{10 \times 4\pi r_2^2}$$

Solution gives $r_2 = 1.406$ m

$\therefore$ Thickness of insulation

$= (1.406 - 1.336)$

$= 0.070$ m $= 7$ cm

EXAMPLE 3.67.

A storage tank fabricated from 20 mm thick pyrex glass (k = 1.4 W/mK) consists of a cylindrical section (length 2 m and inner diameter 1 m) and

two hemispherical end sections. The tank is exposed to ambient air at 25°C and having convective coefficient 50 W/m² K. The tank is used to store heated oil which keeps the inner surface of tank at 125°C. Determine the electric power that needs to be supplied to the heater submerged in the oil so that the prescribed conditions can be maintained. Neglect the radiation effects if any.

Solution : Refer Fig. 3.50 for the geometry of storage tank.

$$r_1 = 0.5 \text{ m}$$

and $r_2 = 0.5 + 0.0.2 = 0.52$ m

Under steady state conditions,

$$Q_{cyl} + Q_{sph} = Q_{total}$$

That is, the heat dissipation from the heater comprises two parts :

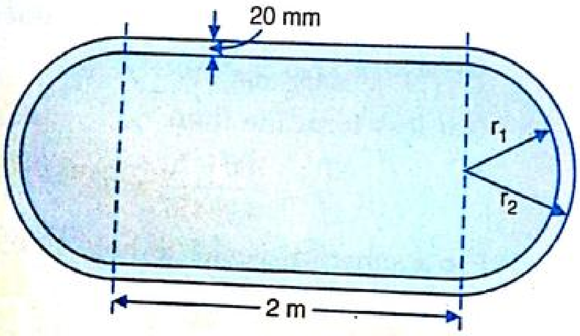


Fig. 3.50.

(*a*) Q_{cyl} = heat flow from the cylindrical section whose length $l = 2$ m

inner radius $r_1 = 0.5$ m and

outer radius $r_2 = 0.52$ m

R_{t1} = resistance to conduction through cylinder

$$= \frac{1}{2\pi kl}\log_e(r_2/r_1)$$

$$= \frac{1}{2\pi\times1.4\times2}\log_e(0.52/0.5)$$

$$= 2.229 \times 10^{-3}$$

and R_{t2} = resistance to convection from cylinder

$$= \frac{1}{h\,A_2} = \frac{1}{h\times2\pi r_2 l}$$

$$= \frac{1}{50\times2\pi\times0.52\times2}$$

$$= 3.062 \times 10^{-3}$$

Then:

$$Q_{cyl} = \frac{t_1 - t_2}{R_{t1} + R_{t2}}$$

$$= \frac{125-25}{(2.229+3.062)\times10^{-3}}$$

$$= 18900 \text{ W}$$

(*b*) Q_{sph} = heat flow from the two hemispherical ends or a sphere with $r_1 = 0.5$ m and $r_2 = 0.52$

R_{t3} = resistance to conduction through a sphere

$$= \frac{r_2 - r_1}{4\pi k r_1 r_2}$$

$$= \frac{0.52-0.5}{4\pi\times1.4\times0.5\times0.52}$$

$$= 4.375 \times 10^{-3}$$

and R_{t4} = resistance to convection through a sphere

$$= \frac{1}{hA_2} = \frac{1}{h\times4\pi r_2^2}$$

$$= \frac{1}{50\times4\pi\times0.52^2}$$

$$= 5.889 \times 10^{-3}$$

Then :

$$Q_{sph} = \frac{t_1 - t_2}{R_{t3} + R_{t4}}$$

$$= \frac{125-25}{(4.375+5.889)\times10^{-3}}$$

$$= 9742.8 \text{ W}$$

∴ Electric power that needs to be supplied to heater

$$= 18900 + 9742.8$$

$= 2862.8$ W ≃ **28.64 kW**

EXAMPLE 3.68

A cylindrical liquid oxygen tank has a diameter of 1.2 m, a length of 6 m and has hemispherical ends. The boiling point of liquid oxygen is –182°C and its heat of vaporisation is 214 kJ/kg. The tank needs to be insulated so as to reduce the boil off

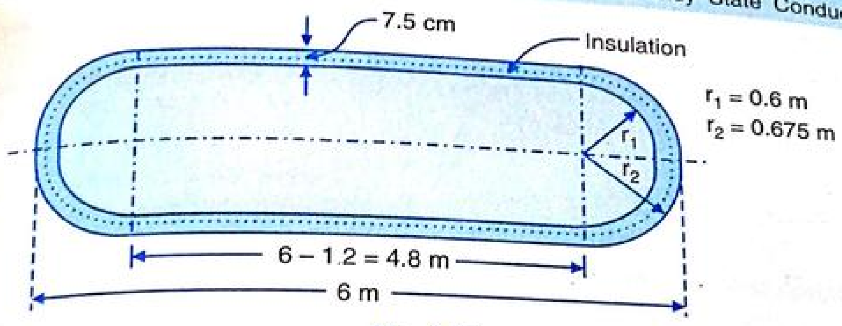


Fig. 3.51.

rate of oxygen in steady state to be no more than 13.85 kg/hr. Workout the thermal conductivity of the insulating material if its maximum thickness is limited to 7.5 cm and the room temperature outside the insulation is 20°C.

Solution : Heat generated during boiling of oxygen

$$Q_{boil} = 13.85 \times 214 = 2964 \text{ kJ/hr}$$
$$= \frac{2964 \times 10^3}{3600}$$
$$= 823.33 \text{ J/s} = 823.33 \text{ W}$$

The dissipation of heat generated during boiling comprises two parts :-

(a) Q_1 = heat flow from an insulated cylindrical tank whose

length l = 4.8 m

inner radius r_1 = 0.6 m and

outer diameter r_2 = 0.6 + 0.075 = 0.675 m

$$Q_1 = \frac{2\pi k l (t_1 - t_2)}{\log_e (r_2/r_1)}$$

(b) Q_2 = heat flow from sphere with r_1 = 0.6 m and r_2 = 0.675 m

$$Q_2 = \frac{4\pi k r_1 r_2 (t_1 - t_2)}{(r_2 - r_1)}$$

Now,

$$Q_{boil} = Q_1 + Q_2$$
$$= \frac{2\pi k l (t_1 - t_2)}{\log_e (r_2/r_1)} + \frac{4\pi k r_1 r_2 (t_1 - t_2)}{r_2 - r_1}$$
$$= k\left[\frac{l}{\log_e (r_2/r_1)} + \frac{2 r_1 r_2}{r_2 - r_1}\right] 2\pi (t_1 - t_2)$$

Sabstituting the given data,

$$823.23 = k\left[\frac{4.8}{\log_e (r_2/r_1)} + \frac{2 \times 0.6 \times 0.675}{0.675 - 0.6}\right] 2\pi[20 - (-182)]$$
$$= k[72.16 + 10.8] \times 2\pi \times 202$$
$$= 105240\, k$$

∴ Thermal conductivity of insulating material,

$$k = \frac{823.23}{105240} = \mathbf{0.0078 \text{ W/m-deg}}$$

EXAMPLE 3.69

A spherical tank of 3 m internal diameter and made of 2 cm thick stainless steel (k = 15 W/m-deg) is used to store ice water at 0°C. The tank loses heat to surroundings at 25°C by natural convection and radiation with a combined heat transfer coefficient of 15.5 W/m²-deg. If the convective coefficient at the inner surface of the tank is 80 W/m²-deg, determine :

(a) *the rate of heat transfer to the iced water in the tank, and*

(b) *the amount of ice that melts during a period of 24 hours (latent heat of ice = 334 kJ/kg)*

Solution : r_1 = 1.5 m

and r_2 = 1.5 + 0.02 = 1.52 m

The inner and outer surface areas of the tank are :

$$A_1 = 4\pi r_1^2$$
$$= 4\pi \times (1.5)^2 = 28.26 \text{ m}^2$$

$$A_2 = 4\pi r_2^2$$
$$= 4\pi \times (1.52)^2 = 29.02 \text{ m}^2$$

The individual thermal resistances are :

R_{t1} (convection at the inner surface)

$$= \frac{1}{h_1 A_1} = \frac{1}{80 \times 28.26}$$
$$= 4.42 \times 10^{-4} \text{ deg/W}$$

R_{t2} (conduction through surface)

$$= \frac{r_2 - r_1}{4\pi k r_1 r_2}$$
$$= \frac{1.52 - 1.5}{4\pi \times 15 \times 1.52 \times 1.5}$$
$$= 0.4666 \times 10^{-4} \text{ deg/W}$$

R_{t3} (convection at the outer surface)

$$= \frac{1}{h_2 A_2} = \frac{1}{15.5 \times 29.2}$$
$$= 22.23 \times 10^{-4} \text{ deg/W}$$

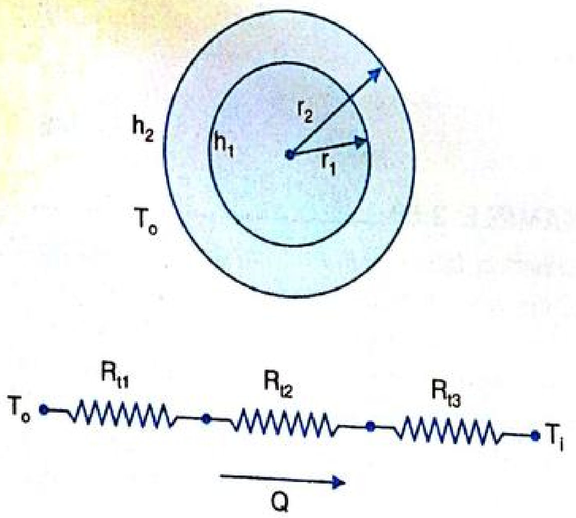


Fig. 3.52.

All the resistances are in series and the total resistance becomes

$$\Sigma R_t = R_{t1} + R_{t2} + R_{t3}$$
$$= (4.42 + 0.4666 + 22.23) \times 10^{-4}$$
$$= 27.116 \times 10^{-4} \text{ deg/W}$$

(*a*) Rate of heat transfer to iced water,

$$Q = \frac{T_o - T_i}{\Sigma R_t} = \frac{25 - 0}{27.116 \times 10^{-4}}$$
$$= \mathbf{9219.65\ W}$$

(*b*) Amount of heat transfer during 24-hours

$$Q = 9219.65 \times 24 \times 3600$$
$$= 7.96 \times 10^8 \text{ J}$$
$$= 7.96 \times 10^5 \text{ kJ}$$

It takes 334 kJ of energy to melt 1 kg of ice at 0°C, and therefore, the amount of ice that melts in a period of 24 hours

$$= \frac{7.96 \times 10^5}{334} = \mathbf{2383\ kg}$$

3.7. SHAPE FACTOR

Recapitulate the Fourier law for steady state heat conduction :

$$Q = -kA\frac{dt}{dx}$$
$$= -k\, dt\left(\frac{A}{dx}\right) \quad \text{...(3.41)}$$

Quite often all the factors relating to geometry of the section are grouped together into a single constant called the ***shape factor***. The shape factors for various sections are given below:

Section	Heat flow rate, Q	Shape factor, S
Plane Wall	$\frac{kA}{\delta}(t_1 - t_2)$	$\frac{A}{\delta}$
Cylinder	$\frac{2\pi k l (t_1 - t_2)}{\log_e (r_2 - r_1)}$	$\frac{2\pi l}{\log_e \frac{r_2}{r_1}}$
Sphere	$\frac{2\pi k r_1 r_2 (t_1 - t_2)}{r_2 - r_1}$	$\frac{4\pi r_1 r_2}{r_2 - r_1}$

Evidently for a plane wall, the shape factor S respresents the ratio of area to length of the heat transfer paths. The shape factor has units of length. The above expressions indicate that for a prescribed temperature difference $(t_1 - t_2)$, bodies with the same shape factor (irrespective of size and configuration) will allow heat conduction proportional to the material thermal conductivity.

The concept of shape factor is advantageously applied in computing heat flow through the walls, edges (connecting the walls) and corners (where the walls meet) of a furnace. For a furnace wall shown in Fig. 3.42, the shape factors for the plane surfaces A and B are $\frac{ab}{dx}$ and $\frac{bc}{dx}$ respectively. From experiments it has been found that shape factor:

For an edge,

$$S_{edge} = 0.54 \text{ times length of edge} \qquad ...(3.42)$$

For a corner,

$$S_{corner} = 0.15\, dx \qquad ...(3.43)$$

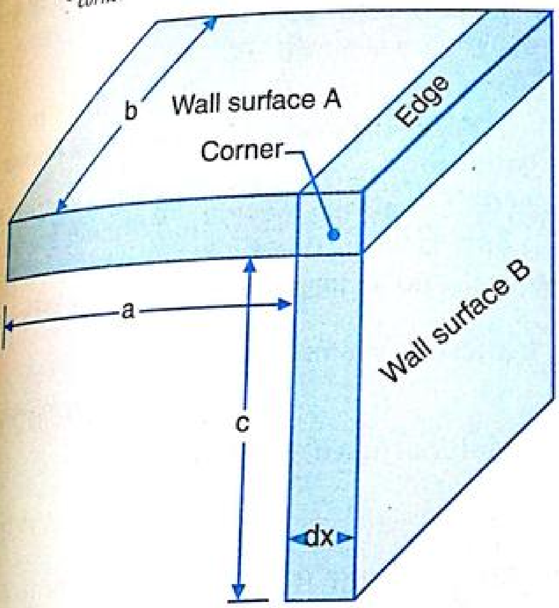


Fig. 3.53. Furnace wall indicating corner and on edge

A complete rectangular furnace has 6 walls, 12 edges and 8 corners. The shape factor for the complete furnace is

$$S_{total} = \frac{2}{dx}(ab + bc + ca) + 4 \times 0.54\,(a + b + c) + 8 \times 0.15\, dx \qquad ...(3.44)$$

where a, b c are the inside dimensions and dx is the wall thickness.

The above relation is true for a furnace which has all the interior dimensions greater than one-fifth of the wall thickness. In case the interior dimensions are less than one-fifth of the wall thickness, the following empirical relation has been suggested to workout the shape factor

$$S_{total} = 0.79\sqrt{A_{inside} \times A_{outside}} \qquad ...(3.45)$$

EXAMPLE 3.70

The annealing furnace for continuous bar stock is open at the ends and has interior dimensions of 0.6 m × 0.6 m × 1.5 m long with a wall 0.3 m thick all around. Calculate the shape factor for the furnace. Compare it with the shape factor of an equivalent furnace, i.e., a cylinder having 0.6 m inner diameter and 1.2 m outer diameter.

Solution : The annealing furnace with open ends has 4 walls and 4 edges.

Shape factor for the 4 walls

$$= 4 \times \frac{\text{area of wall}}{\text{thickness}}$$

$$= 4 \times \frac{1.5 \times 0.6}{0.3} = 12 \text{ m}$$

Shape factor for the 4 edges

$$= 4\,(0.54 \times \text{length of an edge})$$

$$= 4\,(0.54 \times 1.5) = 3.24 \text{ m}$$

$$\therefore \quad S_{total} = 12 + 3.24 = \mathbf{15.24 \text{ m}}$$

(*b*) Shape factor for the equivalent cylinder is equal to

$$\frac{2\pi l}{\log_e \frac{r_2}{r_1}} = \frac{2\pi \times 1.5}{\log_e \frac{0.6}{0.3}} = \mathbf{13.59 \text{ m}}$$

EXAMPLE 3.71

A furnace having inside dimensions

0.8 × 1.0 × 1.5 m

has brick walls 30 cm thick. The temperature drop across the walls is 250°C and thermal conductivity of the refractory brick is 5.25 kJ/m-hr-deg. Workout shape factor for the furnace and determine the hourly heat loss by conduction. What saving in heat loss would occur if the rectangular furnace is replaced by a spherical furnace which has the same inside capacity and walls of same thickness?

Solution : The total shape factor for a complete rectangular furnace consisting of 6 walls, 12 edge and 8 corners will be :

$$S_{total} = \frac{2}{\delta}(ab + bc + ca) + 4 \times 0.54\,(a + b + c) + 8 \times 0.15\,\delta$$

where a, b, c are the inside dimensions and δ is the wall thickness.

$$S_{total} = \frac{2}{0.3}[0.8 \times 1.0 + 1.0 \times 1.5 + 1.5 \times 0.8] + 4 \times 0.54\,(0.8 + 1.0 + 1.5) + 8 \times 0.15 \times 0.3$$

$$= 23.33 + 7.13 + 0.36$$

$$= 30.82 \text{ m}$$

Heat loss Q_1

$$= S\,k\,(\Delta T)$$

$$= 30.82 \times 5.25 \times 250$$

$$= 40451.25 \text{ kJ/hr}$$

(*b*) For a spherical furnace of the same internal capacity

$$\frac{4}{3}\pi r_1^3 = 0.8 \times 1.0 \times 1.5\,;$$

$$r_1 = 0.659\,;$$

$$r_2 = 0.659 + 0.3 = 0.959 \text{ m}$$

Shape factor for the spherical furnace is equal to

$$\frac{4\pi r_1 r_2}{r_2 - r_1} = \frac{4\pi \times 0.659 \times 0.959}{(0.959 - 0.695)} = 26.459 \text{ m}$$

and heat loss

$$Q_2 = 26.459 \times 5.25 \times 250$$

$$= 34727.45 \text{ kJ/hr}$$

Saving in heat loss

$$= \frac{40451.25 - 34727.45}{40451.25}$$

$$= 0.1415 \text{ or } 14.15\%$$

EXAMPLE 3.72

Compare the heat transfer characteristics of three furnaces having a cubical, spherical and cylindrical shape. Each of the furnace has an internal volume of 1 m^3 and uses only 1 m^3 of refractory material in its fabrication.

Solution : Cubical Furnace : Let the furnace have dimensions a, b, c and thickness δ

Inside volume of furnace

$$= 1 \text{ m}^3 \qquad \therefore a = b = c = 1 \text{ m}$$

Material used in fabrication

$$(1 + 2\delta)^3 - 1 = 1 \qquad \therefore \delta = 0.13 \text{ m}$$

Shape factor for 6 faces

$$= \frac{2}{\delta}(ab + bc + ca)$$

$$= \frac{2}{0.13}(1 + 1 + 1) = 46.15 \text{ m}$$

Shape factor for 12 edges

$$= 4\,(a + b + c) \times 0.54$$

$$= 4\,(1 + 1 + 1) \times 0.54 = 6.48 \text{ m}$$

Shape factor for 8 corners

$$= 8 \times 0.15\delta$$

$$= 8 \times 0.15 \times 0.13 = 0.156 \text{ m}$$

$\therefore$ Total shape factor for the cubical furnace is equal to

$$46.15 + 6.48 + 0.156 = \mathbf{52.786 \text{ m}}$$

Spherical Furnace : Let r_1 and r_2 be the internal and external radius respectively

Internal Volume $\frac{4}{3}\pi r_1^3 = 1$

$$\therefore r_1 = 0.62 \text{ m}$$

Material used in fabrication,

$$\frac{4}{3}\pi\left(r_2^3 - r_1^3\right) = 1 \qquad \therefore r_2 = 0.783 \text{ m}$$

Shape factor for the spherical furnace is equal to

$$\frac{4\pi r_1 r_2}{r_2 - r_1} = \frac{4\pi \times 0.62 \times 0.783}{0.783 - 0.62} = \mathbf{37.50 \text{ m}}$$

Cylindrical Furnace : Let the internal and external radius be r_1 and r_2 respectively and the length be unity.

Internal volume $\pi r_1^2 \times 1 = 1$

$$\therefore r_1 = 0.565 \text{ m}$$

Material used in fabrication,

$$\pi\left(r_2^2 - r_1^2\right)l + \pi r_2^2\left(r_2 - r_1\right) = 1$$

$$\pi\left(r_2^2 - 0.565^2\right) \times 1 + \pi r_2^2\left(r_2 - 0.565\right) = 1$$

By hit and trial, external radius

$r_2 = 0.705$ m

Shape factor for the cylindrical portion is equal to

$$\frac{2\pi l}{\log_e \frac{r_2}{r_1}} = \frac{2\pi \times 1}{\log_e \frac{0.705}{0.565}}$$

$= 5.033$ m

Shape factor for the two flat ends,

$$= \frac{2\pi r_1^2}{r_2 - r_1} \quad \left(\frac{\text{inside area}}{\text{thickness}}\right)$$

$$= \frac{2\pi \times 0.565^2}{0.705 - 0.565} = 14.319 \text{ m}$$

Shape factor for 2 edges

$= 2$ (0.54 × length of the edge)

$= 2\ (0.54 \times 2\pi r_1)$

$= 20\ (0.54 \times 2\pi \times 0.565)$

$= 3.832$ m

∴ Shape factor for the cylindrical furnace is equal to

$5.033 + 14.319 + 3.832 = $ **23.184 m**

Comment : For the same material and same temperature difference, the heat flow is proportional to shape factor ($Q = Sk\Delta T$).

∴ Heat flow from

cubical furnace	:	spherical furnace	:	cylindrical furnace
52.786	:	37.50	:	23.184
1	:	0.710	:	0.439

Obviously for the same amount of fabrication material and same iniside capacity, the heat loss is lowest in a cylindrical furnace. This is precisely the reason that cylindrical containers are preferred for storing liquified gases.

3.8. EFFECT OF VARIABLE CONDUCTIVITY

For most materials, the dependence of thermal conductivity on temperature is almost linear

$$k = k_o\,(1 + \beta t) \qquad \text{...(3.46)}$$

where k_0 is the thermal conductivity at zero degree centigrade temperature and β is a constant whose value depends upon the material. This constant may be positive or negative depending upon whether thermal conductivity increases or decreases with temperature. The coefficient β is usually positive for non-metals and insulation materials (exceptions are magnesite bricks) and negative for metallic conductors (exceptions are aluminium and certain non-ferrous alloys).

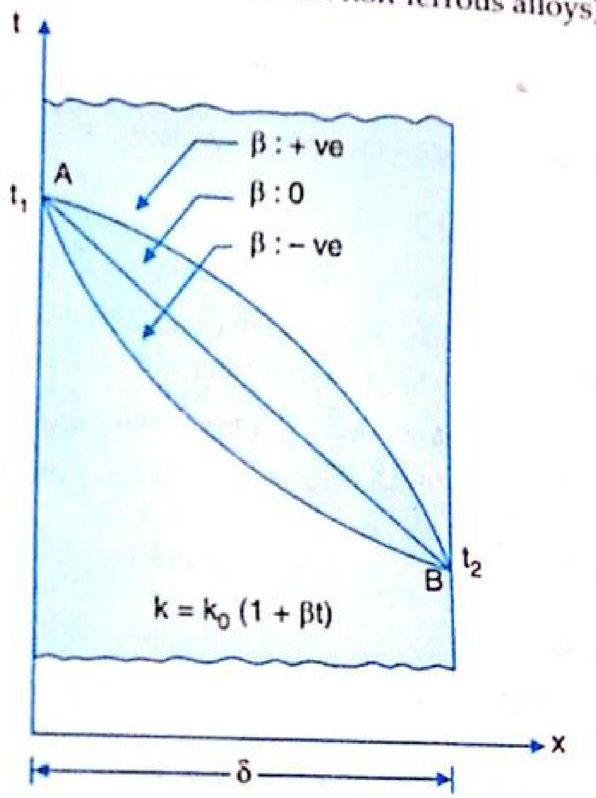


Fig. 3.54. Temperature profile with variable thermal conductivity

With variable thermal conductivity, Fourier law of heat conduction through a plane wall can be expressed as

$$Q = -\,k_o\,(1 + \beta t)\,A\frac{dt}{dx} \qquad \text{...(3.47)}$$

Separating the variables and integrating within the limits $t = t_1$ at $x = 0$ and $t = t_2$ at $x = \delta$

$$Q\int_0^{\delta} dx = -\,k_0 A \int_{t_1}^{t_2} (1 + \beta t)\,dt$$

$$Q\,\delta = -\,k_0 A\left[\left(t_2 + \frac{\beta}{2}t_2^2\right) - \left(t_1 + \frac{\beta}{2}t_1^2\right)\right]$$

$$= k_0 A (t_1 - t_2)\left\{1 + \frac{\beta}{2}(t_1 + t_2)\right\}$$

Thus, the heat flow rate,

$$Q = \frac{k_m A}{\delta}(t_1 - t_2) \quad ...(3.48)$$

The quantity $k_m = k_0\left(1 + \frac{\beta}{2}(t_1 + t_2)\right)$ represents the mean thermal conductivity evaluated at the arithmetic mean temperatures $\frac{1}{2}(t_1 + t_2)$.

Temperature variation: Fourier law of heat induction through a plane wall is

$$Q = -kA\frac{dt}{dx} \; ; \quad \frac{Q}{A} = -\left(k\frac{dt}{dx}\right) \quad ...(3.49)$$

For steady state heat conduction in the infinite wall, Q/A is constant and so has to be the parameter $\left(k\frac{dt}{dx}\right)$. Obviously then, the temperature variation in the wall is governed by the condition that parameter $\left(k\frac{dt}{dx}\right)$ is constant.

Case I $\beta = 0$ **:** We have,

$$k = k_0 (1 + \beta t) = k_0 = \text{constant}$$

The thermal conductivity does not vary with temperature and equals the constant value k_0. Accordingly for the parameter $\left(k\frac{dt}{dx}\right)$ to be constant, the term $\frac{dt}{dx}$ must be constant. Therefore, the slope of temperature curve is constant and the temperature profile is linear.

Case II $\beta > 0$ **:** We have,

$$k = k_0(1 + \beta t) \; ; \; i.e., \; k_0 \propto t$$

The thermal conductivity of the wall material is directly proportional to temperature; k increases with increasing temperature or decreases with decreasing temperature. Since the temperature decreases in x-direction, the thermal conductivity would also decrease. Accordingly to maintain the parameter $\left(k\frac{dt}{dx}\right)$ constant, the term $\frac{dt}{dx}$ must increase. Consequently the value of slope increases from point A and B and that means that the curve should go steeper from A to B. Evidently with positive value of β, the temperature variation curve is of convex nature.

Case III $\beta < 0$ **:** We have,

$$k = k_0 (1 - \beta t), \; i.e., \; k_0 \propto \frac{1}{t}$$

The thermal conductivity of the wall material is inversely proportional to temperature; k decreases with increasing temperature or increases with decreasing temperature. Since the temperature decreases in x-direction, the thermal conductivity would increase. In order to maintain the parameter $\left(k\frac{dt}{dx}\right)$ constant, the term $\frac{dt}{dx}$ must decrease. Consequently the value of the slope of temperature profile decreases. Evidently the temperature profile will be concave for negative value of coefficient β.

EXAMPLE 3.73

The inner and outer surfaces of a furnace wall, 25 cm thick, are at 300°C and 30°C respectively. The thermal conductivity of the wall material varies with temperature and is prescribed by the relation

$$k = (1.45 + 0.5 \times 10^{-5} t^2) \text{ kJ/m-hr-deg}$$

where t is the temperature in degree centigrade. Proceed from the basic principles to calculate the heat loss per square metre of the wall surface area.

Solution : Invoking Fourier's law of heat conduction,

$$Q = -kA\frac{dt}{dx}$$

$$Q\,dx = -kA\,dt$$

$$= -(1.45 + 0.5 \times 10^{-5} t^2)A\,dt$$

Integrating over the wall thickness δ,

$$Q\,\delta = -\left\{1.45\,(t_2 - t_1) + \frac{0.5\times 10^{-5}}{3}\left(t_2^3 - t_1^3\right)\right\}A$$

$$= \left\{1.45\,(t_1 - t_2) + \frac{0.5\times 10^{-5}}{3}\left(t_1^3 - t_1^3\right)\right\}A$$

Inserting the appropriate values,

$$Q \times 0.25$$

$$= \left\{1.45(300 - 30) + \frac{0.5\times 10^{-5}}{3}(300^3 - 30^3)\right\}\times 1$$

$$= 436.45$$

∴ Heat loss from the furnace,

$$Q = \frac{436.45}{0.25} = \mathbf{1745.8\ kJ/m^2\text{-}hr}$$

EXAMPLE 3.74

A plane wall of thickness δ has its surfaces maintained at temperatures T_1 and T_2. The wall is made of a material whose thermal conductivity varies with temperature according to the relation $k = k_0 T^2$. Set up an expression to workout the steady state heat conduction through the wall. Further proceed to calculate the temperature at which mean thermal conductivity be evaluated so as to get the same heat flow by its substitution in the simplified Fourier relation.

Solution: Invoking Fourier's law of heat conduction.

$$Q = -kA\frac{dt}{dx} = k_0 T^2 A\frac{dt}{dx}$$

Separating the variables and integrating within the prescribed boundary conditions, we obtain :

$$Q\int_0^{\delta} dx = -Ak_0\int_{T_1}^{T_2} T^2 dx$$

or $$Q\,\delta = -\frac{Ak_0}{3}\left(T_2^3 - T_1^3\right)$$

∴ $$Q = \frac{Ak_0}{3\delta}\left(T_1^3 - T_2^3\right)$$

which is the required relation

If this heat flow is to be obtained by substituting an average value of thermal conductivity in the simplified Fourier relation, we have

$$\frac{Ak_0}{3\delta}\left(T_1^3 - T_2^3\right) = \frac{k_m A(T_1 - T_2)}{\delta}$$

$$= \frac{k_0 T_m^2 A(T_1 - T_2)}{\delta}$$

Upon simplification, the desired temperature works out as

$$T_m = \sqrt{\frac{T_1^2 + T_2^2 + T_1 T_2}{3}}$$

EXAMPLE 3.75

The temperatures on the two sides of a plane wall are t_1 and t_2, and thermal conductivity of the wall material is prescribed by the relation

$$k = k_0\, e^{(-x/\delta)}$$

where k_0 is constant and δ is the wall thickness. Set up an expression for temperature distribution in the wall.

Solution : From Fourier's law of heat conduction,

$$Q = -\,kA\frac{dt}{dx} = -\,k_0\,A\,e^{(-\,x/\delta)}\,\frac{dt}{dx}$$

Separating the variables and upon integration

$$\frac{Q}{k_0 A}\int e^{(x/\delta)}\,dx = -\int dt$$

The boundary conditions are :

(*i*) At $x = 0$; $t = t_1$ and

(*ii*) At $x = \delta$, $t = t_2$

$$\therefore\ \frac{Q}{k_0 A}\int_0^{\delta} e^{(x/\delta)}\,dx = -\int_{t_1}^{t_2} dt$$

or $$\frac{Q}{k_0 A}\left|\delta\, e^{x/\delta}\right|_0^{\delta} = -\left|\,t\,\right|_{t_1}^{t_2}$$

or $$\frac{Q}{k_0 A}\,\delta(e - 1) = (t_1 - t_2)$$

∴ Heat transfer through the wall,

$$Q = \frac{k_0 A(t_1 - t_2)}{\delta(e-1)} \qquad \text{...}(i)$$

Let the boundary condition (ii) be generalized as

At $x = x$; $t = t_x$

The expression for heat transfer through the wall then can be written as

$$Q = \frac{k_0 A(t_1 - t_x)}{\delta(e-1)} \qquad ...(ii)$$

From identities (i) and (ii), the required expression for temperature distribution in the wall is

$$\frac{t_1 - t_x}{t_1 - t_2} = \frac{x}{\delta}$$

EXAMPLE 3.76

An infinite slab of 50 mm thickness and 0.1 m^2 cross-sectional area has its sides maintained at temperatures of 300°C and 30°C respectively. Measurements indicate that when 1 kW of energy heat flows through it, the temperature at its centre plane is 125°C. Set up an expression for thermal conductivity of the slab material if it varies linearly with temperature.

Solution : Consider a slab of thickness δ and temperatures at the boundary surfaces t_1 and t_2.

From Fourier's law of heat conduction,

$$Q = -kA\frac{dt}{dx}$$

Assuming linear relationship between k and t as $k = mt + c$, we can write

$$Q = -(mt + c)\,A\,\frac{dt}{dx}$$

Separating the variables and integrating between the prescribed limits, we have

$$\frac{Q}{A}\int_0^\delta dx = -\int_{t_1}^{t_2}(mt + c)\,dt$$

or $$\frac{Q}{A}\delta = \frac{m}{2}(t_1^2 - t_2^2) + c(t_1 - t_2).$$

or $$Q = \frac{1}{\delta}\left[\frac{m}{2}(t_1^2 - t_2^2) + c(t_1 - t_2)\right]$$

For left half part of slab :

$t_1 = 300°C$,

$t_2 = 125°C$ and $\delta = 25$ mm

Then :

$$1000 = \frac{0.1}{0.025}\left[\frac{m}{2}(300^2 - 125^2) + c(300 - 125)\right]$$

or $250 = 37187.5\,m + 175\,c$ ...(i)

For right half part of slab :

$t_1 = 125°C$,

$t_2 = 30°C$ and $\delta = 25$ mm

Then :

$$1000 = \frac{0.1}{0.025}\left[\frac{m}{2}(125^2 - 30^2) + C(125 - 30)\right]$$

or $250 = 15175\,m + 95\,c$ ...(ii)

Simultaneous solution of identities (i) and (ii) gives

$m = -0.0228$ and $c = 6.273$

The expression for thermal conductivity may then be written as

$$k = -0.022\,t + 6.273$$

EXAMPLE 3.77

Derive an expression for the heat loss per m^2 of the surface area for a furnace wall of thickness δ when the thermal conductivity varies with temperature according to the relation:

$k = (a + b\,t^2)$ *W/m-deg where t is in °C*

Proceed to calculate the rate of heat transfer through the wall if δ = 0.25 m, $t_1 = 250°C$, $t_2 = 25°C$, and a = 0.3 and b = 5 × 10^{-6}.

Solution : Invoking Fourier's law of heat conduction

$$q = \frac{Q}{A} = -k\frac{dt}{dx} = -(a + b\,t^2)\frac{dt}{dx}$$

Separating the variables and integrating within the prescribed boundary conditions we get

$$q\int_0^\delta dx = -\int_{t_1}^{t_2}(a+bt^2)\,dt$$

or $$q\,\delta = -\left[a\,(t_2-t_1)+\frac{b}{3}(t_2^3-t_1^3)\right]$$

$$= a\,(t_1-t_2) + \frac{b}{3}(t_1^3-t_2^3)$$

$$= (t_1-t_2)\left[a+\frac{b}{3}(t_1^2+t_2^2+t_1t_2)\right]$$

$$\therefore \quad q = \left[\frac{t_1-t_2}{\delta}\right]\left[a+\frac{b}{3}(t_1^2+t_2^2+t_1t_2)\right]$$

which is the required expression.

Substituting the given data, the rate of heat transfer through the wall works out as,

$$q = \frac{250-25}{0.25}\left[0.3+\frac{5\times10^{-6}}{3}(250^2+25^2+250\times25)\right]$$

$$= 900\left[0.3+\frac{5\times10^{-6}}{3}(62500+625+6250)\right]$$

$$= 374.06 \text{ W}$$

EXAMPLE 3.78

A plane brick wall 5 m long × 3 m high × 250 mm thick has temperatures of 800°C and 20°C maintained on its bounding surfaces. Presuming that thermal conductivity of the brick material is related to its temperature in degree celsius by the relation

$$k = (1 + 0.001\,t) \text{ W/m-deg}$$

make calculations for the average thermal conductivity, thermal resistance and heat loss from the wall. What would be the temperature at 100 mm distance from the wall surface at 800 °C ?

Solution : As the dependence of thermal conductivity on temperature is linear, the rate of heat flow can be calculated from the formula for the constant thermal conductivity taken at mean wall temperature.

Average thermal conductivity,

$$k = \left[1+0.001\left(\frac{800+20}{2}\right)\right]$$

$$= 1.41 \text{ W/m-deg}$$

Thermal resistance R_t

$$= \frac{\delta}{kA} = \frac{0.25}{1.41\times5\times3}$$

$$= 0.01182 \text{ deg/W}$$

Heat loss Q

$$= \frac{\Delta t}{R_t} = \frac{800-20}{0.01182} = 6.6\times10^4 \text{ W}$$

(*b*) Let t be the temperature at 100 mm distance from the wall surface at 800°C. For this section of 100 mm thickness :

$$k = \left[1+0.001\left(\frac{800+t}{2}\right)\right]$$

$$= 1.4 + 0.0005t$$

$$R_t = \frac{0.1}{(1.4+0.0005t)\times(5\times3)}$$

$$Q = \frac{(800-t)\times(1.4+0.0005t)\times15}{0.1}$$

For steady state heat conduction, heat passing through each section of the wall is same.

$$\therefore\ 6.60\times10^4 = \frac{(800-t)\times(1.4+0.0005t)\times15}{0.1}$$

$$0.0005\,t^2 + t - 680 = 0$$

Solution of this quadratic equation gives:

$$t = 536.23°\text{C}$$

EXAMPLE 3.79

A furnace is constructed with 10 cm of fire clay and 50 cm of red brick. The inside surface temperature is 1200°C and the outside air temperature is 50°C. Determine the heat loss from 1 m² of the furnace wall and the temperature at the layer interface. Thermal conductivities for the wall material are :

Fire clay :

$$k = 0.28\,(1 + 0.000893\,t) \text{ W/m-deg}$$

Red brick :

$$k = 0.75 \text{ W/m-deg}$$

Solution : Let t_i denote the temperature at the layer interface.

Average thermal conductivity of fire clay

$$= 0.28\left[1+0.000893\left(\frac{1200+t_i}{2}\right)\right]$$

$$= 0.28\,[1+0.0004465\,(1200+t_i)]$$

For a plane wall thermal resistance is δ/kA

$\therefore$ Thermal resistance of the clay

$$= \frac{0.1}{0.28[1+0.0004465(1200+t_i)]\times 1}$$

$$= \frac{1}{2.8+0.00125(1200+t_i)}$$

Thermal resistance of red brick

$$= \frac{0.5}{0.75\times 1} = 0.667$$

Total resistance

$$\Sigma R_t = \frac{1}{2.8+0.00125(1200+t_i)} + 0.667$$

$$\text{Heat loss from the wall} = \frac{\Delta T}{\Sigma R_t}$$

$$Q = \frac{(1200-50)}{1/\{2.8+0.00125(1200+t_i)\}+0.667}$$

Under steady state conditions, the same amount of heat flows through each layer. Then considering heat flow through the red brick

$$Q = \frac{(t_i-50)}{0.667}$$

Equating the two expressions for heat loss,

$$\frac{1150}{1/\{2.8+0.00125(1200+t_i)\}+0.667} = \frac{t_i-50}{0.667}$$

or $$\frac{1150[2.8+0.00125(1200+t_i)]}{2.8675+0.000833(1200+t_i)} = \frac{t_i-50}{0.667}$$

or $$\frac{1150(0.000125t_i+4.3)}{0.000833t_i+3.867} = \frac{t_i-50}{0.667}$$

or $0.000833t_i^2 + 2.86t_i - 3491.6 = 0$

Solution of this quadratic equation gives,

Temperature at the layer interface t_i

$= 954.38°C$

Heat loss from wall

$$= \frac{954.38-50}{0.667} = 1355.89 \text{ W}$$

EXAMPLE 3.80

A furnace wall comprises two layers :

15 cm thick fire brick with

$$k = (0.25 + 0.00025\,t)\ \text{W/m-deg}$$

where t is in °C.

50 cm thick red brick with k = 0.8 W/m-deg

The inside surface temperature of fire brick is 1200°C and the outside of red brick work is at 50°C. At the contact surface, the temperature drops by 20°C due to uneven contact. Work out the heat loss per unit area of the furnace wall and temperature of the red brick surface which is in contact with the fire brick.

Solution : Under steady state conditions and considering unit area of the composite wall

$$q = \frac{t_1-t_2}{\frac{\delta_1}{k_1}} = \frac{t_3-t_4}{\frac{\delta_2}{k_2}}$$

$$= \frac{(t_2-20)-t_4}{\frac{\delta_2}{k_2}} \quad \text{...(i)}$$

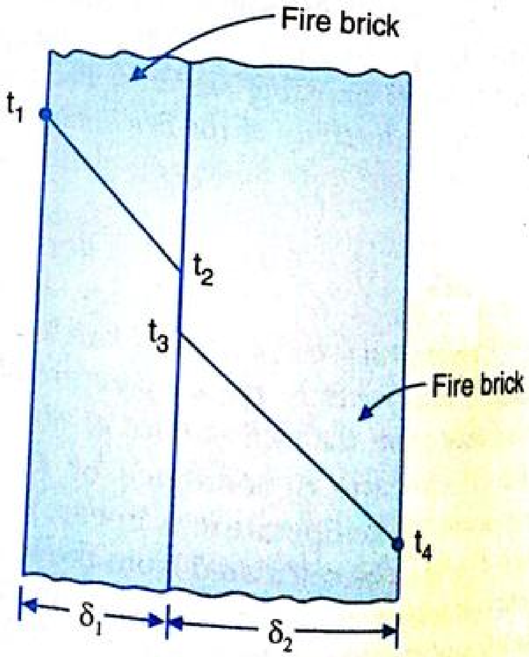


Fig. 3.55.

Given : $t_1 = 1200°C$; $t_4 = 50°C$

$\delta_1 = 0.15$ m; $\delta_2 = 0.5$ m

$$k_1 = k_m = 0.25 + 0.00025\left(\frac{t_1+t_2}{2}\right)$$

$$= 0.25 + 0.00025\left(\frac{1200+t_2}{2}\right)$$

$$k_2 = 0.8 \text{ W/m-deg}$$

Upon substitution of given data in expression (*i*), we get

$$\left[\frac{0.25+0.00025\left(\frac{1200+t_2}{2}\right)\times(1200-t_2)}{0.15}\right]$$

$$= 0.8\left[\frac{(t_2-20)-50}{0.5}\right]$$

or $$\left[0.25+0.00025\left(\frac{1200+t_2}{2}\right)\right]\times(1200-t_2)$$

$$= 0.15 \times \frac{8}{5}\ (t_2 - 70)$$

Simplification gives

$$0.000125 + t_2^2 + 0.49\ t_2 - 496.8 = 0$$

$$t_2 = \frac{-0.49+\sqrt{(0.49)^2-4\times0.000125\times(-496.8)}}{2\times0.000125}$$

$$= \frac{-0.49+0.699}{0.00025} = 836°C$$

Then $t_3 = 83.6 - 20 = 816°C$

Heat loss per unit area of wall,

$$q = \frac{t_3-t_4}{\delta_2/k_2} = \frac{816-50}{0.5/0.8}$$

$$= \mathbf{1225.6\ W/m^2}$$

EXAMPLE 3.81.

An infinite composite slab is made of two layers of materials A and B. The left side layer is 10 cm thick of material with thermal conductivity $k_a = 0.3\ (1 + 0.007t)$, and the right side is 5 cm thick of material B having thermal conductivity $k_b = 0.4\ (1 + 0.001t)$. In these expressions, t denotes the temperature in °C. If the surface temperatures on the left and right side of the wall are 500°C and 25°C respectively, workout the steady state heat flux through the wall.

Solution : Refer Fig. 3.56. for the composite slab. The heat flow through an infinite wall made of a material of thermal conductivity prescribed by the relation

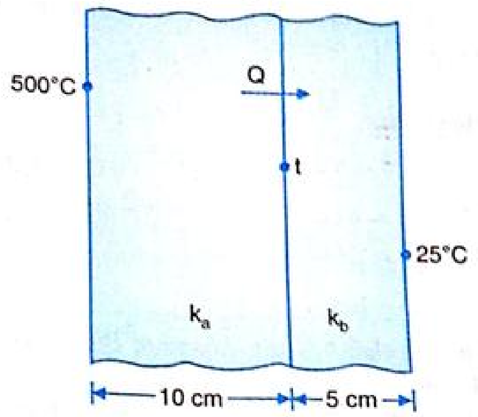


Fig. 3.56.

$k = k_0\ (1 + \beta t)$ is given by

$$Q = k_m \frac{A}{\delta}(t_1 - t_2)$$

$$= \frac{k_0 A}{\delta}\left[1+\frac{\beta}{2}(t_1+t_2)\right]\ (t_1 - t_2)$$

For slab on left :

$\delta_1 = 0.1$ m ; $\beta = 0.007$;

$k_0 = 0.3$; $t_1 = 500°C$

and $t_2 = t$ (temperature at the interface)

Then :

$$\frac{Q}{A} = \frac{0.3}{0.1}\left[1+\frac{0.007}{2}(500+t)\right]\ (500-t)$$

$$= 3[1 + 0.0035\ (500 + t)]\ (500 - t)$$

$$= 3[(500 - t) + 0.0035\ (500^2 - t^2)]$$

$$= 4125 - 3t - 0.0105\ t^2 \quad ...(i)$$

For slab on right :

$\delta_2 = 0.05$ m ; $\beta = 0.001$; $k_0 = 0.4$

$t_1 = t$ (temperature at the interface)

and $t_2 = 25°C$

Then :

$$\frac{Q}{A} = \frac{0.4}{0.05}\left[1+\frac{0.001}{2}(t+25)\right]\ (t - 25)$$

$$= 8t + 0.04t^2 - 202.5 \quad ...(ii)$$

From identities (*i*) and (*ii*)

$$4125 - 3t - 0.0105t^2 = 8t + 0.004t^2 - 202.5$$

or $0.0145\ t^2 + 11t - 4327.5 = 0$

or $t^2 + 758.6t - 298448 = 0$

Solution of this quadratic equation gives :

$$t = 285.7°C$$

Substituting this value of t in expression (*ii*) we get

$$\text{Heat flux, } \frac{Q}{A} = 4125 - 3 \times 285.7 - 0.0105\,(285.7)^2$$

$$= 4125 - 857.1 - 857.06$$

$$= 2410.84 \text{ W/m}^2$$

EXAMPLE 3.82

A composite slab has two layers of thickness 5 cm and 10 cm. The thermal conductivities of the materials of these layers are temperature dependent and are prescribed by the relations

$$k_1 = 0.05\,(1 + 0.006\,t) \text{ W/m-deg}$$

$$k_2 = 0.04\,(1 + 0.007\,t) \text{ W/m-deg}$$

where t is the temperature in degrees centigrade. The inside and outside surface temperatures of the slab are maintained at 500°C and 200°C. Determine steady slate heat flux through the composite slab and the interface temperature.

Solution : Mean thermal conductivity of layer 1,

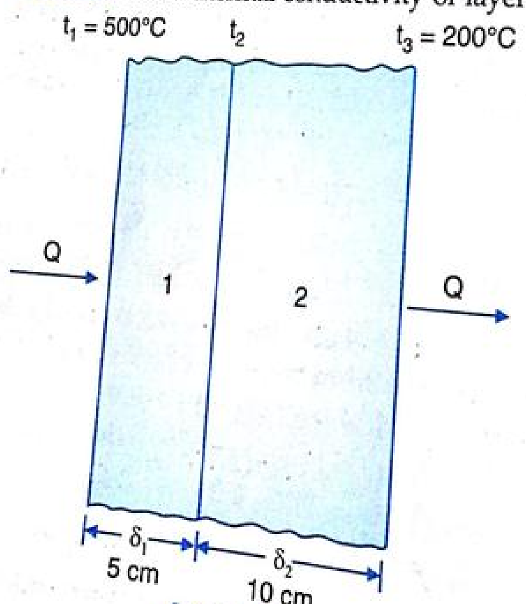


Fig. 3.57.

$$k_1 = 0.05\left[1 + 0.006\left(\frac{t_1 + t_2}{2}\right)\right]$$

$$= 0.05\left[1 + 0.006\left(\frac{500 + t_2}{2}\right)\right]$$

Likewise,

$$k_2 = 0.04\left[1 + 0.007\left(\frac{t_2 + 200}{2}\right)\right]$$

Under steady conditions, heat flows through each layer is the same. Thus

$$q = \frac{Q}{A} = \frac{t_1 - t_2}{\delta_1 / k_1} = \frac{t_2 - t_3}{\delta_2 / k_2}$$

$$\text{or} \quad \frac{k_1\,(t_1 - t_2)}{\delta_1} = \frac{k_2\,(t_2 - t_3)}{\delta_2}$$

Substituting the given values in the above identity, we obtain

$$\frac{0.05\left[1 + 0.006\left(\frac{500 + t_2}{2}\right)\right](500 - t_2)}{0.05}$$

$$= \frac{0.04\left[1 + 0.007\left(\frac{t_2 + 200}{2}\right)\right](t_2 - 200)}{0.1}$$

$$\text{or } (500 - t_2)\left[1 + 0.006\left(\frac{500 + t_2}{2}\right)\right]$$

$$= 0.4\,(t_2 - 200)\left[1 + 0.007\left(\frac{t_2 + 200}{2}\right)\right]$$

$$\text{or } (500 - t_2)\left(\frac{5 + 0.006 t_2}{2}\right)$$

$$= 0.4\,(t_2 - 200)\left(\frac{3.4 + 0.007 t_2}{2}\right)$$

$$\text{or } (500 - t_2)\,(5 + 0.006\,t_2)$$

$$= (t_2 - 200)\,(1.36 + 0.0028\,t_2)$$

$$\text{or } 2500 + 3t_2 - 5t_2 - 0.006 t_2^2$$

$$= 1.36\,t_2 + 0.0028 t_2^2 - 272 - 0.56 t_2$$

$$\text{or } 0.0088 t_2^2 + 2.8\,t_2 - 2772 = 0$$

Solution of this quadratic equation gives

$$t_2 = \frac{-2.8 \pm \sqrt{(2.8)^2 - 4 \times 0.0088(-2772)}}{2 \times 0.0088}$$

$$= 424.27°C$$

$$\therefore k_1 = 0.05\left[1 + 0.006\left(\frac{500 + 424.27}{2}\right)\right]$$

$$= 0.1886 \text{ W/m-deg}$$

Heat flux q

$$= \frac{k_1(t_1 - t_2)}{\delta_1}$$

$$= \frac{0.1886(500 - 424.27)}{0.05}$$

$$= 285.65 \text{ W/m}^2$$

EXAMPLE 3.83

Pressurized water is to be carried through a pipe imbedded in a 1.25 m thick wall whose surfaces are held at constant temperatures of 215°C and 65°C respectively. How far from the hot surface should the pipe be imbedded if it is desired to locate the pipe in the wall where the temperature is 125°C ? The thermal conductivity of the wall material varies with temperature according to the relation :

$$k = 0.3\ (1 + 0.0375\ t)$$

where t is in degrees celsius and k is in W/mK.

(b) How this location would change if the variation of thermal conductivity with temperature is ignored, i.e., the conductivity is assumed to remain constant at k = 0.3 W/mK.

Solution : The heat transfer through a plane wall of variable thermal conductivity is prescribed by the relation :

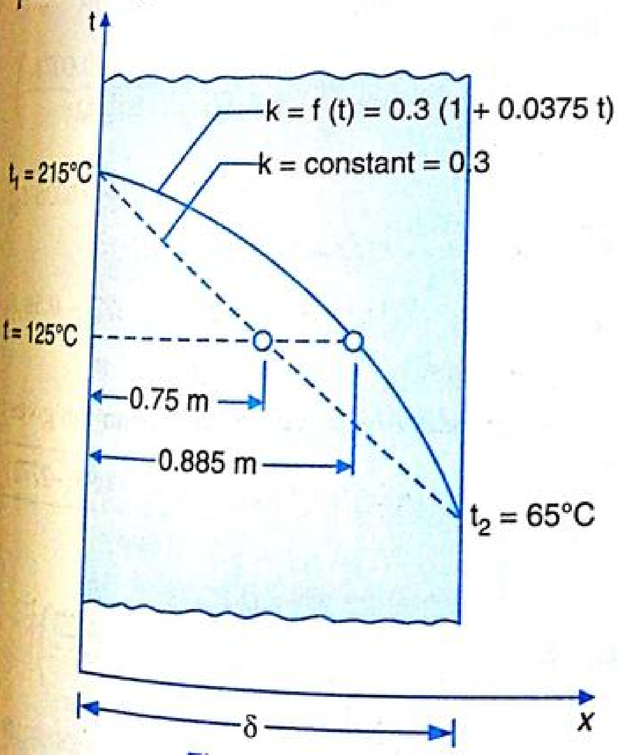


Fig. 3.58.

$$Q = k_m \frac{A}{\delta}(t_1 - t_2)$$

$$= k_0\left[1 + \frac{\beta}{2}(t_1 + t_2)\right]\frac{A}{\delta}(t_1 - t_2)$$

Two situations arise :

(*i*) When t_2 = 65°C, δ = 1.25 m

(*ii*) When t_2 = 125°C, δ = x (unknown)

Inserting the proper values and equating the two expressions for heat transfer through the wall

$$0.3\left[1 + \frac{0.0375}{2}(215 + 65)\right]\frac{A}{1.25}(215 - 65)$$

$$= 0.3\left[1 + \frac{0.0375}{2}(215 + 125)\right]\frac{A}{x}(215 - 125) \quad \text{...}(i)$$

$$225\ A = \frac{199.125A}{x}\ ;$$

$$x = \frac{199.125}{225} = 0.885 \text{ m}$$

Thus the pipe need to be imbedded 0.885 m from the hot wall surface.

(*b*) When dependence of thermal conductivity on temperature is ignored; thermal conductivity takes a constant value of 0.3 W/mK. With that stipulation,

$$0.3 \times \frac{A}{1.25}(215 - 65) = 0.3 \times \frac{A}{x}(215 - 125)$$

$$\frac{150}{1.25} = \frac{90}{x}\ ;\ x = 0.75 \text{ m}$$

Alternatively this location could be determined from the slope of the straight line variation of temperature profile

$$\tan\alpha = \frac{215 - 65}{1.25} = \frac{150}{1.25}$$

$$\therefore \quad \frac{215 - 125}{x} = \frac{150}{1.25}\ ;\quad x = 0.75 \text{ m}$$

The actual temperature at this location of the pipe can be estimated from the expression (*i*)

$$225\ A = 0.3\left[1 + \frac{0.0375}{2}(215 + t)\right]\frac{A}{0.75}(215 - t)$$

$$562.5 = \left[1 - 0.01875\,(215 + t)\right](215 - t)$$

This forms a quadratic equation in t whose solution would give $t \cong 142^\circ C$.

With references to steam tables, saturation pressure at 125°C (where location is desired) is 2.32 bar and at 142°C is 3.574 bar ; a factor of over 1.54 to 1. Obviously a consideration of dependence of thermal conductivity upon temperature is quite significant while imbedding water pipes.

EXAMPLE 3.84.

An infinite composite slab is made of two layers A and B of different materials. The layer A is 5 cm thick, has thermal conductivity $k_a = 0.4\,(1 + 0.07\,t)$ and its exposed surface is insulated. The layer B is 2.5 cm thick, has thermal conductivity 25 W/mK and its outside surface is exposed to a fluid at 20°C where the convective heat transfer coefficient is 32 W/m² K. The temperature at the interface between the two layers is estimated to be 60°C. Determine:

(a) rate of heat flux from the slab to the fluid,

(b) maximum temperature in the system, and

(c) location of the point (from insulated surface) where the temperature is 70°C.

Solution : Refer Fig. 3.59 for the given system of composite, slab and the electrical system corresponding to layer B and the fluid film.

Thermal resistance for layer B,

$$R_b = \frac{\delta_b}{k_b A} = \frac{2.5 \times 10^{-2}}{25 \times A} = \frac{0.001}{A}$$

Thermal resistance for fluid film,

$$R_f = \frac{1}{hA} = \frac{1}{32A} = \frac{0.03125}{A}$$

$$\Sigma R = R_b + R_f$$

$$= \frac{0.001 + 0.03125}{A} = \frac{0.03225}{A}$$

(a) Rate of heat flux from the slab to the fluid

$$Q = \frac{60 - 20}{0.03225/A};$$

$$\frac{Q}{A} = \mathbf{1240.3\ W/m^2}$$

(b) The temperature would be maximum at insulated face of slab A.

From Fourier's heat conduction equation,

$$Q = -kA\frac{dt}{dx}$$

$$= -0.4\,(1 + 0.07t)\,A\,\frac{dt}{dx}$$

Separating the variables and [illegible] integration

$$\frac{Q}{A}\int dx = -\int 0.4\,(1 + 0.07t)\,dt$$

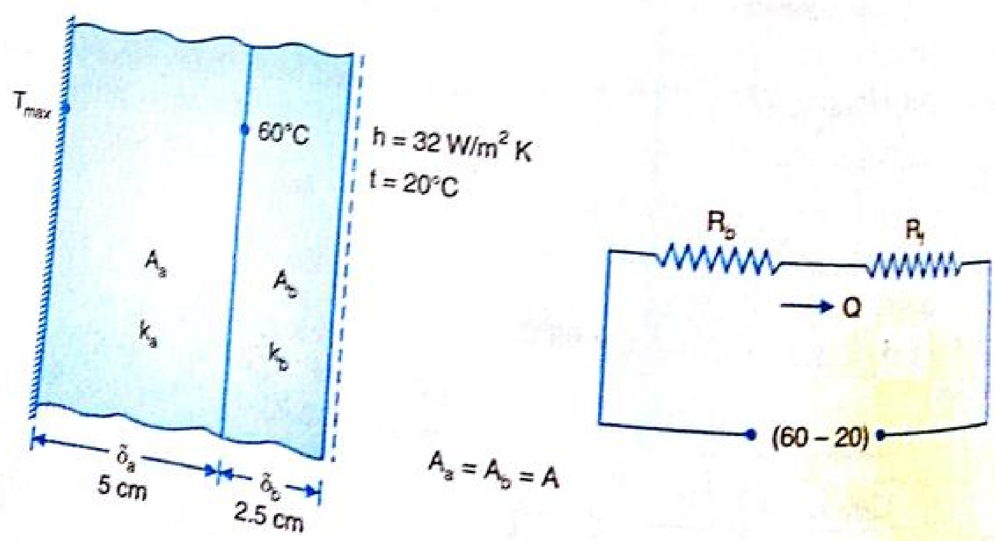


Fig. 3.59.

The appropriate boundary conditions are :

(i) At $x = 0$; $t = t_{max}$

(ii) At $x = \delta_a = 0.05$ m ; $t = 60°C$ (temperature at the interface)

$$\therefore \frac{Q}{A}\int_0^{0.05} dx = -\int_{t_{max}}^{60} 0.4(1+0.07t)\,dt$$

or $\frac{Q}{A} \times 0.05 = 0.4$

$$\left[(t_{max}-60)+\frac{0.07}{2}(t_{max}^2-60^2)\right]$$

or $\frac{Q}{A} = 0.28\,t_{max}^2 + 8t_{max} - 1488$

If the temperature at the interface is to remain steady at 60°C, then the rate of heat transfer through slab A and B must be same. Therefore

$$0.28\,t_{max}^2 + 8t_{max} - 1488 = 1240.3$$

or $0.28\,t_{max}^2 + 8t_{max} - 2728.3 = 0$

Solution of this quadratic equation gives :

t_{max} = **85.45°C**

(c) Let the boundary condition (ii) be generalized as

At $x = x$; $t_x = 80°C$

$$\text{Then } \frac{Q}{A}\int_0^x dx = -\int_{t_{max}}^{t_x} 0.4(1+0.7t)\,dt$$

$$\frac{Q}{A}\,x = 0.4\left[(t_{max}-t_x)+\frac{0.07}{2}(t_{max}^2-t_x^2)\right]$$

or $1240.3x$

$$= 0.4\left[(85.45-70)+\frac{0.07}{2}(85.45^2-70^2)\right]$$

$= 0.4\ [15.45 + 0.035\ (7301.7 - 4900)]$

$= 39.80$

$$\therefore \quad x = \frac{39.80}{1240.3} = 0.0321 \text{ m or } \mathbf{3.21\ cm}$$

Thus the temperature would be 70°C at a distance of 3.21 cm from the insulated surface.

EXAMPLE 3.85

The furnace of a steam boiler comprises two layers a 25 cm thick layer of refractory material and a layer of diatomite. The furnace is stated to operate with the following conditions:

Temperature of gases in the furnace 1200°C

Temperature of air in the boiler room 30°C

Convective heat transfer coefficient from gases to refractory wall 25 W/m² K

Convective heat transfer coefficient from diatomite wall to surrounding air 10 W/m² K

Thermal conductivity of refractory material

$k = 0.25\ (1 + 09.0008\ t)$ W/mK

Thermal conductivity of diatomite layer

$k = 0.12\ (1 + 0002\ t)$ W/mK

If the heat loss from the furnace gases to air in the boiler room is restricted to 650 W/m², determine the thickness of diatomite layer.

Solution : Refer Fig. 3.60 for schematic of the furnace wall and the corresponding electrical circuit.

Under steady state conditions, the heat flow through each section of the composite wall is same. That is

$$Q = \frac{t_1-t_2}{R_{t1}} = \frac{t_2-t_3}{R_{t2}} = \frac{t_3-t_4}{R_{t3}} = \frac{t_4-t_5}{R_{t4}}$$

$$= \frac{t_1-t_2}{1/h_1A} = \frac{t_2-t_3}{\delta_1/k_1A}$$

$$= \frac{t_3-t_4}{\delta_2/k_2A} = \frac{t_4-t_5}{1/h_2A}$$

or $$\frac{Q}{A} = \underset{(i)}{h_1(t_1-t_2)} = \underset{(ii)}{\frac{k_1}{\delta_1}(t_2-t_3)}$$

$$= \underset{(iii)}{\frac{k_2}{\delta_2}(t_3-t_4)} = \underset{(iv)}{h_2(t_4-t_5)}$$

From identity (i) :

$$650 = 25(1200 - t_2)$$

or $$t_2 = 1200 - \frac{650}{25} = 1174°C$$

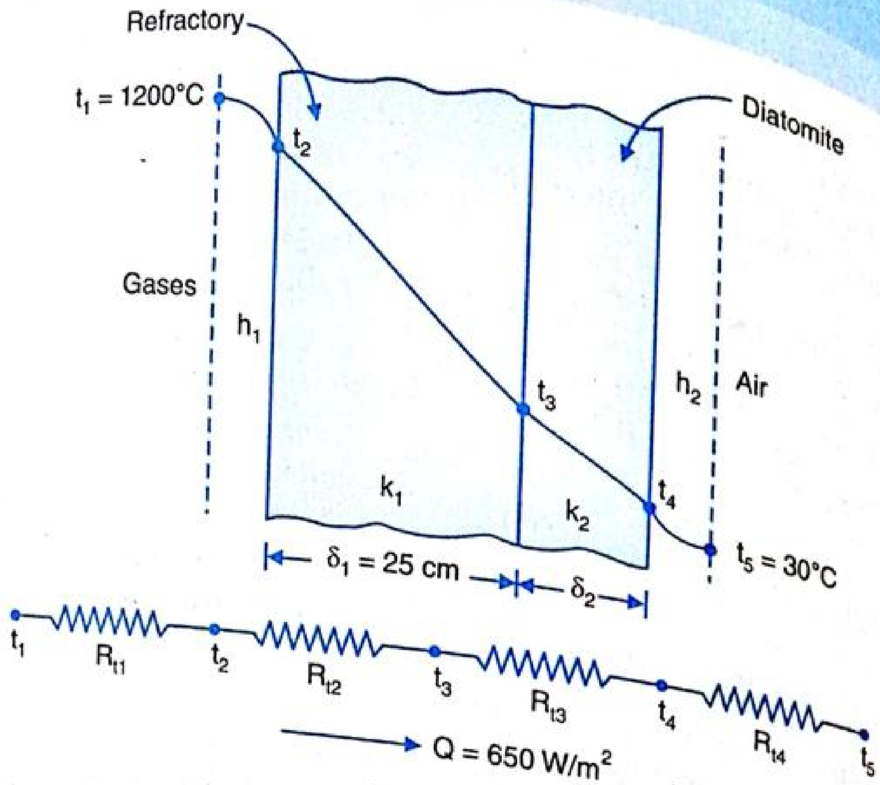


Fig. 3.60.

From identify (*iv*) :

$$650 = 10(t_4 - 30)$$

or $\quad t_4 = \dfrac{650}{25} + 30 = 45°C$

Using the identities (*i*) and (*ii*)

$$h_1(t_1 - t_2) = \frac{k_1}{\delta_1}(t_2 - t_3),$$

where $k_1 = 0.25\left[1 + 0.0008\left(\dfrac{1174 + t_3}{2}\right)\right]$

$\therefore$ 25(1200 – 1174) =

$$\frac{0.25\left[1+0.0008\left(\dfrac{1174+t_3}{2}\right)\right]}{0.25}(1174 - t_3)$$

$$\text{or } 650 = [1 + 0.0004\,(1174 + t_3)]\,(1174 - t_3)$$
$$= 1174 - t_3 + 0.0004\,(1174^2 - t_3^2)$$
$$= 1174 - t_3 + 551 - 0.0004\,t_3^2$$

$$\text{or } 0.0004\,t_3^2 + t_3 - 1075 = 0$$

Solution of this quadratic equation gives

$$t_3 = \frac{-1 \pm \sqrt{1 + 4 \times 0.0004 \times 1075}}{2 \times 0.0004}$$
$$= 811.55°C$$

Then :

$$k_1 = 0.25\left[1 + 0.0008\left(\frac{1174 + 811.55}{2}\right)\right]$$
$$= 0.4485 \text{ W/m-deg}$$

$$k_2 = 0.12\left[1 + 0.0002\left(\frac{811.55 + 95}{2}\right)\right]$$
$$= 0.1309 \text{ W/m-deg}$$

Using the identities (*ii*) and (*iii*)

$$\frac{k_1}{\delta_1}(t_2 - t_3) = \frac{k_2}{\delta_2}(t_3 - t_4)$$

$$\text{or } \frac{0.4485}{0.25}(1174 - 811.55) = \frac{0.1309}{\delta_2}(811.55 - 95);$$

$$650.23 = \frac{93.79}{\delta_2}$$

$\therefore$ Thickness of diatomite layer δ_2

$$= \frac{93.79}{650.23} = 0.144 \text{ m} = 14.4 \text{ cm}$$

EXAMPLE 3.86

The thermal conductivity of an insulating material used over a 20 cm diameter pipe carrying a hot fluid varies as

$$k = 0.065\,(1 + 15 \times 10^{-4}\, t)$$

where t is in °C and k is in W/m-deg.

The pipe surface is at 250°C and the temperature at the outside of insulation is 60°C. If the thickness of insulation is 6 cm, determine:

(a) heat flow from the hot fluid and temperature at mid thickness of insulation,

(b) slopes of temperature profile at inside surface, mid plane and outside surface.

Solution : As the dependence of thermal conductivity on temperature is linear, the mean thermal conductvity corresponds to thermal conductivity taken at mean wall temperature.

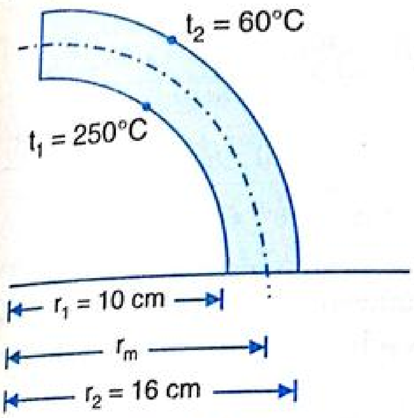


Fig. 3.61.

$$k_m = 0.065\left[1 + 15 \times 10^{-4}\left(\frac{250+60}{2}\right)\right]$$

$$= 0.08 \text{ W/m-deg}$$

The heat flow through a cylindrical surface with radii

r_1 and r_2 is given by

$$Q = \frac{2\pi k l\,(t_1 - t_2)}{\log_e \dfrac{r_2}{r_1}}$$

$$= \frac{2\pi \times 0.08 \times 1\,(250 - 60)}{\log_e \dfrac{0.16}{0.10}}$$

$$= 203.1 \text{ W/m length}$$

Let t be the temperature at mid plane (r_m = 13 cm = 0.13 m)

$$k_m = 0.065\left[1 + 15 \times 10^{-4}\left(\frac{250+t}{2}\right)\right]$$

$$Q = \frac{2\pi k_m l\,(t_1 - t)}{\log_e \dfrac{r_m}{r_1}}$$

$$= \frac{2\pi \times 0.065\left[1 + 15 \times 10^{-4}\left(\dfrac{250+t}{2}\right) \times 1 \times (250 - t)\right]}{\log_e \dfrac{0.13}{0.1}}$$

For steady state heat conduction, heat passing through each section of the pipe is same

$\therefore$ 203.1 =

$$\frac{2\pi \times 0.065\left[1 + 15 \times 10^{-4}\left(\dfrac{250+t}{2}\right)(250 - t)\right]}{\log_e \dfrac{0.13}{0.1}}$$

Simplification gives :

$7.5 \times 10^{-4}\, t^2 + t - 166.3 = 0$

Solution of this quadratic equation gives,

$$t = \frac{-1 + \sqrt{1^2 - 4 \times 7.5 \times 10^{-4}(-166.3)}}{2 \times 7.5 \times 10^{-4}}$$

$$= \frac{-1 + 1.224}{2 \times 7.5 \times 10^{-4}} = \textbf{149.53°C}$$

(*b*) Slopes of temperature profile are worked out from the relation

$$\frac{dt}{dx} = -\frac{Q}{kA} \quad \text{(Fourier law)}$$

(*i*) Inside surface :

area = $2\pi\, r_1 l = 2\pi \times 0.1 \times 1$

$= 0.628 \text{ m}^2$

$k = 0.065\,(1 + 15 \times 10^{-4} \times 250)$

$= 0.0894$ W/m-deg

$$\frac{dt}{dx} = \frac{-203.1}{0.0894 \times 0.628}$$

$$= \mathbf{-\ 3617.53\text{°C/m}}$$

(*ii*) Mid-plane :

area = $2\pi\, r_m l = 2\pi \times 0.13 \times 1$

$= 0.8164 \text{ m}^2$

$$k = 0.065\,(1 + 15 \times 10^{-4} \times 149.53)$$
$$= 0.0796 \text{ W/m-deg}$$

$$\frac{dt}{dx} = \frac{-203.1}{0.0796 \times 0.8164}$$
$$= -3125.3°\text{C/m}$$

(*iii*) Outside surface :

$$\text{area} = 2\pi\, r_2\, l = 2\pi \times 0.16 \times 1$$
$$= 1.0048 \text{ m}^2$$

$$k = 0.0065\,(1 + 15 \times 10^{-4} \times 60)$$
$$= 0.07085 \text{ W/m-deg}$$

$$\frac{dt}{dx} = \frac{-203.1}{0.07085 \times 1.0048}$$
$$= -2852.92°\text{C/m}$$

EXAMPLE 3.87.

A spherical shaped container of inner radius 15 cm stores a cryogenic substance at – 180°C. The inflow of heat is checked by covering it with 10 cm thick insulation material which has thermal conductivity prescribed by the relation : $k = 0.03\,(1 + 0.005t)$ where t is in degree centigrade and k is in W/mK. If the outer surface of insulation layer is at 15°C, determine the heat inflow, the temperature at mid-radius and the radius at which the temperature is – 180°C.

Solution : $r_1 = 15$ cm $= 0.15$ m and

$$r_2 = (15 + 10) = 25 \text{ cm} = 0.25 \text{ m}$$

$$k_m = 0.03\left[1 + 0.005 \times \frac{15 + (-180)}{2}\right]$$
$$= 0.0176 \text{ W/mK}$$

Heat flow through a spherical body is given by

$$Q = \frac{4\pi\, k_m r_1 r_2 \Delta t}{r_2 - r_1}$$

$$= \frac{4\pi \times 0.0176 \times 0.15 \times 0.25 \times [15 - (-180)]}{0.25 - 0.15}$$

$$= 15.75 \text{ W}$$

The heat flow will be radially inwards.

(*b*) Mean radius

$$r_m = \frac{0.15 + 0.25}{2} = 0.2 \text{ m}$$

Let t be the temperature at mean radius. Then

$$k_m = 0.03\left[1 + 0.005\,\frac{t + (-180)}{2}\right]$$
$$= 0.0165 + 7.5 \times 10^{-5} t$$

Under steady state conditions, heat flow through section of the sphere is same. That gives

$$Q = \frac{4\pi\,(0.0165 + 7.5 \times 10^{-5} t) \times 0.15 \times 0.2\,[t - (-180)]}{0.25 - 0.2}$$

or $\quad 15.75 = 7.536(0.0165 + 7.5 \times 10^{-5} t) \times (t + 180)$

$$= 7.536\,[0.0165t - 7.5 \times 10^{-5} t^2 + 2.97 + 0.0135t]$$

$$= 56.52 \times 10^{-5} t^2 + 0.226t + 22.3$$

or $56.52 \times 10^{-5} t^2 + 0.226t + 6.63 = 0$

Solution of this quadratic equation gives :

$$t = \frac{-0.226 \pm \sqrt{(0.226)^2 - 4 \times 56.25 \times 10^{-5} \times 6.63}}{2 \times 56.25 \times 10^{-5}}$$

$$= \frac{-0.226 \pm 0.1895}{2 \times 56.25 \times 10^{-5}}$$

$$= -32.44°\text{C or } -369.33°\text{C}$$

The solution $t = -369.33$°C is not aceptable and as such the mid-radius temperature is – 32.44°C.

(*c*) Let r be the radius at which $t = -40$°C. Then

$$Q = \frac{4\pi k_m\, r_1 r \Delta t}{r - r_1} = \frac{4\pi k_m\, \Delta t}{\left(\frac{1}{r_1} - \frac{1}{r}\right)}$$

where $\quad k_m = 0.03\left[1 + 0.005 \times \frac{-40 + (-190)}{2}\right]$

$$= 0.01275 \text{ W/mK}$$

Then :

$$15.75 = \frac{4\pi \times 0.01275 \times [-40 - (-90)]}{\frac{1}{0.15} - \frac{1}{r}}$$

$$= \frac{24.02}{6.67 - \frac{1}{r}}$$

$$\frac{1}{r} = 6.67 - \frac{24.02}{15.75} = 5.145$$

or

$$r = \frac{1}{15.145} = \mathbf{0.194\ m}$$

∴

EXAMPLE 3.88

Derive an expression for the heat flow rate through a hollow sphere of inside radius r_1 and outside radius r_2, whose internal and external surfaces are maintained at temperatures t_1 and t_2 respectively. The thermal conductivity of the sphere material has a quadratic variation with temperature:

$$k = k_0(1 + \alpha t + \beta t^2)$$

Solution : Invoking Fourier law of heat condition,

$$Q = -kA\frac{dt}{dr}$$

where A is the area normal to radial direction r.

Substituting $A = 4\pi r^2$ and $k = k_0(1 + \alpha t + \beta t^2)$, we obtain

$$Q = -k_0(1 + \alpha t + \beta t^2) \times 4\pi r^2 \times \frac{dt}{dr}$$

$$Q\frac{dr}{r^2} = -4\pi k_0(1 + \alpha t + \beta t^2)\,dt$$

In steady state, Q is constant and hence integration through the spherical walls yields

$$Q\int_{r_1}^{r_2}\frac{dr}{r^2} = -4\pi k_0\int_{t_1}^{t_2}\left(1+\alpha t+\beta t^2\right)dt$$

$$\text{or } -Q\left[\frac{1}{r}\right]_{r_1}^{r_2} = -4\pi k_0\left[\left(t+\frac{\alpha t^2}{2}+\frac{\beta t^3}{3}\right)\right]_{t_1}^{t_2}$$

$$\text{or } Q\frac{r_2-r_1}{r_1 r_2} = 4\pi k_0\left[(t_1-t_2)+\frac{\alpha}{2}\left(t_1^2-t_2^2\right)+\frac{\beta}{3}\left(t_1^3-t_2^3\right)\right]$$

$$\text{or} \quad Q = \frac{4\pi k_0 r_1 r_2}{r_2-r_1}(t_1-t_2)\left[1+\frac{\alpha}{2}(t_1+t_2)+\frac{\beta}{3}\left(t_1^2+t_2^2+t_1 t_2\right)\right]$$

which is the required expression.

Comparing this expression with the heat flow equation through a sphere where temperature dependence of thermal conductivity is ignored, *i.e.*,

$$Q = \frac{4\pi k\, r_1 r_2}{r_2-r_1}(t_1-t_2)$$

the constant thermal conductivity k works out as

$$k = k_0\left[1+\frac{\alpha}{2}(t_1+t_2)+\frac{\beta}{3}\left(t_1^2+t_2^2+t_1 t_2\right)\right]$$

EXAMPLE 3.89

A hollow sphere of inside and outside radii r_1 and r_2 respectively is heated such that its inner and outer surfaces are maintained at uniform temperatures t_1 and t_2. If the material of which the sphere is composed has a thermal conductivity which varies with temperature according to the expression

$$k = k_1 + (k_2 - k_1)\left(\frac{t-t_1}{t_2-t_1}\right)$$

find the heat flow rate through the sphere.

Solution : Invoking Fourier's law for uni-direction steady state heat conduction;

$$Q = -kA\frac{dt}{dr}$$

where A is the area normal to radial direction r.

Substituting $A = 4\pi r^2$ and

$$k = k_1 + (k_2 - k_1)\left(\frac{t-t_1}{t_2-t_1}\right)$$

$$Q = -\left[k_1+(k_2-k_1)\frac{t-t_1}{t_2-t_1}\right] \times 4\pi r^2 \times \frac{dt}{dr}$$

$$\text{or } Q\frac{dr}{r^2} = -4\pi\left[k_1+(k_2-k_1)\left(\frac{t-t_1}{t_2-t_1}\right)\right]dt$$

In steady state, Q is constant and hence integration through the spherical wall gives :

$$Q\int_{r_1}^{r_2}\frac{dr}{r^2} = -4\pi\int_{t_1}^{t_2}\left[k_1+(k_2-k_1)\left(\frac{t-t_1}{t_2-t_1}\right)\right]dt$$

$$\text{or } -Q\left|\frac{1}{r}\right|_{r_1}^{r_2}$$

$$= -4\pi\left[k_1 t + \frac{k_2 - k_1}{t_2 - t_1}\left(\frac{t^2}{2} - tt_1\right)\right]_{t_1}^{t_2}$$

$$Q\frac{r_2 - r_1}{r_1 r_2} = -4\pi\left[k_1(t_2 - t_1) + \frac{k_2 - k_1}{t_2 - t_1}\left(\frac{t_2^2 - t_1^2}{2}\right) - t_1\left(\frac{k_2 - k_1}{t_2 - t_1}\right)(t_2 - t_1)\right]$$

$$= -4\pi\left[k_1(t_2 - t_1) + \frac{(k_2 - k_1)}{2}(t_2 + t_1) - t_1(k_2 - k_1)\right]$$

$$= -4\pi\left[k_1(t_2 - t_1) + \frac{k_2 - k_1}{2}(t_2 + t_1 - 2t_1)\right]$$

$$= -4\pi\left[k_1(t_2 - t_1) + \frac{k_2 - k_1}{2}(t_2 - t_1)\right]$$

$$= -4\pi(t_2 - t_1)\left[k_1 + \frac{k_2 - k_1}{2}\right]$$

$$= 4\pi(t_1 - t_2)\left(\frac{k_1 + k_2}{2}\right)$$

$$\therefore \quad Q = 4\pi r_1 r_2\left(\frac{k_1 + k_2}{2}\right) \times \left(\frac{t_1 - t_2}{r_2 - r_1}\right)$$

EXAMPLE 3.90

A thin spherical container of 100 cm inside radius is covered with 20 cm layer of asbestos insulation. The temperature at the inside and outside surfaces are maintained at – 185°C and 5°C respectively. The container stores liquid oxygen which boils at – 185°C and has latent heat of vaporization equal to 212.5 kJ/kg. Determine the rate of evaporation of liquid oxygen.

The thermal conductivity of asbestos insulation varies linearly with temperature and its values are stated as follows :

0.156 W/mK at 5°C
and 0.125 W/mK at – 185°C

Solution : Let the linear correlation between k and t be prescribed as $k = mt + c$. Then from the given data,

$$0.156 = 5m + c$$

and $$0.125 = -185\,m + c$$

Simultaneous solution of these expressions gives

$$m = 1.63 \times 10^{-4} \text{ and } c = 0.155$$

For a hollow sphere, the Fourier's conduction equation is

$$Q = -kA\frac{dt}{dr} = -(mt + c) \times 4\pi r^2\frac{dt}{dr}$$

Separating the variables and upon integration within the given boundary conditions, we have

$$\frac{Q}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{t_1}^{t_2}(mt + c)\,dt$$

$$\text{or} \quad \frac{Q}{4\pi}\left|-\frac{1}{r}\right|_{r_1}^{r_2} = -\left|\frac{mt^2}{2} + ct\right|_{t_1}^{t_2}$$

$$\text{or} \quad \frac{Q}{4\pi}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{m}{2}(t_1^2 - t_2^2) + c(t_1 - t_2)$$

Substituting the appropriate values, we obtain

$$\frac{Q}{4\pi}\left[\frac{1}{1} - \frac{1}{1.2}\right] = \frac{1.63 \times 10^{-4}}{2}[(-185)^2 - 5^2] + 0.155(-185 - 0)$$

$$\text{or} \quad \frac{0.1667Q}{4\pi} = 2.787 - 28.675$$

$$\therefore \quad Q = \frac{2.787 - 28.675 \times 4\pi}{0.1667}$$

$$= -1950 \text{ W}$$

The negative sign is indicative of the fact that heat flow is from outer surface to inner surface.

Rate of evaporation $\dot{m}$

$$= \frac{Q}{L} = \frac{1950 \times 10^{-3}}{212.5}$$

$= 9.176 \times 10^{-3}$ kg/s

$= 33.03$ kg/hr

EXAMPLE 3.91

A hollow cylinder has inside radius r_1, outside radius r_2 and length l. The cylinder is subjected to steady heat transfer which results in constant surface temperature t_1 and t_2 at r_1 and r_2 respectively. For a linear dependence of thermal conductivity upon temperature expressed as $k = k_0 (1 + \beta t)$, obtain an expression for the heat transfer from the cylinder.

Workout this heat transfer per unit length of a hollow cylinder of inside radius $r_1 = 1.25$ cm outside radius $r_2 = 2$ cm. The corresponding temperatures on the surfaces are 310°C and 290°C. The thermal conductivity of the cylinder material varies depending on temperature, obeying the following equation

$$k = (372 - 0.0344\ t) \text{ W/m-deg}$$

Solution : Invoking Fourier law of heat conduction,

$$Q = -kA\frac{dt}{dr}$$

where A is the area normal to radius r. Substituting $k = k_0 (1 + \beta t)$ and $A = 2\pi rl$, where l is the cylinder length, we obtain

$$Q = -k_0 (1 + \beta t)\ 2\pi rl\frac{dt}{dr}$$

$$\frac{Q}{2\pi l}\frac{dr}{r} = -k_0 (1 + \beta t)dt$$

In steady state $Q/2\pi l$ is constant and hence integration through the cylinder wall yields

$$\frac{Q}{2\pi l}[\log_e r]_{r_1}^{r_2} = -k_0\left[t + \frac{\beta}{2}t^2\right]_{t_1}^{t_2}$$

$$\frac{Q}{2\pi l}\log_e\frac{r_2}{r_1} = -k_0\left[(t_2 - t_1) + \frac{\beta}{2}(t_2^2 - t_1^2)\right]$$

$$= k_0\left[1 + \frac{\beta}{2}(t_1 + t_2)\right](t_1 - t_2)$$

$$= k_m (t_1 - t_2)$$

where $k_m = k_0\left[1 + \frac{\beta}{2}(t_1 + t_2)\right]$ defines the average thermal conductivity at the mean temperature $\frac{1}{2}(t_1 + t_2)$ of cylinder surfaces.

∴ Heat transfer from the cylinder

$$Q = \frac{t_1 - t_2}{\frac{1}{2\pi k_m l} \times \log_e\frac{r_2}{r_1}}$$

$$= \frac{t_1 - t_2}{R_t}$$

where R_t is the thermal resistance of the cylinder wall.

(*b*) For a linear dependence of thermal conductivity upon temperature,

$$k_m = 372 - 0.0344 \times \left(\frac{310 + 290}{2}\right)$$

$$= 361.68 \text{ W/m-deg}$$

$$R_t = \frac{1}{2\pi \times 361.68 \times 1}\log_e\frac{2.0}{1.25}$$

$$= 2.069 \times 10^{-4} \text{ deg/W}$$

∴ Heat loss Q

$$= \frac{310 - 290}{2.069 \times 10^4} = 96.665 \text{ kW}$$

EXAMPLE 3.92

Setup an equation representing temperature variation in terms of surface temperature for one-dimensional steady state heat conduction through a plane wall. There is no internal heat generation and the thermal conductivity has a linear variation with temperature.

Solution : A linear dependence of thermal conductivity upon temperature can be expressed as $k = k_0 (1 + \beta t)$

Invoking Fourier equation for heat conduction,

$$Q = -kA\frac{dt}{dx} = -k_0 (1 + \beta t) A\frac{dt}{dx}$$

Upon integration

$$Qx = -k_0\left(t + \frac{\beta}{2}t^2\right)A + C$$

The constant of integration C is evaluated by applying the condition $t = t_1$ at $x = 0$. That gives:

$$C = k_0\left(t_1 + \frac{\beta}{2}t_1^2\right)A$$

$$\therefore\ Qx = -k_0\left(t + \frac{\beta}{2}t^2\right)A + k_0\left(t_1 + \frac{\beta}{2}t_1^2\right)A$$

$$Q = -\frac{k_0 A}{x}\left[\left(t + \frac{\beta}{2}t^2\right) - \left(t_1 + \frac{\beta}{2}t_1^2\right)\right] \quad ...(i)$$

Applying the 2nd boundary value, *i.e.*,

$$t = t_2 \quad \text{at} \quad x = l$$

$$Q = -\frac{k_0 A}{l}\left[\left(t_2 + \frac{\beta}{2}t_2^2\right) - \left(t_1 + \frac{\beta}{2}t_1^2\right)\right] \quad ...(ii)$$

Equating the expressions (*i*) and (*ii*) and upon rearrangement, we obtain the following quadratic equation in t,

$$\frac{\beta}{2}t^2 + t + \frac{x}{l}(t_1 - t_2) + \frac{\beta x}{2l}(t_1^2 - t_2^2) - \left(t_1 + \frac{\beta}{2}t_1^2\right) = 0$$

Solution for t works out to be :

$$t = \frac{-1 + \left[1 - 2\beta\left\{\frac{x}{l}(t_1 - t_2) + \frac{\beta x}{2l}(t_1^2 - t_2^2) - \left(t_1 + \frac{\beta}{2}t_1^2\right)\right\}\right]^{\frac{1}{2}}}{\beta}$$

$$= -\frac{1}{\beta} + \frac{1}{\beta}\left[1 - \frac{2\beta x}{l}(t_1 - t_2) - \frac{\beta^2 x}{l}(t_1^2 - t_2^2) + 2\beta\left(t_1 + \frac{\beta}{2}t_1^2\right)\right]^{\frac{1}{2}}$$

$$= -\frac{1}{\beta} + \frac{1}{\beta}\left[(1 + \beta^2 t_1^2 + 2\beta t_1) + \frac{x}{l}(1 + \beta^2 t_2^2 + 2\beta t_2) - \frac{x}{l}(1 + \beta^2 t_1^2 + 2\beta t_1)\right]^{\frac{1}{2}}$$

$$= -\frac{1}{\beta} + \frac{1}{\beta}\left[(1 + \beta t_1)^2 + \frac{x}{l}\left\{(1 + \beta t_2)^2 - (1 + \beta t_1)^2\right\}\right]^{\frac{1}{2}}$$

which is the required expression for temperature distribution in terms of surface temperatures for steady state heat conduction through a plane wall of thermal conductivity having linear variation with temperature.

EXAMPLE 3.93

Set up a generalised equation representing temperature variation in terms of heat flux for one-dimensional steady state heat conduction through a plane wall, a cylinder and a sphere. There is no internal heat generation and the thermal conductivity has a linear variation with temperature.

Solution : A linear dependence of thermal conductivity upon temperature can be expressed as

$$k = k_0\,(1 + \beta t)$$

Case I Plane Wall : Invoking Fourier equation for heat conduction,

$$Q = -kA\frac{dt}{dx} = -k_0\,(1 + \beta t)\,A\frac{dt}{dx}$$

Upon integration,

$$Qx = -k_0\left(t + \frac{\beta}{2}t^2\right)A + C$$

The constant of integration C is evaluated by applying the condition $t = t_1$ at $x = 0$. That gives :

$$C = k_0\left(t_1 + \frac{\beta}{2}t_1^2\right)A$$

$$\therefore\ Qx = -k_0\left(t + \frac{\beta}{2}t^2\right)A + k_0\left(t_1 + \frac{\beta}{2}t_1^2\right)A$$

Dividing both sides by k_0A and rearranging, we obtain the following quadratic equation in t,

$$\frac{\beta}{2}t^2 + t + \frac{Qx}{k_0 A} - \left(t_1 + \frac{\beta}{2}t_1^2\right) = 0$$

Solution for t works out to be :

$$t = \frac{-1 + \sqrt{1 - 4\frac{\beta}{2}\left[\frac{Qx}{k_0 A} - t_1 + \frac{\beta}{2} \times t_1^2\right]}}{2\frac{\beta}{2}}$$

$$= -\frac{1}{\beta} + \sqrt{\frac{1}{\beta^2} - \frac{2}{\beta}\left(\frac{Qx}{k_0 A} - t_1 - \frac{\beta}{2}t_1^2\right)}$$

Further simplification gives :

$$t = -\frac{1}{\beta} + \sqrt{\frac{1}{\beta^2} + \frac{2}{\beta}t_1 + t_1^2 - \frac{Qx}{\beta k_0 A}}$$

$$= -\frac{1}{\beta} + \sqrt{\left(t_1 + \frac{1}{\beta}\right)^2 - \frac{Q}{\beta k_0}\frac{2x}{A}} \quad ...(i)$$

Case II. Cylindrical Wall : Invoking Fourier equation for heat conduction,

$$Q = -kA\frac{dt}{dr} = -k_0(1 + \beta t)\,2\pi r l\frac{dt}{dr}$$

$$Q\frac{dr}{r} = -2\pi k_0 l\,(1 + \beta t)dt$$

Upon Integration

$$Q \log_e r = -2\pi k_0 l\left(t + \frac{\beta t^2}{2}\right) + C$$

The constant of integration is evaluated by applying the boundary condition

$$t = t_1 \quad \text{at} \quad r = r_1$$

That gives

$$C = 2\pi k_0 l\left(t_1 + \frac{\beta t_1^2}{2}\right) + Q\log_e r_1$$

$$\therefore Q\log_e r = -2\pi k_0 l\left(t + \frac{\beta t^2}{2}\right) + 2\pi k_0 l\left(t_1 + \frac{\beta t_1^2}{2}\right) + Q\log_e r_1$$

or

$$Q\log_e\frac{r}{r_1} = 2\pi k_0 l\left[\left(t_1 + \frac{\beta t_1^2}{2}\right) - \left(t + \frac{\beta t^2}{2}\right)\right]$$

Upon rearrangement we get the following quadratic equation in t,

$$\frac{\beta t^2}{2} + t + \frac{Q\log_e(r/r_1)}{2\pi k_0 l} - \left(t_1 + \frac{\beta t_1^2}{2}\right) = 0$$

Solution for t works out to be :

$$t = \frac{-1 + \sqrt{1 - 4\frac{\beta}{2}\left[Q\log_e\frac{r}{r_1}/2\pi k_0 l - t_1 + \frac{\beta}{2}t_1^2\right]}}{2\left(\frac{\beta}{2}\right)}$$

$$= -\frac{1}{\beta} + \sqrt{\frac{1}{\beta^2} - \frac{2}{\beta}\left[\frac{Q\log_e(r/r_1)}{2\pi k_0 l} - \left(\frac{t_1 + \beta t_1^2}{2}\right)\right]}$$

Further simplification gives :

$$t = -\frac{1}{\beta} + \sqrt{\frac{1}{\beta^2} + \frac{2}{\beta}t_1 + t_1^2 - \frac{Q\log_e(r/r_1)}{\beta k_0 \pi l}}$$

$$= -\frac{1}{\beta} + \sqrt{\left(t_1 + \frac{1}{\beta}\right)^2 - \frac{Q}{\beta k_0}\frac{\log_e(r/r_1)}{\pi l}} \quad ...(ii)$$

Adopting a similar approach, we would obtain the following expression for temperature distribution through a sphere,

$$t = -\frac{1}{\beta} + \sqrt{\left(t_1 + \frac{1}{\beta}\right)^2 - \frac{Q}{\beta k_0}\frac{1}{2\pi}\left(\frac{1}{r_1} - \frac{1}{r}\right)} \quad ...(iii)$$

From the above derived expressions (*i*) to (*iii*), we observe that when the higher temperature t_1 and heat flow Q are known, the general equation for temperature distribution is given by

$$t = -\frac{1}{\beta} + \sqrt{\left(t_1 + \frac{1}{\beta}\right)^2 - \frac{Q}{\beta k_0}f}$$

where, $f = \frac{2x}{A}$ for a plate

$= \frac{1}{\pi l}\log_e\frac{r}{r_1}$ for a cylinder

$= \frac{1}{2\pi}\left(\frac{1}{r_1} - \frac{1}{r}\right)$ for a sphere

3.9. CRITICAL THICKNESS OF INSULATION

Contrary to the common belief that addition of insulating material on a surface always brings about a decrease in the heat transfer rate, there are instances when the addition of insulation to the outside surfaces of cylindrical or spherical walls (geometries which have non-constant cross-sectional areas) does not reduce the heat loss. In fact, under certain circumstances, it actually increases the heat flow upto a certain thickness of insulation. To establish this fact, consider a thin-walled metallic cylinder of length l, radius r_i and transporting a fluid at temperature t_i which is higher than the ambient temperature t_o. Surrounding this cylinder is an annular section of insulating sheathing of thickness $(r - r_i)$ and thermal conductivity k.

With stipulations of :

(*i*) steady state conditions

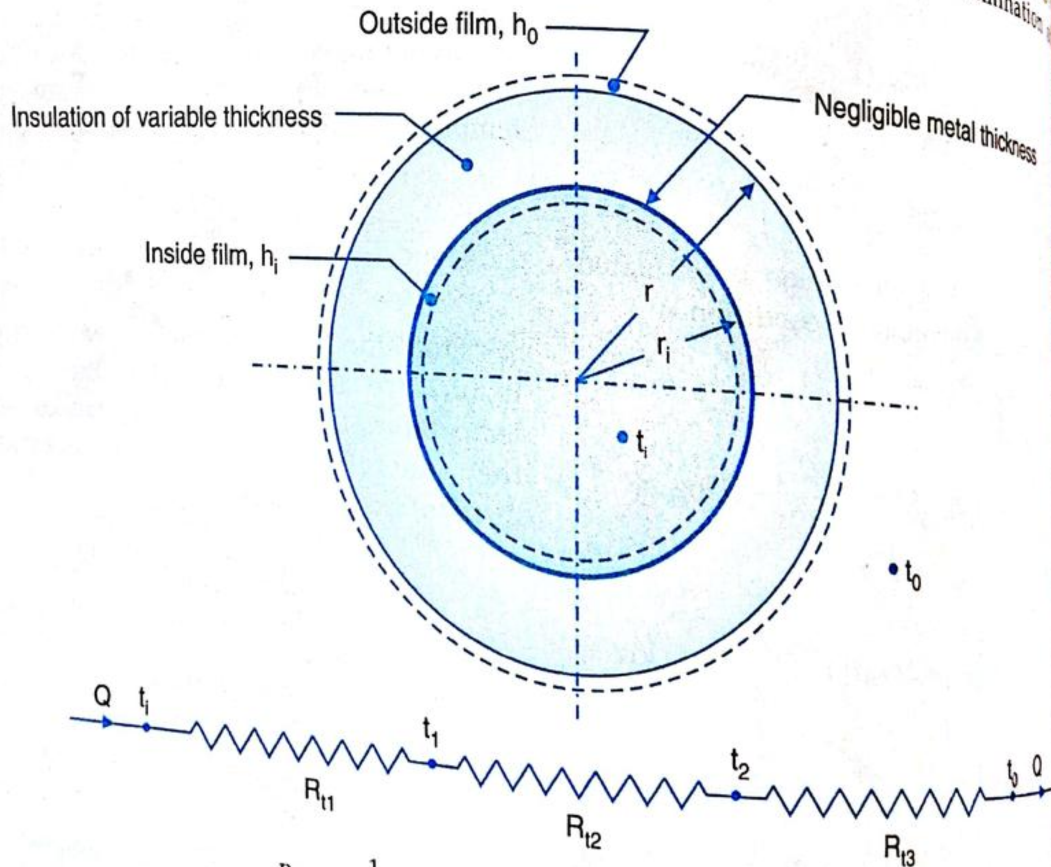
(*ii*) one-dimensional heat flow only in radial direction

(*iii*) negligible thermal resistance d... cylinder wall

(*iv*) negligible radiation exchange betw... outer surface of insulation and surround... the heat transmission can be expressed...

$$Q = \frac{(t_i - t_o)}{\frac{1}{2\pi r_i l h_i} + \frac{1}{2\pi k l}\log_e \frac{r}{r_i} + \frac{1}{2\pi r l h_o}}$$

where h_i and h_o are the film coefficie... the inner and outer surface respectively... denominator represents the sum of the... resistance to heat flow. The values of k, r_i... and h_o are constant; therefore the total the... resistance will depend upon thickness... insulation which specifies the outer radi... of the arrangement. An examination...



$$R_{t_1} = \frac{1}{2\pi r_i l h_i}, \quad R_{t_2} = \frac{1}{2\pi k l}\log_e \frac{r}{r_i}, \quad R_{t_3} = \frac{1}{2\pi r l h_0}$$

Fig. 3.62. Critical thickness of pipe insulation

...equation 3.51 would reveal that with increase in r (i.e., thickness of insulation), the thermal resistance of insulating increases but the resistance due to convection coefficient at the outer surface drops. The thermal resistance due to inside film coefficient remains unaffected with change in radius r. Obviously, addition of insulation material can either increase or decrease the rate of heat transmission depending upon a change in the total resistance with outer radius r.

The effect of insulation thickness can be studied by differentiating the total resistance R_t with respect to r and setting the derivative equal to zero.

$$\frac{dR_t}{dr} = \frac{d}{dr}\left[\frac{1}{2\pi r_i l h_i} + \frac{1}{2\pi k l}\log_e \frac{r}{r_i} + \frac{1}{2\pi r l h_o}\right]$$

$$= \frac{1}{2\pi k l}\frac{1}{r} - \frac{1}{2\pi r^2 l h_o}$$

$$\therefore \frac{1}{2\pi k l} \times \frac{1}{r} - \frac{1}{2\pi r^2 l h_o} = 0$$

which gives $r = \dfrac{k}{h_o}$

To determine whether the foregoing result maximises or minimises the total resistance, the second derivative needs to be calculated

$$\frac{d^2R_t}{dr^2} = -\frac{1}{2\pi k l}\frac{1}{r^2} + \frac{1}{\pi r^3 l h_o} \qquad ...(3.51)$$

at $r = \dfrac{k}{h_o}$

$$\frac{d^2R_t}{dr^2} = -\frac{1}{2\pi k l}\left(\frac{h_o^2}{k^2}\right) + \frac{1}{\pi l h_o}\left(\frac{h_o^3}{k^3}\right)$$

$$= \frac{h_o^2}{2\pi k^3 l}$$

which is indeed positive. Then $r = k/h_o$ represents the condition for minimum resistance and consequently maximum heat flow rate. The insulation radius at which resistance to heat flow is minimum is called the ***critical radius***. The critical radius, designated by r_c is dependent only on the thermal quantities k and h_o. Thus

$$r = r_c = \frac{k}{h_o}$$

The fact that heat flow rate attains a maximum at $r = r_c$ is the result of the above mentioned opposing effects; increasing r increases the thermal resistance of the insulation layer but decreases the thermal resistance of the surface area. At $r = r_c$ the total resistance reaches a minimum. Apparently, a pipe carrying a high temperature fluid will lose more heat (compared to a bare pipe) if the conductivity and thickness of insulation are improperly chosen. Dependence of heat loss on the thickness of insulation has been shown in Fig. 3.63.

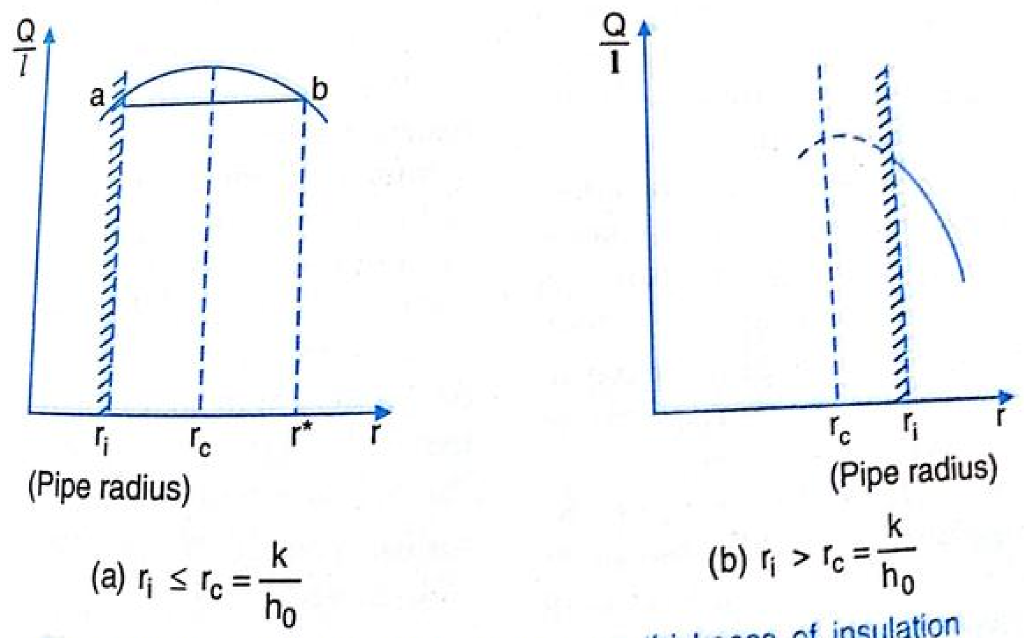


Fig. 3.63. Dependence of heat loss on thickness of insulation

Two cases of practical interest are

(i) $r_i < r_c$: *the addition of insulation to a bare pipe (point a in Fig. 3.63) leads to increasing heat transfer untill the outer radius of insulation becomes equal to the critical radius. This may be attributed to the fact that in the range $r_i < r_c$, the progressive decrease in the convection resistance with addition of insulation predominates over the correspondence increase in conduction resistance. The net result is drop in total resistance and consequently the heat loss increases.*

Any further increase in insulation thickness causes the heat loss to decrease from this peak value. However until a certain amount of insulation (r^* represented by point b) is added, the heat loss is still greater than that for the bare pipe. Evidently an insulation thickness greater than $(r^* - r_i)$ must be added to reduce the heat loss below the uninsulated rate.

The phenomenon of increase in heat transmission with addition of insulation is most likely to occur when insulating materials of poor quality are applied to pipes and wires of small radius. Such a situation is used to advantage in the insulation of electrical wires and cables. The electrical wires are given a coating of insulation with the prime objective to provide protection from electrical hazards. However an increase in the rate of heat dissipation can be made feasible and the conductors maintained within safe temperature limits by a proper choice of the insulation thickness. That permits some increase in the current carrying capacity of the cable.

(ii) $r_i > r_c$: The effect of wall thickness dominates and the overall thermal resistance increases. Sheathing of insulation then acts as lagging and that obstructs the flow of heat.

Heat insulation is the main objective in steam and refrigeration pipes. For insulation to be properly effective in restricting heat transmission, the outer radius r_o must be greater than or equal to the critical radius. If $r_o < r_c$ no useful purpose will be served with the chosen material for insulation.

Following a similar approach the critical radius of insulation for a sphere can be worked out as :

$$R_t = \frac{1}{4\pi k}\left[\frac{1}{r_1}-\frac{1}{r}\right]+\frac{1}{4\pi r^2 h_o}$$

$$\frac{dR_t}{dr} = \frac{1}{4\pi k r^2}-\frac{2}{4\pi r^3 h_o}$$

$$= 0 \text{ for maximum heat flow rate}$$

or $\quad r^3\, h_o = 2k\, r^2$

or $r\ (= r_c) = \dfrac{2k}{h_o}$

EXAMPLE 3.94

"Addition of insulating material does not always bring about a decrease in the heat transfer rate for geometries with non-constant cross-sectional area." Comment upon the validity of this statement.

A pipe of outside diameter 20 mm is to be insulated with asbestos which has a mean thermal conductivity of 0.1 W/m-deg. The local coefficient of convective heat to the surroundings is 5 W/m²-deg. Comment upon the utility of asbestos as the insulating material.

What should be the minimum value of thermal conductivity of insulating material to reduce heat transfer ?

Solution : The critical radius of insulation for optimum heat transfer from pipe is given by

$$r_c = \frac{k}{h_o} = \frac{0.1}{5} = 0.02 \text{ m or } 20 \text{ mm}$$

$$\therefore \quad r_o = 10 \text{ mm}$$

For insulation to be properly effective in restricting heat transmission, the pipe radius r_o must be greater than or equal to the critical radius r_c. Here $r_o < r_c$ and as such there is no point in using absestos as the insulating material. Addition of asbestos insulation will increase the heat transfer rate and that is not desirable. An insulating material with smaller thermal conductivity need to be employed.

For insulation to be effective, the pipe radius should be greater than the critical radius, *i.e.*,

$r_o > r_c : 0.01 \geq \frac{k}{5}$

or $\quad k \leq 5 \times 0.01 = 0.05$ W/m-deg

Apparently, the maximum conductivity of insulation premitted is 0.05 W/m-deg

EXAMPLE 3.95

A cable of 10 mm outside is to be laid in an atmosphere of 25°C (h_o = 12.5 W/m²-deg) and its surface temperature is likely to be 75°C due to heat generated within it. How would the heat flow from the cable be affected if it is insulated with rubber having thermal conductivity k = 0.15 W/m-deg ?

Solution : The critical radius of insulation for the given cable is,

$$r_c = \frac{k}{h_o} = \frac{0.15}{12.5}$$

$$= 0.012 \text{ m} = 12 \text{ mm}$$

Radius of the cable, r_o = 5 mm

Since $r_c > r_o$, the heat flow from the cable would be increased by insulating the cable.

The heat flow from the cable when provided with critical insulation is given by

$$Q = \frac{2\pi \Delta t}{\frac{1}{k} \log_e \frac{r_c}{r_o}}$$

for unit length of cable

$$= \frac{2\pi\,(75.25)}{\frac{1}{0.15} \log_e \frac{12}{5}}$$

$$= \frac{2\pi \times 50}{5.836}$$

= **53.80 W per metre length**

EXAMPLE 3.96

Explain the concept of critical insulation. How do you decide the thickness of insulation for electric wires and steam pipes ?

A heat exchanger shell of outside radius 15 cm is to be insulated with glass wool of thermal conductivity 0.0825 W/m-deg. The temperature at the surface of shell is 280°C and it can be assumed to remain constant after the layer of insulation has been applied to the shell. The convective film coefficient between the outside surface of glass wool and the surrounding air is estimated to be 8 W/m²-deg. Further, it is specified that the temperature at the outer surface of insulation must not exceed 30°C and the loss of heat per metre length of the shell should not be greater than 200 W. Would the glass wool serve the intended purpose to restric heat loss ? If so, what should be the thickness of insulating material to suit the prescribed conditions ?

Solution: The critical radius of insulation is given by

$$r_c = \frac{k}{h_o} = \frac{0.0825}{8}$$

$$= 0.01031 \text{ m} = 10.31 \text{ mm}$$

The critical radius r_c = 10.31 mm is considerably smaller than the radius r = 15 cm of the shell and as such the use of glass wool as insulating material would serve the intended purpose to restrict heat loss.

(*b*) Heat loss from a cylindrical shell is equal to

$$\frac{(t_i - t_o)}{\frac{1}{2\pi r_o h_o} + \frac{1}{2\pi k} \times \log_e \frac{r_o}{r_i}} \text{ for unit length}$$

Inserting the appropriate values,

$$200 = \frac{(280-30)}{\frac{1}{2\pi r_o \times 8} + \frac{1}{2\pi \times 0.0825} \times \log_e \frac{r_o}{0.15}}$$

$$200 = \frac{250}{\frac{0.0199}{r_o} + 1.92 \log_e \frac{r_o}{0.15}}$$

$$\text{or} \quad \frac{0.0199}{r_o} + 1.92 \log_e \frac{r_o}{0.15} = \frac{250}{200} = 1.25$$

Through trial and error, r_o = 27.5 cm

∴ Thickness of insulation

= 27.5 − 15 = 12.5 cm

EXAMPLE 3.97

A 2 mm diameter wire with 0.8 mm thick layer of insulation (k = 0.15 W/m-deg) is used in a certain electric heating application. The insulated

surface is exposed to atmosphere with convective heat transfer coefficient 40 W/m²-deg. What percentage change in heat transfer rate would occur if critical thickness of insulation is used? It may be assumed that temperature difference between surface of the wire and surrounding air remains unchanged.

Solution: Let Δt be the temperature difference between the surface of wire and air surrounding the insulation.

Case I: $r_1 = 1$ mm; $r_2 = 1 + 0.8 = 1.8$ mm

Heat loss from the wire,

$$Q_1 = \frac{\Delta t}{\frac{1}{2\pi kl}\log_e\frac{r_2}{r_1} + \frac{1}{2\pi r_2 l h_o}}$$

$$= \frac{2\pi l \Delta t}{\frac{1}{k}\log_e\frac{r_2}{r_1} + \frac{1}{r_2 h_o}}$$

$$= \frac{2\pi l \Delta t}{\frac{1}{0.15}\log_e\frac{1.8}{1} + \frac{1}{0.0018\times 40}}$$

$$= \frac{2\pi l \Delta t}{3.92 + 13.88}$$

$= 0.0562 \times (2\pi\ l\ \Delta t)$

Case II: The critical radius of insulation for the pipes,

$$r_c = \frac{k}{h_o} = \frac{0.15}{40}$$

$= 0.00375$ m $= 3.75$ mm

Heat loss from the wire when provided with critical layer of insulation,

$$Q_2 = \frac{2\pi l \Delta t}{\frac{1}{k}\log_e\frac{r_c}{r_1} + \frac{1}{r_c h_o}}$$

$$= \frac{2\pi l \Delta t}{\frac{1}{0.15}\log_e\frac{3.75}{1} + \frac{1}{0.00375\times 40}}$$

$$= \frac{2\pi l \Delta t}{8.81 + 6.67}$$

$= 0.0646 \times (2\pi\ l\ \Delta t)$

% age increase in heat loss

$$= \frac{0.0646 - 0.0562}{0.0562} \times 100$$

$= \mathbf{14.95\%}$

EXAMPLE 3.98

A wire of radius 3 mm and 1.25 m length is to be maintained at 60°C by insulating it by a material of thermal conductivity 0.175 W/mK. The temperature of surrounding air is 20°C with heat transfer coefficient 8.5 W/m²K. For maximum heat dissipation, determine:

(a) minimum thickness of insulation and the heat loss

(b) percentage increase in heat loss due to insulation.

Solution: For maximum heat dissipation, the thickness of insulation corresponds to critical radius of insulation.

$$r_c = \frac{k}{h} = \frac{0.175}{8.5}$$

$= 0.0206$ m $= 20.6$ mm

$\therefore$ Minimum thickness of insulation

$= r_c - r_i = 20.6 - 3 = 17.6$ mm

Heat loss with insulation $Q_1 = \dfrac{t_1 - t_0}{R_{t1} + R_{t2}}$

where R_{t1} (resistance to conduction)

$$= \frac{1}{2\pi kl}\log_e\frac{r_c}{r_1}$$

$$= \frac{1}{2\pi\times 0.175\times 1.25}\log_e\frac{20.6}{3}$$

$= 1.403$ K/W

R_2 (resistance to convection)

$$= \frac{1}{2\pi r_c l h_0}$$

$$= \frac{1}{2\pi\times 0.0206\times 1.25\times 8.5}$$

$= 0.727$ K/W

$$\therefore \quad Q_1 = \frac{60-20}{1.403+0.727} = 18.78 \text{ W}$$

(b) Heat loss without insulation

$$Q_2 = hA(t_1 - t_0)$$
$$= 8.5 \times (2\pi \times 0.003 \times 1.25)(60 - 20)$$
$$= 8.01 \text{ W}$$

Then percentage increase in heat dissipation due to insulation,

$$= \frac{18.78 - 8.01}{8.01} \times 100 = \mathbf{134.46\%}$$

EXAMPLE 3.99

An electric cable of 5 mm radius is applied a uniform sheathing of plastic insulation (k = 0.175 W/m-deg). The convective film coefficient on the surface of bare cable as well as insulated cable was estimated as 11.65 W/m²-deg and a surface temperature of 55°C was noted when the cable was directly exposed to ambient air at 15°C. For keeping the wire as cool as possible, find the thickness of insulation. Also determine the surface temperature of insulated cable if the intensity of current carried by the conductor remains unchanged.

Solution : For keeping the wire as cool as possible, the required condition corresponds to that for critical radius of insulation, that is

$$r_o = \frac{k}{h} = \frac{0.175}{11.65} = 0.015 \text{ m} = 15 \text{ mm}$$

∴ Thickness of insulation

$$= (r_o - r_i) = 15 - 5 = 10 \text{ mm}$$

(b) For a bare (un-insulated) wire, the heat flow is

$$Q_1 = hA\,\Delta t$$
$$= 11.65 \times (2\pi \times 0.005 \times 1) \times (55 - 15)$$
$$= 14.63 \text{ W per metre length of cable}$$

For the sheathed (insulated) cable

$$Q_2 = \frac{\Delta t}{\frac{1}{2\pi kl} \times \log_e \frac{r_0}{r_1} + \frac{1}{2\pi r_0 l h_0}}$$

$$= \frac{(t_o - 15)}{\frac{1}{2\pi \times 0.171 \times 1}\log_e \frac{15}{5} + \frac{1}{2\pi \times 0.0151 \times 11.65}}$$

$$= \frac{(t_o - 15)}{1.9108} \text{ W per meter length of cable}$$

If the intensity of current carried by the conductor remains unchanged, then

$$Q_1 = Q_2 = I^2R$$

$$\therefore \quad 14.63 = \frac{(t_0 - 15)}{1.9108}$$

Hence surface temperature of the insulated cable

$$t_o = 14.63 \times 1.9108 + 15 = \mathbf{45.95°C}$$

EXAMPLE 3.100

An electrical conductor of 10 mm diameter and having 2 mm thick insulation (k = 0.18 W/mK) covering is located in air at 25°C having convection heat transfer coefficient of 8 W/m²K. If the base conductor has resistivity of 72 μΩ cm and its surface temperature is 80°C, determine :

(a) current capacity of conductor

(b) critical thickness of insulation, and

(c) maximum current carrying capacity

Solution : For conduction through insulation,

$$R_{t1} = \frac{1}{2\pi kl}\log_e \frac{r_2}{r_1}$$

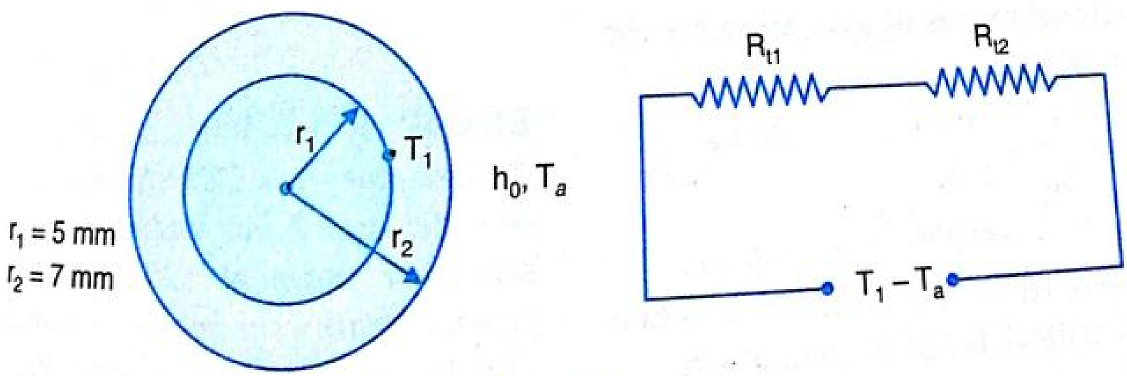


Fig. 3.64.

$$= \frac{1}{2\pi \times 0.18 \times l} \log_e \frac{7}{5}$$

$$= \frac{0.2976}{l} \text{ deg/W}$$

For outside convection,

$$R_{t2} = \frac{1}{h_o A} = \frac{1}{h_o \times 2\pi r_2 l}$$

$$= \frac{1}{8 \times 2\pi \times 7 \times 10^{-3} \times l}$$

$$= \frac{2.843}{l} \text{ deg/W}$$

Total thermal resistance,

$$R_t = R_{t1} + R_{t2}$$

$$= \frac{0.2976}{l} + \frac{2.843}{l}$$

$$= \frac{3.1406}{l} \text{ deg/W}$$

Then : $Q = \dfrac{\Delta t}{R_t} = \dfrac{80 - 25}{3.1406 / l}$

$$\frac{Q}{l} = \frac{80 - 25}{3.1406}$$

= 17.51 W/m length of conductor

From the relation, $Q = I^2 R$,

$$I = \sqrt{\frac{Q}{R}} = \sqrt{\frac{Q}{\rho l / A}} = \sqrt{\frac{Q}{l} \times \frac{A}{\rho}}$$

$$A = \pi \times 0.005^2$$

and $\rho = 72\ \mu\ \Omega\ \text{cm} = 72 \times 10^{-8}\ \Omega\ \text{m}$

$\therefore$ Current capacity of conductor,

$$I = \sqrt{17.51 \times \frac{\pi \times 0.005^2}{72 \times 10^{-18}}}$$

= **43.69 amp**

(*b*) The critical radius of insulation for the electrical conductor,

$$r_c = \frac{k}{h_o} = \frac{0.18}{8}$$

$= 0.225$ m = **22.5 mm**

(*c*) Heat loss from the conductor when provided with critical layer of insulation,

$$Q_{max} = \frac{\Delta t}{R_{t1} + R_{t2}}$$

where $R_1 = \dfrac{1}{2\pi kl} \log_e \dfrac{r_c}{r_1}$

$$= \frac{1}{2\pi \times 0.18 \times l} \log_e \frac{22.5}{5}$$

$$= \frac{1.33}{l} \text{ deg/W}$$

$$R_2 = \frac{1}{h_o A} = \frac{1}{h_o \times 2\pi r_c l}$$

$$= \frac{1}{8 \times 2\pi \times 0.0225 \times l}$$

$$= \frac{0.8846}{l} \text{ deg/W}$$

$$\therefore \quad Q_{max} = \frac{80 - 25}{\frac{1.33}{l} + \frac{0.8846}{l}}$$

or $\dfrac{Q_{max}}{l} = \dfrac{55}{1.33 + 0.8846}$

= 24.835 W/m length of conductor

Then the maximum current that can flow through the conductor,

$$I_{max} = \sqrt{\frac{Q_{max}}{l} \times \frac{A}{\rho}}$$

$$= \sqrt{24.835 \times \frac{\pi \times 0.005^2}{72 \times 10^{-8}}}$$

= **52.03 amp**

Percentage increase in current carrying capacity,

$$= \frac{I_{max} - I}{I} = \frac{52.03 - 43.69}{43.69}$$

= 0.1908 or **19.08%**

EXAMPLE 3.101

A steel pipe (k = 72 W/m-deg) of 34 mm outer diameter and 2 mm radial thickness carries dry saturated steam at 120°C. The pipe has been provided with asbestos insulation (k = 0.3 W/m-deg) to check and minimise the rate of steam con-

...ation. The pipe is located in surroundings at 25°C. Taking unit length of pipe, calculate

(a) thickness of asbestos insulation for which the rate of steam condensation is same as that when the pipe is uninsulated,

(b) mass flow rate of condensation when the above insulation is provided, and

(c) highest rate of condensation and the corresponding insulation thickness.

Take surface conductances on air-side and steam-side as 13 W/m²-deg and 500 W/m²-deg respectively and h_{fg} at 120°C = 2300 kJ/kg.

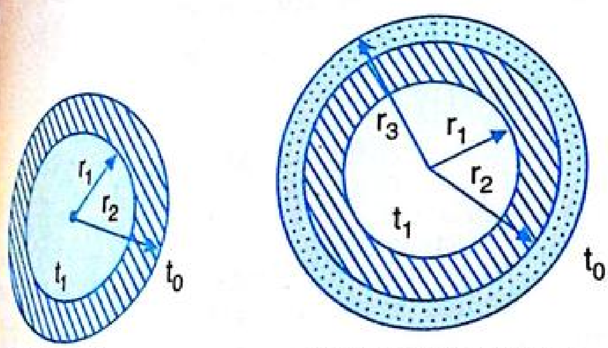
Solution : Heat flow from the bare (uninsulated pipe),

$$Q_1 = \frac{t_1 - t_o}{\dfrac{1}{2\pi r_1 l h_i} + \dfrac{1}{2\pi k_1 l}\log_e\dfrac{r_2}{r_1} + \dfrac{1}{2\pi r_2 l h_o}}$$

$$= \frac{2\pi l(t_1 - t_o)}{\dfrac{1}{h_i r_1} + \dfrac{1}{k_1}\log_e\dfrac{r_2}{r_1} + \dfrac{1}{h_o r_2}} \qquad ...(i)$$

Heat flow from the pipe when insulated,

$$Q_2 = \frac{2\pi l(t_1 - t_o)}{\dfrac{1}{h_i r_1} + \dfrac{1}{k_1}\log_e\dfrac{r_2}{r_1} + \dfrac{1}{k_2}\log_e\dfrac{r_3}{r_2} + \dfrac{1}{h_o r_3}} \qquad ...(ii)$$



(a) Bare pipe (b) Insulated pipe

Fig. 3.65.

As per the given condition :

$Q_1 = Q_2$ and therefore

$$\frac{1}{h_i r_1} + \frac{1}{k_1}\log_e\frac{r_2}{r_1} + \frac{1}{h_o r_2}$$

$$= \frac{1}{h_i r_1} + \frac{1}{k_1}\log_e\frac{r_2}{r_1} + \frac{1}{k_2}\log_e\frac{r_3}{r_2} + \frac{1}{h_o r_3}$$

or $$\frac{1}{h_o r_2} = \frac{1}{k_2}\log_e\frac{r_3}{r_2} + \frac{1}{h_o r_3}$$

or $$\frac{1}{r_2} = \frac{h_o}{k_2}\log_e\frac{r_3}{r_2} + \frac{1}{r_3}$$

Substituting the given values :

$$\frac{1}{0.017} = \frac{15}{0.30}\log_e\frac{r_3}{0.017} + \frac{1}{r_3}$$

Through trial and error :

$$r_3 = 0.0235 = 23.5 \text{ mm}$$

∴ Thickness of insulation

$$= 23.5 - 17.0 = 6.5 \text{ mm}$$

(b) $r_1 = 15$ mm; $r_2 = 17$ mm

and $r_3 = 23.5$ mm

As per the given condition, the heat flow may be calculated either from expression (i) or from expression (ii). Using expression (i),

$$Q = \frac{2\pi(t_1 - t_o)}{\dfrac{1}{h_i r_1} + \dfrac{1}{k_1}\log_e\dfrac{r_2}{r_1} + \dfrac{1}{h_o r_2}}$$

taking unit length

$$= \frac{2\pi(125 - 25)}{\dfrac{1}{500 \times 0.015} + \dfrac{1}{72}\log_e\dfrac{0.017}{0.015} + \dfrac{1}{15 \times 0.017}}$$

$$= \frac{200\pi}{0.1333 + 0.001738 + 3.9215}$$

$$= 154.813 \text{ W (J/s)}$$

$$= \frac{154.813 \times 3600}{1000}$$

$$= 557.32 \text{ kJ/hr}$$

∴ Condensate flow

$$= \frac{557.32}{2300} = 0.2423 \text{ kg/hr}$$

(c) Maximum heat flow will take place when critical thickness of insulation is applied.

Critical radius, r_3

$$= r_c = \frac{k}{h_o} = \frac{0.3}{15} \text{ m}$$

$$= \frac{0.3}{15} \times 1000 = 20 \text{ mm}$$

$$\text{Heat flow} = \frac{2\pi(t_i - t_o)}{\frac{1}{h_i r_1} + \frac{1}{k_1}\log_e \frac{r_2}{r_1} + \frac{1}{k_2}\log_e \frac{r_3}{r_2} + \frac{1}{h_o r_3}}$$

$$= \frac{2\pi(125-25)}{\frac{1}{500 \times 0.015} + \frac{1}{72}\log_e \frac{0.017}{0.015} + \frac{1}{0.3}\log_e \frac{0.02}{0.017} + \frac{1}{15 \times 0.02}}$$

$$= \frac{200\pi}{0.1333 + 0.001738 + 0.5417 + 3.333}$$

$$= 156.62 \text{ W (J/s)}$$

$$= \frac{156.62 \times 3600}{1000} = 563.83 \text{ kJ/hr}$$

∴ Condensate flow

$$= \frac{563.83}{2300} = \mathbf{0.245 \text{ kg/hr}}$$

EXAMPLE 3.102

A tube with 20 mm outside diameter is covered with an insulation (k = 0.18 W/mK) to reduce heat loss so as to maintain the tube at a prescribed uniform temperature. The dissipation of heat from the outside surface of insulation occurs by convection into the ambient air with convection coefficient 15 W/m²K. Determine :

(a) the critical thickness of insulation

(b) the ratio of heat loss from the tube with insulation to that without solution for the thickness of insulation equal to the critical thickness, and for the thickness of insulation 20 mm thicker than the critical thickness.

Solution : Critical radius of insulation,

$$r_c = \frac{k}{h_o} = \frac{0.18}{15} = 0.012 \text{ m} = 12 \text{ mm}$$

Critical thickness of insulation

$$= r_c - r_i = 12 - 10 = \mathbf{2 \text{ mm}}$$

(a) Heat loss from the tube with insulation

$$Q_{with} = \frac{T - T_o}{\frac{1}{2\pi kl}\log_e \frac{r_o}{r_i} + \frac{1}{2\pi r_o l h}}$$

$$= \frac{2\pi r_o l h (T - T_o)}{\frac{r_o h}{k}\log_e \frac{r_o}{r_i} + 1}$$

where T is the uniform temperature of the tube and T_o is the temperature of ambient air.

Heat loss from the tube without insulation

$$Q_{without} = 2\pi r_i l h (T - T_\infty)$$

$$\therefore \quad \frac{Q_{with}}{Q_{without}} = \frac{r_o}{r_i}\left[\frac{r_o h}{k}\log_e \frac{r_o}{r_i} + 1\right]^{-1}$$

- When $r_o = r_c = \frac{k}{h}$

$$\frac{Q_{with}}{Q_{without}} = \frac{r_c}{r_i}\left[\log_e \frac{r_c}{r_i} + 1\right]^{-1}$$

$$= \frac{12}{10}\left(\log_e \frac{12}{10} + 1\right)^{-1} = 1.015$$

The heat loss is increased by 1.5 percent despite the fact that 2 cm thick insulation cover is provided.

- When thickness of insulation is 20 mm more than the critical insulation. That is

$$r_0 = 10 + (2 + 20) = 32 \text{ mm}$$

Then :

$$\frac{Q_{with}}{Q_{without}} = \frac{32}{10}\left[\frac{0.032 \times 15}{0.18}\log_e\left(\frac{32}{10}\right) + 1\right]^{-1}$$

$$= 3.2 \times (4.102)^{-1} = 0.78$$

The heat loss decreases by 22 percent when 22 mm thick insulation is provided.

EXAMPLE 3.103

Establish the following relation for heat dissipation from an electric cable which has been provided with critical insulation

$$Q = \frac{2\pi k l (t_i - t_o)}{1 + \log_e (k / h_o r_i)}$$

The symbols have their usual meanings and subscripts i and o refer to the conditions at inner and outer surface respectively.

An electric cable of 5 mm outer radius has a sheathing of rubber insulation for which thermal conductivity is 0.16 W/mK. When current flows through the cable, heat is generated and surface temperature of 70°C is anticipated for the cable. This cable is laid in an environment having a temperature 20°C and the total coefficient associated with convection and radiation between the cable and environment is approximately 8 W/m²K. Make calculations for the most economical thickness and the corresponding increase in heat dissipation due to insulation. Also find out the increase in current carrying capacity of the cable by providing critical thickness of insulation.

Solution : The heat transmission from the cable to the enviornment can be expressed as

$$Q = \frac{\Delta t}{R_t}$$

$$= \frac{(t_i - t_o)}{\frac{1}{2\pi r_o l h_o} + \frac{1}{2\pi k l} \times \log_e \frac{r_o}{r_i}}$$

When critical thickness (a thickness which gives the maximum heat dissipation from the cable surface) is provided; $r_o = k/h_o$

$$\therefore \quad Q = \frac{(t_i - t_o)}{\frac{1}{2\pi \frac{k}{h_0} \times l h_0} + \frac{1}{2\pi k l} \log_e \frac{k}{h_0 r_i}}$$

$$= \frac{2\pi k l (t_i - t_o)}{1 + \log_e \frac{k}{h_0 r_i}}$$

(*b*) Inserting the appropriate values

$$\left(\frac{Q}{l}\right)_{insulated} = \frac{2\pi \times 0.16 (70 - 20)}{1 + \log_e \frac{0.16}{8 \times 0.005}}$$

= 21 W per metre length of the cable

For a bare cable, the heat dissipation would be

$$\left(\frac{Q}{l}\right)_{bare} = 2\pi\, r_i\, h\, (t_i - t_o)$$

Presuming that the surface temperature of the cable and the outside convective coefficient in the bare condition are same as those in insulated condition,

$$\left(\frac{Q}{l}\right)_{bare} = 2\pi \times 0.005 \times 8\,(70 - 20)$$

= 12.56 W per metre length of the cable

Percentage increase in heat dissipation due to insulation

$$= \frac{21 - 12.56}{12.56} \times 100 = 67.19\%$$

This increase in heat dissipation would occur when critical insulation has been applied to the cable.

Critical radius r_c

$$= \frac{k}{h_o} = \frac{0.16}{8} = 0.02 \text{ m} = 20 \text{ mm}$$

∴ Thickness of insulation

$$= 20 - 5 = 15 \text{ mm}$$

(*c*) The heat loss Q is equivalent to I^2R where I is the current capacity and R is the electrical resistance of the cable,

$$Q_1 = I_1^2 R \;; \quad \text{Bare cable}$$

$$Q_2 = I_2^2 R \;; \quad \text{Insulated cable}$$

$$\therefore \quad \frac{I_2}{I_1} = \sqrt{\frac{Q_2}{Q_1}} = \sqrt{\frac{21}{12.56}} = 1.293$$

Hence percentage increase in current carrying capacity

$$k = \frac{1.293 I_1 - I_1}{I_1} \times 100 = 29.3\%$$

Obviously rubber insulated wires can carry more current than a bare wire for the same rise in temperature.

EXAMPLE 3.104

Explain why an insulated small diameter wire has a high current carrying capacity than an uninsulated wire ?

A copper wire of radius 0.5 mm is insulated uniformly with plastic (k = 0.5 W/mK) sheathing 1 mm thick. The wire is exposed to atmosphere at 30°C and the outside surface coefficient is 8 W/m²K. Find the maximum safe current carried by

the wire so that no part of the insulated plastic is above 75°C. For copper :

Thermal conductivity = 400 W/mK

Specific electrical resistance = 2 × 10^{-8} ohm-m

Would the capacity of the wire to carry more current (dissipate more heat) increase or decrease with further addition of insulation ?

Solution : The electrical resistance of the wire,

$$R = \frac{\rho l}{A} = \frac{2\times 10^{-8}\times 1}{\pi(0.5\times 10^{-3})^2}$$

$$= 0.02548\ \Omega \text{ per metre length}$$

The heat to be dissipated per unit length of the conductor,

$$Q = I^2 R = 0.02548\ I^2$$

Under steady state conditions, this must equal the heat flow through the plastic sheathing per metre length of the conductor. Thus :

$$0.02548\ I^2 = \frac{(t_i - t_o)}{\dfrac{1}{2\pi r_0 h_0} + \dfrac{1}{2\pi k \log_e \dfrac{r_0}{r_i}}}$$

$$= \frac{2\pi\,(t_i - t_0)}{\dfrac{1}{r_0 h_0} + \dfrac{1}{k \log_e \dfrac{r_0}{r_i}}}$$

Now: $$\frac{1}{r_0 h_0} = \frac{1}{1.5\times 10^{-3}\times 8} = 83.33$$

$$\frac{1}{k\log_e \dfrac{r_0}{r_i}} = \frac{1}{0.5\times \log_e \dfrac{1.5\times 10^{-3}}{0.5\times 10^{-3}}}$$

$$= 2.197$$

Thus, $$0.02548\ I^2 = \frac{2\pi\,(75-30)}{83.33+2.197}$$

$$= 3.304$$

∴ Maximum safe current permissible,

$I =$ = 11.39 amperes

(*b*) Critical radius of plastic sheathing,

$$r_c = \frac{k}{h_o} = \frac{0.5}{8}$$

$$= 0.0625 \text{ m} = 62.5 \text{ mm}$$

Thickness of insulation

$$= 62.5 - 0.5 = 62 \text{ mm.}$$

Since the insulation is only upto 1.5 mm, the current carrying capacity of the wire can be substantially increased by adding more thickness of plastic insulation. A limit would, however, be imposed by the unacceptable temperature which may develop at the centre of the wire and for that an analysis of temperature distribution within the wire needs to be made.

EXAMPLE 3.105

Establish the following relation which will keep the insulated wire at the same temperature as if it were bare (uninsulated)

$$\log \frac{r_o}{r_i} = \frac{k}{h_o\, r_i}\left(1-\frac{r_i}{r_o}\right)$$

The symbols have their usual meanings and the subscripts i and o refer to the condition at the inside and outside surfaces respectively.

An electric cable of 6 mm radius is applied a uniform sheathing 2 mm thick of plastic insulation (k = 0.15 W/m-deg). The convective film coefficient between the surface of plastic and surrounding air is estimated as 10 W/m²-deg. Does the insulation serve to augment heat loss and thus help in the cooling of wire ? Workout the outer radius of insulation which will keep the insulation at the same temperature as if it were bare.

Solution : For a bare (un-insulated) wire, the heat flow is

$$Q_1 = 2\pi\, r_i l\, h_0\,(t_i - t_o) = \frac{(t_i - t_o)}{\dfrac{1}{2\pi r_i l h_o}}$$

For the insulated wire,

$$Q_2 = \frac{(t_i - t_o)}{\dfrac{1}{2\pi\, r_0 l\, h_0} + \dfrac{1}{2\pi\, k\, l \log_e \dfrac{r_0}{r_i}}}$$

The condition to keep the insulated wire at the same temperature as if it were bare inherently stipulates that heat flow from the wire is same in both the situations. Thus

$$\frac{1}{2\pi r_i l h_i} = \frac{1}{2\pi r_o l h_o} + \frac{1}{2\pi k l}\log_e\frac{r_o}{r_i}$$

$$\frac{1}{r_i h_i} = \frac{1}{r_o h_o} + \frac{1}{k}\log_e\frac{r_o}{r_i}$$

$$\log\frac{r_o}{r_i} = k\left(\frac{1}{r_i h_o} - \frac{1}{r_o h_o}\right) = \frac{k}{h_o r_i}\left(1 - \frac{r_i}{r_o}\right)$$

(b) The critical radius of insulation,

$$r_c = \frac{k}{h_o} = \frac{0.15}{10} = 0.015 \text{ m} = 15 \text{ mm}$$

Since r_c is greater than the outer radius $r_i = 3$ mm of the insulated wire, the sheathing helps to dissipate more heat and thus help in the cooling of the wire.

Inserting appropriate values in the relation outlined above

$$\log_e\frac{r_o}{r_i} = \frac{0.15}{10\times 0.006}\left(1 - \frac{r_i}{r_o}\right)$$

$$= 2.5\left(1 - \frac{r_i}{r_o}\right)$$

By trial and error, this equality is satisfied

when $\frac{r_o}{r_i} = 9.25$

∴ Outer radius r_o

$= 9.25 \times 0.006 = 0.0555$ m

$=$ **55.5 mm**

Thickness of insulation

$= 55.5 - 6 =$ **49.5 mm**

EXAMPLE 3.106

A 3 mm diameter and 5 m long electric wire is tightly wrapped with a 2 mm thick plastic cover of thermal conductivity 0.15 W/m-deg. Electrical measurements indicate that a current of 10 ampere passes through the wire and there is a voltage drop of 8 volts along the wire. If the insulated wire is exposed to a medium at 30°C with a heat transfer coefficient 12 W/m²-deg, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also state whether doubling the thickness of the plastic cover will increase or decrease this interface temperature. Comment on the findings

Solution : $r_1 = 1.5$ mm

and $r_2 = 1.5 + 2 = 3.5$ mm

Due to current flow in the wire, heat is generated and it is transferred to the surrounding medium in radial direction.

$$Q = VI = 8 \times 10 = 80 \text{ W}$$

Conduction resistance for the plastic cover,

$$R_{t1} = \frac{1}{2\pi k l}\log_e\frac{r_2}{r_1}$$

$$= \frac{1}{2\pi \times 0.15 \times 5}\log_e\frac{3.5}{1.5}$$

$$= 0.1798 \text{ K/W}$$

Convection resistance for the outer surface,

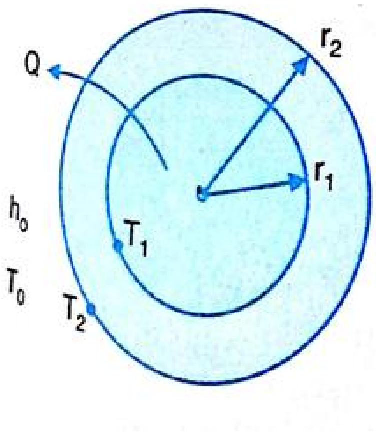
$$R_{t2} = \frac{1}{h_o A_2} = \frac{1}{h_o \times 2\pi r_2 l}$$

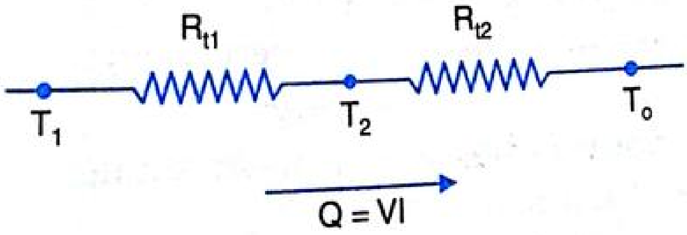
$$= \frac{1}{12\times(2\pi\times 0.0035\times 5)}$$

$$= 0.7583 \text{ deg/W}$$

These two resistances are in series and therefore total thermal resistance for the system is

$$\Sigma R_t = R_{t1} + R_{t2} = 0.1798 + 0.7583$$
$$= 0.9381 \text{ deg/W}$$





Heat flow from the wire to the surrounding medium,

$$Q = \frac{T_1 - T_o}{\Sigma R_t}$$

∴ Temperature at the interface of wire and the plastic cover

$$T_1 = Q \times R_t + T_o$$
$$= 80 \times 0.9381 + 30 = 105.03°C$$

(b) $r_1 = 1.5$ mm

and $r_2 = 1.5 + 4 = 5.5$ mm

Then: $R_{t1} = \frac{1}{2\pi \times 0.15 \times 5} \log_e \frac{5.5}{1.5}$

$= 0.2759$ deg/W

$$R_{t2} = \frac{1}{12 \times (2\pi \times 0.0055 \times 5)}$$

$= 0.4825$ deg/W

$\Sigma R_t = R_{t1} + R_{t2} = 0.2759 + 0.4825$

$= 0.784$ deg/W

$T_1 = 80 \times 0.7584 + 30 = 90.67°C$

Critical radius of insulation r_c

$$= \frac{k}{h_o} = \frac{0.15}{12}$$

$= 0.0125$ m $= 12.5$ mm

Since $r_c > r_2$, T_1 should decrease when Q is held constant and this happens for the given situation. The interface temeperature T_1 would reach the minimum value when the outer radius of plastic covering equals the critical radius.

EXAMPLE 3.107

A long copper rod of 3 cm diameter conducts a current of 1000 amperes and has an electrical resistance of 20 × 10^{-6} ohm per metre length. The rod is insulated to a radius of 1.8 cm with fibrous cotton having a thermal conductivity of 0.06 W/m-deg which is further covered by a layer of plastic of thermal conductivity 0.4 W/m-deg. The surroundings are at a temperature of 20°C and the heat transfer coefficient between the plastic and the surroundings is 20 W/m²-deg.

Determine the thickness of the layer of plastic which gives the minimum temperature in cotton insulation. For this condition, find the temperature of copper rod and the maximum temperature in the plastic layer.

Solution : The expressions for the steady state heat transfer rate for the given physical system is

$$Q = \frac{\Delta t}{R_{t_1} + R_{t_2} + R_{t_3}}$$

$$= \frac{\Delta t}{\left[\frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}\right]}$$

The effect of insulation (plastic covering) can be studied by differentiating the total thermal resistance R_t with respect to r_3 and setting the derivative equal to zero

$$\frac{dR_t}{dr_3} = \frac{d}{dr_3}\left[\frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}\right] = 0$$

or $\frac{1}{2\pi k_2 l r_3} - \frac{1}{2\pi h_o l r_3^2} = 0$

which gives :

$$r_3 = k_2 / h_o$$

To determine whether the foregoing result maximises or minimises the total resistance, the second derivative needs to be evaluated

$$\frac{d^2 R_t}{dr_3^2} = -\frac{1}{2\pi k_2 l r_3^2} + \frac{1}{\pi h_o l r_3^3}$$

At $r_3 = k_2/h_o$

$$\frac{d^2 R_t}{dR_3^2} = -\frac{1}{2\pi k_2 l}\left(\frac{h_0^2}{k_2^2}\right) + \frac{1}{\pi h_o l}\left(\frac{h_o^3}{k_2^3}\right)$$

$$= \frac{h_o^2}{2\pi k_2^2 l}$$

which is indeed positive.

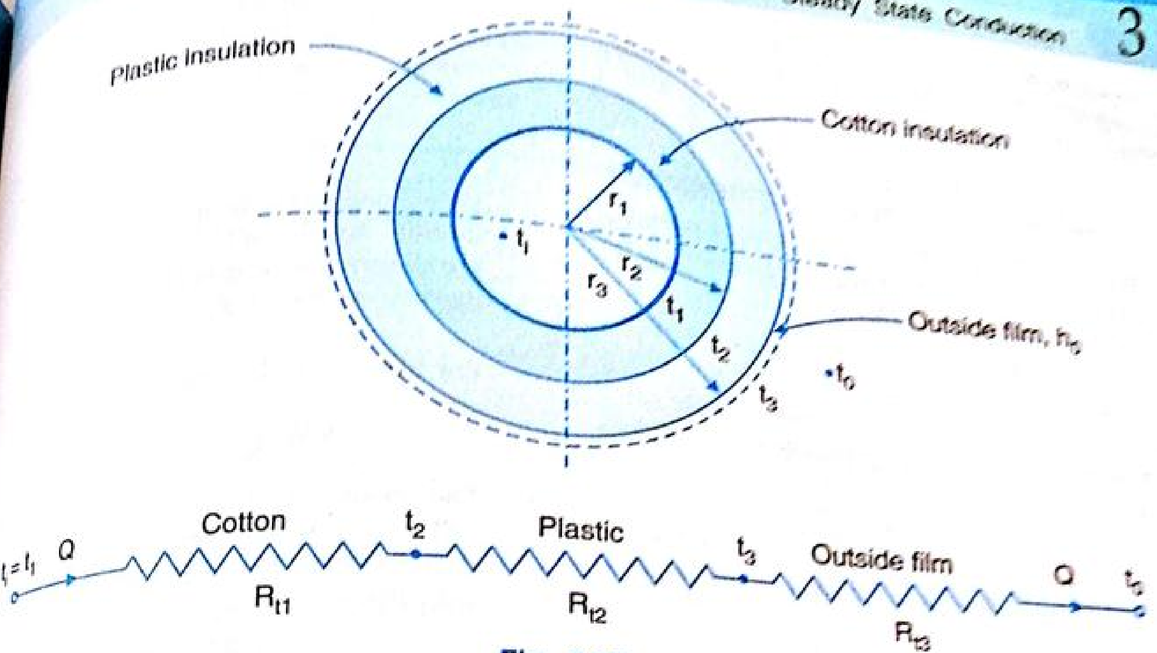


Fig. 3.67.

Then $r_3 = k_2/h_o$ represents the condition for minimum thermal resistance (*i.e.*, maximum heat flow rate) and hence minimum temperature in the cotton insulation.

$$\therefore \quad r_3 = \frac{k_2}{h_o} = \frac{0.4}{20} = 0.02 = 2.0 \text{ cm}$$

Hence, thickness of plastic insulation is equal to

$$r_3 - r_2 = 2.0 - 1.8 = 0.2 \text{ cm}$$

Heat generated in the copper rod due to flow of current

$$= I^2 R = 1000^2 \times (20 \times 10^{-6})$$
$$= 20 \text{ W per metre length}$$

Total thermal resistance,

$$\Sigma R_t = R_{t_1} + R_{t_2} + R_{t_3}$$

$$= \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_o}$$

$$= \frac{1}{2\pi \times 0.06 \times 1} \log_e \frac{1.8}{1.5} + \frac{1}{2\pi \times 0.4 \times 1} \log_e \frac{2.0}{1.8} + \frac{1}{2\pi \times 0.02 \times 1 \times 20}$$

$$= 0.484 + 0.042 + 0.398$$
$$= 0.924 \text{ deg/W}$$

Heat flow through the composite system,

$$Q = \frac{\Delta t}{\Sigma R_t} = \frac{t_1 - 20}{0.924}$$

Under steady state conditions, this heat flow equals the heat generated in the copper rod due to flow of current. Therefore,

$$20 = \frac{t_1 - 20}{0.924}$$

and hence minimum temperature in the cotton insulation,

$$t_1 = 20 \times 0.924 + 20 = 38.48°\text{C}$$

The same heat passes through each layer of the insulated copper rod. The maximum temperature in the plastic layer corresponds to the minimum temperature of cotton

insulation and its value can be worked out by considering the heat passing through the cotton covering.

$$20 = \frac{38.48 - t_2}{0.484}$$

$$t_2 = 38.48 - 20 \times 0.484 = 28.8°C$$

SALIENT POINTS

1. For a plane wall of homogenous material with constant thermal conductivity, the rate of conduction heat transfer is

$$Q = \frac{kA(t_1 - t_2)}{\delta} = \frac{\Delta t}{\frac{\delta}{kA}} = \frac{\Delta t}{R_t}$$

The temperature distribution is linear and is prescribed as

$$t = t_1 + \frac{t_2 - t_1}{\delta}x$$

For a n-layer composite wall,

$$Q = \frac{t_1 - t_{n+1}}{\sum_1^n \frac{\delta}{kA}}$$

where $\sum_1^n \frac{\delta}{kA}$ is the sum of thermal resistances of different layers comprising the composite wall.

2. The rate of convective heat transfer between a solid boundary and adjacent fluid is given by

$$Q = hA(t_s - t_f) = \frac{t_s - t_f}{R_t}$$

where h is the convective heat transfer coefficient or the film coefficient between the surface and surrounding fluid.

The heat transfer through a wall separating two moving fluids would be

$$Q = \frac{t_a - t_b}{\frac{1}{h_a A} + \frac{\delta}{kA} + \frac{1}{h_b A}}$$

Quite often, the above expression is written as

$$Q = UA(t_a - t_b)$$

Then: $$\frac{1}{U} = \frac{1}{h_a} + \frac{\delta}{k} + \frac{1}{h_b}$$

The parameter U is called the overall heat transfer coefficient, it has dimensions of W/m^2-deg and it denotes the reciprocal of the thermal resistance to heat flow.

3. For thermal paths connected in series, the total thermal resistance is

$$R = \Sigma R_i$$

For thermal paths connected in parallel

$$\frac{1}{R} = \Sigma \frac{1}{R_i}$$

4. The steady state heat conduction through a hollow cylinder of inner radius r_1 at temperature t_1 and other radius r_2 at temperature t_2 is

$$Q = \frac{2\pi kl(t_1 - t_2)}{\log_e \frac{r_2}{r_1}} = \frac{\Delta t}{R_t}$$

where l is the length of the cylinder. The thermal resistance R_t equals

$$R_t = \frac{\log_e \frac{r_2}{r_1}}{2\pi kl}$$

The temperature distribution is logarithmic and is given by

$$t = t_1 - \frac{t_1 - t_2}{\log_e \frac{r_2}{r_1}} \log_e \frac{r}{r_1}$$

Writing the expression for heat flow rate through cylinder as

$$Q = k\,A_m \frac{t_1 - t_2}{r_2 - r_1}$$

We get:

$$A_m = \frac{2\pi(r_2 - r_1)l}{\log_e \frac{r_2}{r_1}} = \frac{A_2 - A_1}{\log_e \frac{A_2}{A_1}}$$

A_m is called the logarithmic mean area. Further

$$A_m = 2\pi\, r_m l = \frac{2\pi(r_2 - r_1)l}{\log_e \frac{r_2}{r_1}}$$

Then logarithmic mean radius of the cylindrical tube is

$$r_m = \frac{r_2 - r_1}{\log_e \frac{r_2}{r_1}}$$

In a composite cylinder having two different layers, the rate of conduction heat flow is

$$Q = \frac{\Delta t}{\frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2}}$$

If the internal and external film coefficients for the two layer composite cylinder are h_i and h_o respectively, then

$$Q = \frac{\Delta t}{\frac{1}{2\pi r_1 l h_i} + \frac{1}{2\pi k_1 l} \log_e \frac{r_2}{r_1} + \frac{1}{2\pi k_2 l} \log_e \frac{r_3}{r_2} + \frac{1}{2\pi r_3 l h_0}}$$

Writing the heat flow rate as $U_0 A_0 \ \Delta t$, the overall heat transfer coefficient U_0 based on outer area will be given as

$$U_0 = \frac{1}{\frac{r_3}{r_1 h_i} + \frac{r_3}{k_1} \log_e \frac{r_2}{r_1} + \frac{r_3}{k_2} \log_e \frac{r_3}{r_2} + \frac{1}{h_0}}$$

The steady state heat conduction through a spherical sheel of inner radius r_1 and outer radius r_2 is

$$Q = \frac{4\pi l (t_1 - t_2) r_1 r_2}{r_2 - r_1} = \frac{\Delta t}{R_t}$$

The thermal resistance R_t equals

$$R_t = \frac{r_2 - r_1}{4\pi k r_1 r_2}$$

The temperature distribution associated with radial conduction through a sphere is hyperbola represented by

$$t = t_1 + \frac{(t_1 - t_2) r_1 r_2}{r_1 - r_2}\left[\frac{1}{r_1} - \frac{1}{r}\right]$$

Writing the expression for heat flow through a sphere as

$$Q = \frac{k A_m (t_1 - t_2)}{r_2 - r_1}$$

We have: $A_m = 4\pi \ r_1 r_2$ and geometric mean radiusing the shell is

$$r_m = \sqrt{r_1 r_2}$$

In a composite sphere having two different layers, the rate of conduction heat flow is

$$Q = \frac{t_1 - t_2}{\frac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_2 r_2 r_3}}$$

Further, if the convective film coefficients at the inner and outer surfaces of the composite spherical shell are also considered

$$Q = \frac{t_1 - t_2}{\frac{1}{4\pi r_1^2 h_i} + \frac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_2 r_2 r_3} + \frac{1}{4\pi r_3^2 h_0}}$$

6. For most materials, the dependence of thermal conductivity on temperature is almost linear

$$k = k_0 (1 + \beta t)$$

Depending upon + ve or – ve sign of the constant β, the thermal conductivity may increase or decrease with temperature. The coefficient β is usually + ve for non-metals and insulation materials, and – ve for metallic conductors.

(*i*) When $\beta = 0$, the slope of temperature curve is constant and the temperature profile is linear.

(*ii*) When β is + ve, the value of the slope of temperature profile increases and the temperature variation curve is of convex nature.

(*iii*) When β is – ve, the value of the slope of temperature profile decreases and the temperature profile is of concave nature.

7. For geometrics having non-constant cross-sectional area (cylindrical or spherical walls), the addition of insulation to the outside surface may increase the heat flow upto a certain thickness of insulation.

The insulation radius at which resistance to heat flow is minimum and consequently the heat flow rate is maximum is called critical radius.

The critical radius is dependent only on the thermal quantities k (thermal conductivity of pipe material) and h_0 (outside film coefficient).

$$r = r_c = \frac{k}{h_0} \quad \text{for a cylinder}$$

and $$r = r_c = \frac{k}{2h_0} \quad \text{for a sphere}$$

19(c):

Heat flowing per second across the cylinder of radius r is given by

$$= \frac{k_1 (\pi r^2)(t_1 - t_2)}{\delta}$$

where t_1 and t_2 are the temperatures at the two ends ($t_1 > t_2$) and δ is the distance between them

Heat flow across the cylindrical shell is

$$k_2 \frac{\pi\{(2r)^2 - r^2\}(t_1 - t_2)}{\delta}$$

Total heat flow per second would be given by the sum of above two heat flows and that works out as

$$\frac{t_1 - t_2}{\delta} \pi (k_1 r^2 + 3 k_2 r^2)$$

If k is the equivalent thermal conductivity of the system, then

$$\frac{t_1 - t_2}{\delta} \pi (k_1 r^2 + 3 k_2 r^2)$$

$$= \frac{k \pi (2r)^2 + (t_1 - t_2)}{\delta}$$

i.e., $\quad k = \frac{k_1 + 3k_2}{4}$

20(b):

Heat dissipation would be small when the material with low thermal conductivity (better insulation) is placed inside, *i.e.*, next to the steam pipe.

25(b):

For a bare pipe upto critical radius, the heat loss increases. At critical radius, it is maximum and beyond that it decreases.

26(a):

For a cylindrical pipe, the critical radius of insulations is given by

$$r_c = \frac{k}{h_o} = \frac{0.15}{6}$$

$$= 0.025 \text{ m} = 25 \text{ mm}$$

27(c):

For a spherical sheath, the critical radius of insulation is

$$r_c = \frac{2k}{h_o} = \frac{2 \times 0.01}{10}$$

$$= 0.008 \text{ m} = 8 \text{ mm}$$

$\therefore$ Diameter of sheath = 16 mm.

30(d):

The situation corresponds to minimum thermal resistance and consequently maximum heat flow rate.

❏❏❏

Chapter

4

Conduction With Heat Generation

Learning objectives : After having made a study of the subject matter presented in this chapter, the reader will be able to

- *list the situations where heat is generated internally at uniform rate within the conducting medium itself*
- *analyse the conduction in a plane wall with uniform heat generations*
- *analyse the heat conduction through a solid and hollow cylindrical rod with uniform heat generation*
- *determine the heat flow and temperature distribution through a solid and hollow cylinder with uniform heat generation*
- *discuss dielectric heating, heat transfer through the piston crown, nuclear fuel elements with and without cladding*

In many situations of practical importance, heat is generated internally at uniform rate within the conducting medium itself. Notable examples are :

(*i*) resistance heating in electrical appliances; essentially it is the conversion of electrical energy into thermal energy in the current carrying medium

(*ii*) energy generated in the fuel element of a nuclear reactor

(*iii*) liberation of energy due to some exothermic chemical reactions occuring within the medium

(*iv*) drying and setting of concrete

The rate of heat generation has to be controlled one; otherwise the resulting temperature growth might result in the failure of the medium. Undoubtedly temperature distribution within the medium and the rate of heat dissipation to the surroundings assume great importance in the design of thermal units.

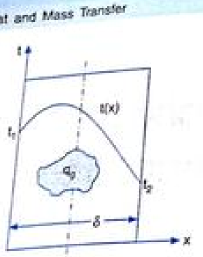
4.1. PLANE WALL WITH UNIFORM HEAT GENERATION

Consider heat conduction through a plane wall in which heat sources are uniformly distributed over the entire volume. The wall surfaces are maintained at temperature t_1 and t_2, and the wall thickness δ is small in comparison with other dimensions.

Assumptions:

1. Steady state conditions
2. One-dimensional heat flow
3. Constant thermal conductivity k
4. Uniform volumetric heat generation (q_g per unit volume) within the wall.

The differential equation describing the temperature distribution can be set up by making an energy balance on an elemental strip of thickness dx at a distance x from the left hand face of the wall.



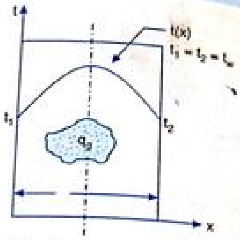


Fig. 4.1. Conduction in a plane wall with uniform heat generation

Q_x (heat conducted in at distance x)

$$= -kA\frac{dt}{dx}$$

Q_g (heat generated in the element)

$$= A\,dx\,q_g$$

Q_{x+dx} (heat conducted out at a distance $x + dx$)

$$= Q_x + \frac{d}{dx}(Q_x)dx$$

From steady state condition of heat flow,

$$Q_x + Q_g = Q_{x+dx}$$

$$= Q_x + \frac{d}{dx}(Q_x)dx$$

or $$Q_g = \frac{d}{dx}(Q_x)dx$$

or $$A\,dx\,q_g = \frac{d}{dx}\left(-kA\frac{dt}{dx}\right)dx$$

$$= -kA\frac{d^2t}{dx^2}\,dx$$

That gives appropriate heat flow equation as

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0 \qquad ...(4.1)$$

Integrating equation 4.1, we obtain the general solution as

$$\frac{dt}{dx} = -\frac{q_g}{k}x + C_1 \qquad ...(4.2)$$

$$t = -\frac{q_g}{2k}x^2 + C_1x + C_2 \qquad ...(4.3)$$

A. Both surfaces are maintained at a common temperature :

From the prescribed boundary conditions,

$$t = t_1 = t_w \text{ at } x = 0 \text{ and}$$
$$t = t_2 = t_w \text{ at } x = \delta$$

the constants of integration take the values

$$C_2 = t_w\,;\quad C_1 = \frac{q_g}{2k}\delta$$

Substituting these values of C_1 and C_2 in equation 4.3, the expression for temperature profile becomes

$$t = -\frac{q_g}{2k}x^2 + \frac{q_g}{2k}\delta x + t_w$$

$$= \frac{q_g}{2k}(\delta - x)x + t_w \qquad ...(4.4)$$

Maximum temperature and its location within the wall can be worked out by differentiating the equation for temperature profile with respect to x and equating the derivative to zero.

$$\frac{dt}{dx} = \frac{q_g}{2k}(\delta - 2x)$$

Since the factor $\frac{q_g}{2k} \neq 0$

$$\delta - 2x = 0 \quad \text{or} \quad x = \frac{\delta}{2}$$

The temperature distribution as prescribed by equation 4.4. is thus parabolic and symmetrical about the mid plane. Maximum value of temperature occurs at $x = \delta/2$ and it equals

$$t_{max} = \left[\frac{q_g}{2k}(\delta - x)x\right]_{x=\delta/2} + t_w$$

$$= \frac{q_g}{8k}\delta^2 + t_w \qquad ...(4.5)$$

Heat transmission then occurs towards both surfaces, and for each surface it is given by

$$Q = -kA\left(\frac{dt}{dx}\right)_{x=0 \text{ or } x=\delta}$$

$$= -kA\left[\frac{q_g}{2k}(\delta - 2x)\right]_{x=0 \text{ or } x=\delta}$$

$$= \frac{A\delta}{2}q_g \qquad ...(4.6)$$

For both surfaces

$$Q = A\,\delta\,q_g$$

= (volume of conducting medium) (heat generating capacity)

Heat conducted to the wall surface is finally dissipated to the surrounding atmosphere at temperature t_a. Then for each surface,

$$\frac{A\delta}{2}q_g = hA(t_w - t_a)$$

or $$t_w = t_a + \frac{q_g}{2h}\delta \qquad ...(4.7)$$

Substituting this value of wall temperature in equation 4.4, one gets the temperature distribution in term of temperature t_a of the surrounding atmosphere

$$t = t_a + \frac{q_g}{2h}\delta + \frac{q_g}{2k}(\delta - x)x \qquad ...(4.8)$$

Equation 4.8. applies equally well to plane walls which are perfectly insulated at one face and maintained at a fixed temperature t_w on the other face. The full hypothetical slab with thickness 2δ can be considered with the same bounding conditions :

$$t = t_w \text{ at } x = 2\delta$$

and $$\frac{dt}{dx} = 0 \text{ at } x = \delta$$

The location $x = \delta$ corresponds to mid-plane of the hypothetical full wall or insulated face of the given wall.

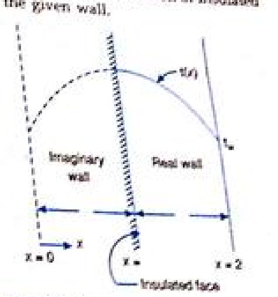


Fig. 4.2. Conduction in a insulated wall

The equation 4.4. for temperature distribution and the equation 4.5. for maximum temperature at the mid-plane (insulated end of given wall) may be written as

$$t = \frac{q_g}{2k}(2\delta - x)x + t_w$$

$$t_{max} = \frac{q_g}{2k}\delta^2 + t_w$$

B. Temperatures of both the surfaces are different :

When the boundary conditions

$$t = t_1 \text{ at } x = 0 \text{ and}$$
$$t = t_2 \text{ at } x = \delta$$

are applied to equation 4.3, the constants of integration take the values

$$C_2 = t_1\,;\quad C_1 = \frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta$$

Substituting these values of integration constants in equation 4.3, the expression for temperature profile becomes :

$$t = -\frac{q_g}{2k}x^2 + \frac{(t_2 - t_1)}{\delta}x + \frac{q_g}{2k}\delta x + t_1$$

$$= \frac{q_g}{2k}\delta x - \frac{q_g}{2k}x^2 + \frac{x}{\delta}(t_2 - t_1) + t_1$$

or $$t = \left[\frac{q_g}{2k}(\delta - x) - \frac{t_1 - t_2}{\delta}\right]x + t_1 \quad ...(4.9)$$

The following transformations may be made to obtain the temperature distribution in non-dimensional form

$$t - t_2 = \frac{q_g}{2k}\delta^2\left[\frac{x}{\delta} - \left(\frac{x}{\delta}\right)^2\right] + \frac{x}{\delta}(t_2 - t_1) + (t_1 - t_2)$$

$$\therefore \frac{t - t_2}{t_1 - t_2} = \frac{q_g}{2k}\frac{\delta^2}{(t_1 - t_2)}\left[\frac{x}{\delta} - \left(\frac{x}{\delta}\right)^2\right] - \frac{x}{\delta} + 1$$

$$= \frac{q_g}{2k}\frac{\delta^2}{(t_1 - t_2)}\frac{x}{\delta}\left(1 - \frac{x}{\delta}\right) + \left(1 - \frac{x}{\delta}\right)$$

The parameter $\frac{q_g}{2k}\frac{\delta^2}{(t_1 - t_2)}$ is constant and can be replaced by a factor B

$$\frac{t - t_2}{t_1 - t_2} = B\frac{x}{\delta}\left(1 - \frac{x}{\delta}\right) + \left(1 - \frac{x}{\delta}\right)$$

$$= \left(1 - \frac{x}{\delta}\right)\left(\frac{Bx}{\delta} + 1\right) \quad ...(4.10)$$

Maximum temperature and its location within the wall can be worked out by differentiating equation for temperature profile with respect to (x/δ) and setting the derivative to zero :

$$\frac{dt}{d(x/\delta)} = \left(1 - \frac{x}{\delta}\right)B + \left(\frac{Bx}{\delta} + 1\right)(-1) = 0$$

or $B - B\frac{x}{\delta} - B\frac{x}{\delta} - 1 = 0$

or $2B\frac{x}{\delta} = B - 1$

or $$\frac{x}{\delta} = \frac{B - 1}{2B} \quad ...(4.11)$$

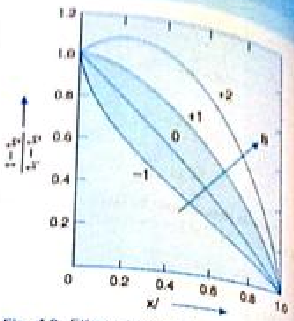


Fig. 4.3. Effect of factor B on temperature distribution

Thus maximum value of temperature occurs at

$$\frac{x}{\delta} = \frac{B - 1}{2B}$$

and it equals

$$\frac{t_{max} - t_2}{t_1 - t_2} = \left(1 - \frac{B - 1}{2B}\right)\left(B \times \frac{B - 1}{2B} + 1\right)$$

$$= \frac{B + 1}{2B} \times \frac{B + 1}{2} = \frac{(B + 1)^2}{4B} \quad ...(4.12)$$

The effect of factor

$$B = \frac{q_g}{2k}\frac{\delta^2}{(t_1 - t_2)}$$

on temperature distribution in the wall has been shown in Fig. 4.3. With an increase in the value B, there occurs a change in the slope of the curve. Apparently a sufficiently large value of q_g can reverse the direction of heat flow. The temperature distribution is linear corresponding to $B = 0$ and that pertains to the situation when there is no internal heat generation. A negative value of B results when q_g represents the heat absorption within the body.

Further, rate of heat flow from any face of the wall is;

$$Q = -kA\frac{dt}{dx}$$

$$= -kA\left[C_1 - \frac{q_g}{k}x\right]$$

$$= -kA\left[\frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta - \frac{q_g x}{k}\right]$$

$$= kA\left[\frac{t_1 - t_2}{\delta} - \frac{q_g}{2k}(\delta - 2x)\right]$$

Heat flow from the left hand face, *i.e.*, at $x = 0$

$$Q_1 = kA\left[\frac{t_1 - t_2}{\delta} - \frac{q_g}{2k}\delta\right] \quad ...(4.13a)$$

Heat flow from the right hand face, *i.e.*, at $x = \delta$

$$Q_2 = kA\left[\frac{t_1 - t_2}{\delta} + \frac{q_g}{2k}\delta\right] \quad ...(4.13b)$$

Let us now examine two cases :

(i) Maximum temperature occurs within the wall; the heat flow will then be from both the surfaces and the total heat flow will become

$$Q_t = Q_1 + Q_2$$

(ii) Maximum temperature occurs at the left hand face, *i.e.*, t_1 is maximum; the heat flow will then be only towards the right, *i.e.*, in the direction of falling temperature

$$Q_t = Q_2 \text{ (only)}$$

$$= kA\left[\frac{t_1 - t_2}{\delta} + \frac{q_g}{2k}\delta\right]$$

If there is no internal heat generation (*i.e.*, $q_g = 0$), the above expression reduces to heat conduction equation $Q = kA\,(t_1 - t_2)/\delta$ for a plane wall without any internal generation of energy.

C. Current carrying electrical conductor :

When heat generated within a material is due to passage of electric current, q_g can be expressed in electrical terms

$$Q_g = I^2R\;;\quad R = \frac{\rho L}{A}$$

and heat generated per unit volume of conductor

$$q_g = \frac{Q_g}{AL}$$

where I is the current, R is the electrical resistance, ρ is resistivity, L and A are the length and cross-sectional area of the conductors. Combining these relations, we get

$$q_g = I^2 \times \frac{\rho L}{A} \times \frac{1}{AL}$$

$$= \left(\frac{I}{A}\right)^2 \rho = i^2\rho = \frac{i^2}{k_e} \quad ...(4.14)$$

Here i is the current density and electrical conductivity k_e is the reciprocal of the resistivity ρ of the medium.

EXAMPLE 4.1

The rear window of an automobile is made of 5 cm thick glass of thermal conductivity 0.8 W/m-deg. To defrost this window, a thin transparent film type heating element has been fixed to its inner surface. For the conditions given below, determine the electric power that must be provided per unit area of window if a temperature 5°C is maintained at its outer surface.

Interior air temperature and the corresponding surface coefficient, = 20°C and 12 W/m²-deg.

Surrounding air temperature and the corresponding surface coefficient, = − 15°C and 70 W/m²-deg.

Electric heater provides uniform heat flux.

Solution : $t_i = 20°C$; $h_i = 12\ W/m^2\text{-deg}$
$t_o = -15°C$; $h_o = 70\ W/m^2\text{-deg}$; $t_s = 5°C$

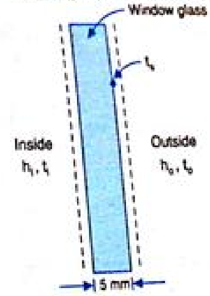


Fig. 4.4.

For unit area, the heat balance provides

$$\frac{(t_i - t_s)}{\frac{1}{h_i} + \frac{\delta}{k}} + q_g = h_o\,(t_s - t_o)$$

Substituting the given data,

$$\frac{(20-5)}{\frac{1}{12} + \frac{0.005}{0.8}} + q_g = 70 \times [5 - (-15)]$$

$$\text{or} \quad \frac{15}{0.0833 + 0.00625} + q_g = 1400$$

$\therefore$ Electric power to be provided,

$$q_g = 1400 - \frac{15}{0.08955}$$
$$= 1400 - 167.50 = \mathbf{1232.5\ W/m^2}$$

EXAMPLE 4.2

A composite slab consists of 5 cm thick layer of steel (k = 146 kJ/m – hr-deg) on the left side and a 6 cm thick layer of brass (k = 276 kJ/m-hr-deg) on the right hand side. The outer surfaces of the steel and brass layer are maintained at 100°C and 50°C respectively. The contact between the two slabs is perfect and heat is generated at the rate of 4.2 × 10⁵ kJ/m²-hr at the plane of contact. The heat thus generated is dissipated from both sides of composite slab for steady state conditions. Calculate the temperature at the interface and heat flow through each slab.

Solution : Let t_i be the temperature at the interface. Under stipulation for heat dissipation from both sides,

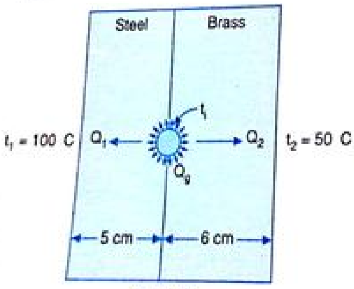


Fig. 4.5.

$$t_i > t_1 > t_2$$

Accordingly we may write

$$Q_1 + Q_2 = Q_g$$

$$\frac{k_1 A_1 (t_i - t_1)}{\delta_1} + \frac{k_2 A_2 (t_i - t_2)}{\delta_2} = Q_g$$

Considering unit surface area

$$\frac{146 \times 1 \times (t_i - 100)}{0.05} + \frac{276 \times 1 \times (t_i - 50)}{0.06} = 4.2 \times 10^5$$

$$\text{or } 2920\,(t_i - 100) + 4600\,(t_i - 50) = 4.2 \times 10^5$$

$$\text{or } 7520\, t_i = 4.2 \times 10^5 + 2.92 \times 10^5 + 2.3 \times 10^5$$
$$= 9.42 \times 10^5$$

$\therefore$ Temperature at the interface,

$$t_i = \frac{9.42 \times 10^5}{7520} = 125.26°C$$

Heat transfer through the steel layer,

$$Q_1 = \frac{146 \times 1 \times (125.26 - 100)}{0.05}$$
$$= 73759\ \text{kJ/m}^2\text{-hr}$$

Heat transfer through the brass layer,

$$Q_2 = \frac{276 \times 1 \times (125.26 - 50)}{0.06}$$
$$= \mathbf{346196\ kJ/m^2\text{-}hr}$$

EXAMPLE 4.3

A square plate heater measuring 16 cm × 16 cm and of rating 1 kW is inserted between two slabs. Slab A is 2 cm thick (k = 60 W/m-deg) and slab B is 1 cm thick (k = 0.25 W/m-deg). The outside heat transfer coefficients on side A and side B are 200 W/m²-deg and 50 W/m²-deg respectively. If the surrounding air is at 20°C, make calculations for the maximum temperature in the system and outer surface temperature of two slabs. Also calculate the heat transfer through each slab.

Solution : Let t_{max} be the maximum temperature at the heater section. Then for steady state heat flow, one may write

Q = heat flow through slab A (Q_1) + heat flow through slab B (Q_2)

$$= \frac{t_{max} - t_o}{\frac{\delta_1}{k_1 A} + \frac{1}{h_1 A}} + \frac{t_{max} - t_o}{\frac{\delta_2}{k_2 A} + \frac{1}{h_2 A}}$$

$$= A\,(t_{max} - t_o)\left[\frac{1}{\frac{\delta_1}{k_1} + \frac{1}{h_1}} + \frac{1}{\frac{\delta_2}{k_2} + \frac{1}{h_2}}\right]$$

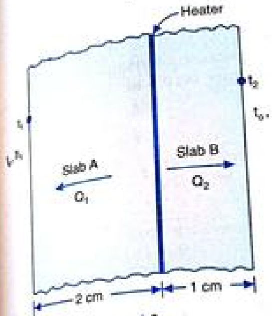


Fig. 4.6.

Substituting the given data,

$$1000 = (0.16 \times 0.16)\,(t_{max} - 20) \times \left[\frac{1}{\frac{0.02}{60} + \frac{1}{200}} + \frac{1}{\frac{0.01}{0.25} + \frac{1}{50}}\right]$$

$$= 0.0256\,(t_{max} - 20) \times \left[\frac{1}{0.00033 + 0.005} + \frac{1}{0.04 + 0.02}\right]$$

$$= 0.0256\,(t_{max} - 20)\,[187.617 + 16.667]$$

$$= 5.2296\,(t_{max} - 20)$$

$$\therefore \quad t_{max} = \frac{1000}{5.2296} + 20 = \mathbf{211.22°C}$$

Considering the heat flow through slab A,

$$\frac{Q_1}{A} = \frac{k_1 (t_{max} - t_1)}{\delta_1} = h_1 (t_1 - t_0)$$

$$\therefore \frac{60(211.22 - t_1)}{0.02} = 200\,(t_1 - 20)$$

$$633660 - 3000\, t_1 = 200\, t_1 - 4000$$

That gives :

$$t_1 = \frac{633600 + 4000}{3200} = \mathbf{199.25°C}$$

$$Q_1 = h_1 A\,(t_1 - t_0)$$
$$= 200 \times (0.16 \times 0.16) \times (199.25 - 20)$$
$$= \mathbf{917.76\ W}$$

Considering the heat flow through slab B,

$$\frac{Q_2}{A} = \frac{k_2 (t_{max} - t_2)}{\delta_2} = h_2 (t_2 - t_0)$$

$$\therefore \frac{0.25(211.22 - t_2)}{0.01} = 50\,(t_2 - 20)$$

$$5280.5 - 25\, t_2 = 50\, t_2 - 1000$$

That gives :

$$t_2 = \frac{5280.5 + 1000}{75} = \mathbf{83.74°C}$$

$$Q_2 = h_2 A\,(t_2 - t_0)$$
$$= 50 \times (0.16 \times 0.16) \times (83.74 - 20)$$
$$= \mathbf{81.59\ W}$$

Check :

$$Q = Q_1 + Q_2$$
$$= 917.36 + 81.59 = 999.35\ \text{W}$$

= 1 kW which equals the rating of heater.

EXAMPLE 4.4

A slab of 12 cm thickness and generating heat uniformly at 10⁶ W/m³ has thermal conductivity of 200 W/m-deg. Both surfaces of the slab are maintained at 150°C. Determine: (a) the temperature, temperature gradients and heat flow rate at the quarter planes, (b) maximum temperature and its location.

Solution : With uniform heat generation, and both sides of the slab at the same temperature, the temperature distribution is prescribed by the relation :

$$t = \frac{q_g}{2k}\,(\delta - x)\,x + t_w$$

$$\frac{dt}{dx} = \frac{q_g}{2k}(\delta - 2x)$$

Heat flow rate $Q = -kA\frac{dt}{dx}$

At the quarter plane AB :

$$x = 3 \text{ cm} = 0.03 \text{ m}$$

$$t = \frac{10^6}{2 \times 200}(0.12 - 0.03) \times 0.03 + 150$$

$$= 6.75 + 150 = 156.75°C$$

$$\frac{dt}{dx} = \frac{10^6}{2 \times 200}(0.12 - 2 \times 0.03)$$

$$= 150°C/m$$

$$Q = -200 \times 1 \times 150$$

$$= -30{,}000 \text{ W/m}^2 \text{ (unit area)}$$

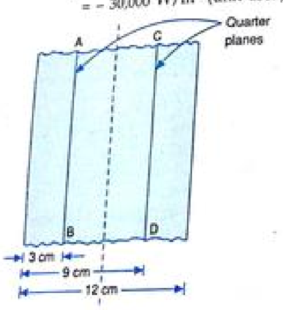


Fig. 4.7.

The negative sign signifies that the heat flow rate at the quarter plane AB is in a direction opposite to that of measurement of distance (– ve x-direction).

The heat conducted at a section equals the heat generated from mid section to that section.

∴ Heat conduction at section AB

= Heat generated from mid section to section AB

= q_g × (A × distance from mid section to section AB)

= $10^6 \times (1 \times 0.03)$

$$= 30{,}000 \text{ W/m}^2$$

At the quarter plane CD :

x = 9 cm = 0.09 m and the different parameters would work out to be

$$t = 156.75°C;$$

$$\frac{dt}{dx} = -150°C/W$$

and $Q = 30{,}000 \text{ W/m}^2$

(*b*) Because of symmetry, the maximum temperature occurs at the mid-plane, *i.e.*, at

$$x = \frac{\delta}{2}$$

$$\therefore \quad t_{max} = \frac{q_g}{2k}\left(\delta - \frac{\delta}{2}\right)\frac{\delta}{2} + t_w = \frac{q_g}{8k}\delta^2 + t_w$$

$$= \frac{10^6}{8 \times 200} \times (0.12)^2 + 150$$

$$= 9 + 150 = \mathbf{159°C}$$

EXAMPLE 4.5

Consider a 1.2 m thick slab of poured concrete (k = 1.148 W/m-deg) with both of side surfaces maintained at a temperature of 20°C. During its curing, chemical energy is released at the rate of 80 W/m³. Presuming that temperature does not vary with time, workout the maximum temperature of concrete.

(b) What maximum thickness of concrete can be poured without causing the temperature gradient to exceed 98.5°C per metre any where in the slab?

Solution : With uniform heat generation and both sides of the slab maintained at the same temperature, the temperature distribution is prescribed by the relation,

$$t = \frac{q_g}{2k}(\delta - x)x + t_w \quad \text{...(i)}$$

Because of symmetry, the maximum temperature occurs at the mind plane, *i.e.*, at $x = \delta/2$

$$\therefore \quad t_{max} = \frac{q_g}{2k}\left(\delta - \frac{\delta}{2}\right)\frac{\delta}{2} + t_w = \frac{q_g}{8k}\delta^2 + t_w$$

$$= \frac{80}{8 \times 1.48} \times (1.2)^2 + 20 = 29.73°C$$

(*b*) Differentiating expression (*i*) with respect to x,

$$\frac{dt}{dx} = \frac{q_g}{2k}(\delta - 2x)$$

Apparently the temperature gradient is largest at $x = 0$, and

$$\left(\frac{dt}{dx}\right)_{max} = \frac{q_g}{2k}\delta$$

Substituting the given values,

$$98.5 = \frac{80}{2 \times 1.48}\delta = 27.03\,\delta$$

∴ Maximum permissible thickness of slab,

$$\delta = \frac{98.5}{27.03} = \mathbf{3.64 \ m}$$

EXAMPLE 4.6

A 25 mm thick meat slab (k = 1 W/m-deg) is roasted with the help of microwave heating. For good quality roasting, it is desired that centre temperature of the slab be maintained at 100°C when the surrounding temperature is 25°C. What should be the heating capacity in W/m³ of the microwave if the heat transfer coefficient on the surface of meat slab is 20 W/m²-deg? Also calculate the surface temperature of the slab.

Solution : Maximum temperature occurs at the centre of slab and it is prescribed by the relation

$$t_{max} = t_a + \frac{q_g\delta}{2h} + \frac{q_g\delta^2}{8k}$$

$$= t_a + q_g\left[\frac{\delta}{2h} + \frac{\delta^2}{8k}\right]$$

Inserting the appropriate values,

$$100 = 25 + q_g\left[\frac{0.025}{2 \times 20} + \frac{0.025^2}{8 \times 1}\right]$$

$$= 25 + q_g\,(0.00625 + 0.0000781)$$

$$= 25 + 0.0007031\,q_g$$

∴ Heating capacity of microwave, q_g

$$= \frac{100 - 25}{0.0007031}$$

$$= 102599 \text{ W/m}^3 = 102.6 \text{ kW/m}^3$$

(*b*) Heat conducted to a section equals the heat generated from mid section to that section.

Heat conducted to the surface

$$= q_g \times \left(A \times \frac{\delta}{2}\right)$$

The heat conducted to the wall surface is finally dissipated to the surroundings at temperature t_a. Then for each surface

$$q_g A\frac{\delta}{2} = hA(t_w - t_a)$$

or

$$t_w = t_a + \frac{q_g}{2h}\delta$$

$$= 25 + \frac{102599}{2 \times 20} \times 0.025 = \mathbf{64.12°C}$$

EXAMPLE 4.7

A long stainless steel bar 20 mm × 20 mm in square cross-section is perfectly insulated on three sides and is maintained at a temperature of 400°C on the remaining side. Determine the maximum temperature in the bar when it is conducting a current of 1000 ampere. Take thermal and electrical conductivities of steel as 16 W/m-deg and 1.5 × 10⁴/ohm-cm and neglect the edge effects. Also work out the heat flow from the bar.

Solution : The heat generated per unit volume due to flow of electric current is worked out from the relation.

$$q_g = \left(\frac{I}{A}\right)^2 \rho$$

where ρ is resistivity in ohm-cm, *i.e.*, reciprocal of electrical conductivity

$$q_g = \left(\frac{1000}{2 \times 2}\right)^2 \times \frac{1}{1.5 \times 10^4}$$

$$= 4.167 \text{ W/cm}^3$$

$$= 4.167 \times 10^6 \text{ W/m}^3$$

The temperature distribution through the bar is prescribed by the relation

$$t = \left[\frac{q_g}{2k}(\delta - x) + \frac{t_2 - t_1}{\delta}\right]x + t_1$$

Maximum temperature occurs at the centre, *i.e.*, at $x = \delta/2$. Further under the given

boundary conditions : $t_1 = t_2 = 400°C$.
Therefore,

$$t_{max} = \frac{q_g}{2k}\left(\delta - \frac{\delta}{2}\right)\frac{\delta}{2} + t_1$$

$$= \frac{q_g}{8k}\delta^2 + t_1$$

$$= \frac{4.167 \times 10^6}{8 \times 16}(0.02)^2 + 400$$

$$= 413.02°C$$

Under steady state conditions, the heat flow through the bar equals the heat generated within it

$$= q_g \times \text{volume of the bar}$$

$$= (4.167 \times 10^6) \times (0.02 \times 0.02 \times 1)$$

$$= 1.67 \times 10^3 \text{ W/m length of the bar}$$

EXAMPLE 4.8.

A metallic rod of 6.5 mm diameter and 1.25 m length runs between two large bus bars which are at 25°C temperature. The lateral surface of the rod is insulated against the flow of heat and electric current. Determine the maximum current the rod can carry if its temperature is not to exceed 180°C at any point. For the material of rod,

electric resistivity = 1.75×10^{-6} ohm cm

thermal conductivity = 250 W/mK

Solution : With one-dimensional heat flow through the rod with both ends maintained at the same temperature, the temperature distribution is prescribed by the relation

$$t = \frac{q_g}{2k}(\delta - x)x + t_w$$

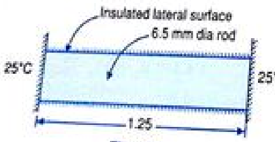


Fig. 4.8.

Further, due to symmetry the maximum temperature would occur at the mid plane, *i.e.*, at $x = \delta/2$

$$t_{max} = \frac{q_g}{2k}\left(\delta - \frac{\delta}{2}\right)\frac{\delta}{2} + t_w = \frac{q_g}{8k}\delta^2 + t_w$$

Substituting the appropriate values, we get

$$180 = \frac{q_g}{8 \times 250} \times 1.5^2 + 25$$

Then heat generated per unit volume due to flow of current,

$$q_g = \frac{(180 - 25) \times 8 \times 250}{1.5^2}$$

$$= 198400 \text{ W/m}^3$$

Rate of heat generation Q_g

$$= q_g \times \text{volume}$$

$$= 198400 \times \frac{\pi}{4}(0.0065)^2 \times 1.25$$

$$= 8.22 \text{ W}$$

In terms of current flowing and resistance of rod,

$$Q_g = I^2 R = I^2 \times \frac{\rho l}{A}$$

Given : Resistivity ρ

$$= 1.75 \times 10^{-6} \text{ ohm cm}$$

$$= 1.75 \times 10^{-8} \text{ ohm m}$$

$$\text{Then } Q_g = I^2 \times \frac{1.75 \times 10^{-8} \times 1.25}{\frac{\pi}{4}(0.0065)^2}$$

$$= 6.595 \times 10^{-4} I^2$$

$\therefore$ Maximum current the rod can carry,

$$= \sqrt{\frac{8.22}{6.595 \times 10^{-4}}} = 111.64 \text{ amp}$$

EXAMPLE 4.9

A stack of laminated sheets is fabricated from 3 mm thick single sheets of a plastic (Thermal conductivity k = 0.2 W/m-deg) by using suitable adhesive. Any distortion resulting during hardening of the adhesive is prevented by clamping the assembled sheets between steel plates. The heat evolved during the hardening process is equivalent to a steady uniform heat generation of 100 W/m³ and this heat is suitably dissipated to maintain the steel plates at a temperature of 170°C. If the maximum permissible temperature in the stack is limited to 180°C, calculate the number of laminated sheets that can be processed at any one time. Heat transfer in the stack may be assumed to be one dimensional.

Solution : The temperature distribution through the stack of laminated sheets is prescribed by the relation,

$$t = \left[\frac{q_g}{2k}(\delta - x) + \frac{t_2 - t_1}{\delta}\right]x + t_1$$

Maximum temperature occurs at the centre, *i.e.*, at $x = \delta/2$. Further under the given boundary conditions : $t_1 = t_2 = 170°C$.
Therefore,

$$t_{max} = \frac{q_g}{2k}\left(\delta - \frac{\delta}{2}\right)\frac{\delta}{2} + t_1 = \frac{q_g}{8k}\delta^2 + t_1$$

Substituting the given values,

$$180 = \frac{100}{8 \times 0.2}\delta^2 + 170$$

$$\delta^2 = \frac{10 \times 8 \times 0.2}{100} \; ; \; \delta = 0.4 \text{ m}$$

Therefore the maximum thickness of the stack is 0.4 m. The number of laminated sheets in the stack would be

$$\frac{0.4}{3 \times 10^{-3}} = 133.3$$

Obviously the maximum number of sheets should be limited to 133.

EXAMPLE 4.10

A flat plate fuel element for a nuclear reactor is 6 mm thick and is clad on each face with aluminium 1 mm thick. The rate of heat generation is uniform within the element and has a magnitude of 5.6×10^8 W/m³. If the coolant temperature is 140°C, make calculations for the temperature at the free surface of the aluminium, the aluminium/uranium interface, and at the centre of the fuel element.

Thermal conductivity for uranium and aluminium are 24.5 W/mK and 200 W/mK respectively, and the heat transfer coefficient at the aluminium/coolant interface is 2.85×10^4 W/m²K.

Solution : A sketch of the arrangement of the fuel element is shown in Fig. 4.9. Let Q_g be the total heat generated within the uranium per unit area of the plate.

$$Q_g = 5.6 \times 10^8 \times (1 \times 0.006)$$

$$= 33.6 \times 10^5 \text{ W per unit area}$$

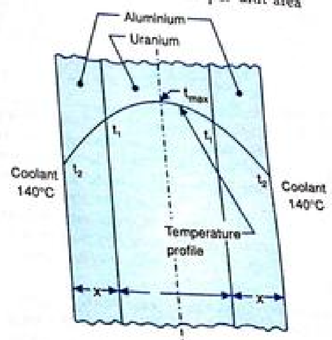


Fig. 4.9.

Then heat flow in each normal direction is 16.8×10^5 W/m². Under steady state conditions, this heat is first conducted through the aluminium cladding and then convected to the coolant.

$$16.85 \times 10^5 = \frac{kA}{x}(t_1 - t_2) = hA(t_2 - 140)$$

where t_2 is the aluminium/coolant interface temperature, t_1 is the aluminium/uranium interface temperature and x is the thickness of aluminium cladding.

Taking $A = 1$ m², we obtain,

$$t_2 = \frac{16.85 \times 10^5}{2.85 \times 10^4 \times 1} + 140 = 199.12°C$$

$$t_1 = \frac{16.85 \times 10^5 \times 0.2 \times 10^{-2}}{200 \times 1} + 199.12$$

$$= 215.97°C$$

Since the uranium fuel is subject to uniform internal heat generation, the maximum temperature within this material will be described by the equation :

$$t_{max} = \frac{q_g}{8k}\delta^2 + t$$

Here $\delta = 6$ mm $= 6 \times 10^{-3}$ m and t is the common temperature on both surfaces of the uranium fuel $= 215.97°C$.

$$\therefore \quad t_{max} = \frac{5.6 \times 10^8}{8 \times 24.5}(6 \times 10^{-3})^2 + 215.97$$
$$= \mathbf{318.82°C}$$

EXAMPLE 4.11

A 2 cm thick steel plate of thermal conductivity 50 W/m-deg has a uniform volumetric heat generation of 40×10^{-6} W/m³. The temperature at one surface of the plate is 160°C and that at the other is 100°C. Workout the temperature distribution across the plate, value and position of the maximum temperature, and flow of heat from each surface of the plate. Neglect the end effects.

Solution : (*a*) This is essentially a one-dimensional problem with uniform heat generation throughout the plate. The temperature distribution is prescribed by the relation

$$t = \left[\frac{q_g}{2k}(\delta - x) + \frac{t_2 - t_1}{\delta}\right]x + t_1$$

$$= \left[\frac{40 \times 10^6}{2 \times 50}(0.02 - x) + \frac{100 - 160}{0.02}\right]x + 160$$

$$= 160 + 5 \times 10^3 x - 4 \times 10^5 x^2$$

The temperature distribution is parabolic. The temperature t is in degrees celsius and the distance x is in metres.

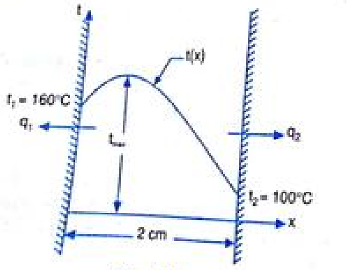


Fig. 4.10.

(*b*) The position of maximum temperature determined by differentiating the expression for temperature distribution with respect to x and setting the result equal to zero. That is,

$$\frac{dt}{dx} = 5 \times 10^3 - 8 \times 10^5 x = 0$$

and therefore,

$$x = \frac{5 \times 10^3}{8 \times 10^5} = 0.00625 \text{ m}$$

The value of maximum temperature is

$$t_{max} = 160 + 5 \times 10^3 \times 0.00625 - 4 \times 10^5 (0.00625)^2$$
$$= 160 + 31.25 - 15.625$$
$$= \mathbf{175.625°C}$$

(*c*) The heat flow at the left face ($x = 0$) is

$$q_1 = -kA\left(\frac{dt}{dx}\right)_{x=0}$$
$$= -50 \times 1 \times (5 \times 10^3 - 8 \times 10^5 x)_{x=0}$$
$$= \mathbf{-250 \times 10^3\ W/m^2}$$

The –ve sign signifies that the heat flow at the left face is in a direction opposite to that of measurement of distance (–ve x-direction)

The heat flow at the right face ($x = 0.02$ m) is

$$q_2 = 50 \times 1 \times (5 \times 10^3 - 8 \times 10^5 x)_{x=0.02}$$
$$= \mathbf{+550 \times 10^3\ W/m^2}$$

The positive sign signifies that heat flow at the right face is in the +ve x-direction.

Check : The sum of q_1 and q_2 which is equal to 800×10^3 W/m² must equal the total heat generated per unit length of plate. Thus

$$Q_g = 40 \times 10^6 \times (0.02 \times 1)$$
$$= \mathbf{800 \times 10^3\ W/m^2}$$

EXAMPLE 4.12

A plane wall 0.5 m thick (k = 20 W/m-deg) is insulated on one side while the other side is maintained at a constant temperature of 350°C. The rate of heat generation within the wall is 500 W/m³. Determine the maximum temperature to which the wall will be subjected and location of the plane where it occurs.

Solution : With uniform heat generation and both sides of plane wall maintained at the same temperature, the temperature distribution is prescribed by the relation

$$t = t_w + \frac{q_g}{2k}(\delta - x)x$$

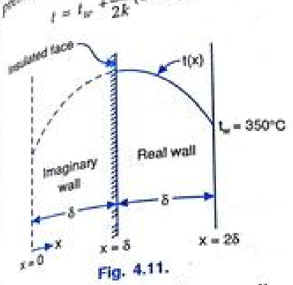


Fig. 4.11.

The above relation applies equally well to a plane wall which is perfectly insulated at one face and maintained at a fixed temperature on the other face. A full hypothetical wall with thickness 2δ can be considered with the same bounding conditions :

$$t = t_w \text{ at } x = 0 \text{ and at } x = 2\delta$$

The location $x = \delta$ corresponds to mid plane of the hypothetical wall or insulated face of the given wall. Thus for an insulated wall of thickness δ, the temperature distribution conforms to the relation

$$t = t_w + \frac{q_g}{2k}(2\delta - x)x$$

The maximum temperature occurs at mid plane (insulated face of the given wall) where $x = \delta$

$$t_{max} = t_w + \frac{q_g}{2k}(2\delta - \delta)\delta$$
$$= t_w + \frac{q_g}{2k}\delta^2$$
$$= 350 + \frac{500}{2 \times 20}(0.5)^2 = \mathbf{353.125°C}$$

EXAMPLE 4.13

An 8 cm thick plane wall generates heat at the rate of 1.2×10^5 W/m³. One side of the wall is exposed to environment at 90°C whilst the other side is insulated. The convective heat transfer coefficient between the wall and environment is 560 W/m²-deg. Proceed from the basic principles to determine the maximum temperature to which the wall will be subjected. The thermal conductivity of the wall material may be taken as 0.15 W/m-deg

Solution : The appropriate form of one-dimensional steady state conduction heat equation is

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0$$

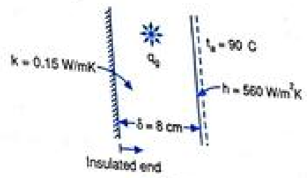


Fig. 4.12.

Upon integration

$$\frac{dt}{dx} = -\frac{q_g}{k}x + C_1 \quad \text{...(i)}$$

Another integration gives the general solution for temperature distribution

$$t = -\frac{q_g}{2k}x^2 + C_1 x + C_2 \quad \text{...(ii)}$$

The constants of integration are determined from the relevant boundary conditions which are :

(*i*) At $x = 0$, the conduction region is perfectly insulated and hence heat flow is zero. From Fourier's law $Q = -kA\,(dt/dx)$, and accordingly the temperature derivative must be zero at $x = 0$. Hence using expression (*i*), we get : $C_1 = 0$.

(*ii*) At $x = \delta$, the heat conduction equals the convective heat flow to the surroundings. That is

$$-kA\left|\frac{dt}{dx}\right|_{x=\delta} = hA\left[t(\delta) - t_a\right]$$

$$\text{or } -\left|\frac{dt}{dx}\right|_{x=\delta} = \frac{h}{k}[t(\delta) - t_a] \quad ...(iii)$$

Again from expression (*i*) and noting that $C_1 = 0$

$$\left|\frac{dt}{dx}\right|_{x=\delta} = -\frac{q_g}{k}\delta \quad ...(iv)$$

From expression (*iii*) and (*iv*),

$$t(\delta) = t_a + \frac{q_g}{h}\delta$$

$$\therefore \quad C_2 = t_a + \frac{q_g}{h}\delta + \frac{q_g}{2k}\delta^2 \quad ...(v)$$

With these value of constants, the general solution for temperature distribution becomes

$$t = -\frac{q_g}{2k}x^2 + t_a + \frac{q_g}{h}\delta + \frac{q_g}{2k}\delta^2$$

$$= t_a + \frac{q_g}{h}\delta + \frac{q_g}{2k}(\delta^2 - x^2) \quad ...(vi)$$

The maximum temperature occurs at the insulated wall boundary, *i.e.*, at $x = 0$

$$\therefore \quad t_{max} = t_a + \frac{q_g}{h}\delta + \frac{q_g}{2k}\delta^2$$

$$= 90 + \frac{1.2\times10^5\times0.08}{560} + \frac{1.2\times10^5\times(0.08)^2}{2\times0.15}$$

$$= 2667.15°C$$

EXAMPLE 4.14

An infinite slab of 20 cm thickness and thermal conductivity 20 W/mK separates two fluids having temperatures 35°C and 25°C respectively. The heat transfer coefficient on the hot fluid side is 25 W/m²K and that on the cold fluid side is 50 W/m²K. If the heat generation in the slab is at uniform rate of 6 kW/m³, set up an expression for the temperature distribution in the slab. Proceed to determine :

(a) maximum temperature in the slab and its location,

(b) temperature at the centre of slab and at the two surfaces, and

(c) heat transferred from each surface.

Solution : Refer Fig. 4.13 for conduction in the infinite slab with uniform heat generation and different temperatures at the two surfaces. The appropriate heat flow equation is

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0$$

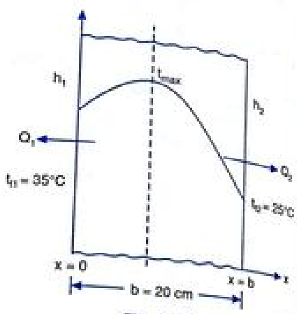


Fig. 4.13.

Integrating this equation, we obtain the general solution as

$$\frac{dt}{dx} = -\frac{q_g}{k}x + C_1 \quad ...(i)$$

$$t = -\frac{q_g}{2k}x^2 + C_1x + C_2 \quad ...(ii)$$

At the left surface, *i.e.*, at $x = 0$

rate of heat conduction

= rate of heat convection

$$k\left.\frac{dt}{dx}\right|_{x=0} = h_1\left[t_{x=0} - t_{f_1}\right] \quad \left[\frac{dt}{dx}\text{ is +ve}\right]$$

Using expressions (*i*) and (*ii*), we get

$$kC_1 = h_1(C_2 - t_{f1})$$

$$\text{or} \quad C_2 = \frac{k}{h_1}C_1 + t_{f1} \quad ...(iii)$$

At the right surface, *i.e.*, at $x = b$

rate of heat conduction

= rate of heat convection

$$-k\left.\frac{dt}{dx}\right|_{x=b} = h_2\left[t_{x=b} - t_{f_2}\right]$$

$$\text{or } -k\left[-\frac{q_g}{k}b + C_1\right] = h_2\left[-\frac{q_g}{2k}b^2 + C_1b + C_2 - t_{f_2}\right]$$

Substituting the value of C_2 as per expression (*iii*), and solving for C_1, we obtain

$$-k\left[-\frac{q_g}{k}b + C_1\right] = h_2\left[-\frac{q_g}{2k}b^2 + C_1b + \frac{k}{h_1}C_1 - t_{f_1} + t_{f_2}\right]$$

$$\text{or } C_1\left(b + \frac{k}{h_1} + \frac{k}{h_2}\right) = \frac{q_g}{h_2}b + \frac{q_g}{2k}b^2 - t_{f1} + t_{f2}$$

$$\text{or } C_1 = \frac{\frac{q_g}{h_2}b + \frac{q_g}{2k}b^2 - t_{f_1} + t_{f_2}}{b + \frac{k}{h_1} + \frac{k}{h_2}}$$

$$= \frac{\frac{6000}{50}\times0.2 + \frac{6000}{2\times20}\times0.2^2 - 35 + 25}{0.2 + \frac{20}{25} + \frac{20}{50}}$$

$$= \frac{24 + 6 - 10}{0.2 + 0.8 + 0.4} = 14.28$$

Then :

$$C_2 = \frac{k}{h_1}C_1 + t_{f1}$$

$$= \frac{20}{25}\times14.28 + 35 = 46.42$$

Accordingly the temperature distribution is prescribed by the relation,

$$t = -\frac{q_g}{2k}x^2 + 14.28x + 46.42 \quad ...(iv)$$

(*a*) Let the maximum temperature in the slab be at $x = x_m$. Then

$$\frac{dt}{dx} = 0 \quad \text{at} \quad x = x_m$$

That is $0 = -\frac{q_g}{k}x_m + C_1$

$$\text{or} \quad x_m = \frac{C_1}{q_g}k = \frac{14.28}{6000}\times20 = 0.0476 \text{ m}$$

Then the maximum temperature t_{max} can be worked and by substituting the value of $x = x_m = 0.0476$ m in the expression for temperature distribution.

$$t_{max} = -\frac{6000}{2\times20}\times(0.0476)^2 + 14.28\times0.0476 + 46.42$$

$$= -0.3398 + 0.6797 + 46.42$$

$$= 46.76°C$$

(*b*) At $x = 0$

$$t = C_2 = 46.42°C$$

At $x = b = 0.2$ m

$$t = -\frac{6000}{2\times20}\times(0.2)^2 + 14.28\times0.2 + 46.42$$

$$= -6 + 2.856 + 46.42$$

$$= 43.276°C$$

At $x = \frac{b}{2} = 0.1$ m

$$t = -\frac{6000}{2\times20}\times(0.1)^2 + 14.28\times0.1 + 46.42$$

$$= -1.5 + 1.428 + 46.42$$

$$= 46.348°C$$

(*c*) Heat transfer from left face, *i.e.*, at $x = 0$

$$Q_1 = kA\left.\frac{dt}{dx}\right|_{x=0} = kAC_1$$

$$\therefore \quad \frac{Q_1}{A} = kC_1 = 20\times14.28$$

$$= 28.56 \text{ J/m}^2$$

Heat transfer from right face, *i.e.*, at $x = b = 0.2$ m

$$Q_2 = kA\left[-\frac{q_g}{k}b + C_1\right]$$

$$\therefore \quad \frac{Q_2}{A} = 20\left[-\frac{6000\times0.2}{20} + 14.28\right]$$
$$= 914.4 \text{ J/m}^2$$

EXAMPLE 4.15.

A solid rod (thermal conductivity 40 W/mK) of 50 cm length and 5 cm² uniform cross-sectional area has been welded to two large plates which consequently stand separated. An insulation is filled in the space between the two plates and that provides insulation to the circumferential surface of the rod. There exists voltage difference between the two plates and as such current flows through the rod and 25 W electrical energy is dissipated. If the temperatures at the plates are 150°C and 75°C, set up an expression for temperature distribution in the rod. Also determine:

(i) the maximum temperature in the rod and its location

(ii) the heat flux at each end

Solution : The appropriate heat flow equation is

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0$$

Fig. 4.14.

Upon integration of this equation, we obtain the general solution as

$$\frac{dt}{dx} = -\frac{q_g}{k}x + C_1$$

and
$$t = -\frac{q_g}{2k}x^2 + C_1 x + C_2$$

Given :

$$q_g = \frac{q}{V} = \frac{q}{AL}$$
$$= \frac{25}{5\times10^{-4}\times0.5}$$
$$= 10^5 \text{ W/m}^3$$

Invoking the boundary conditions,

$t = t_1 = 150°C$ at $x = 0$

and $t = t_2 = 75°C$ at $x = l$, we get

$$C_2 = t_1 = 150$$

and
$$C_1 = \frac{t_2 - t_1}{l} + \frac{q_g l}{2k}$$
$$= \frac{75-150}{0.5} + \frac{10^5\times0.5}{2\times40} = 475$$

Accordingly, the temperature distribution in the rod is prescribed by the relation

$$t = -\frac{q_g}{2k}x^2 + 475\,x + 150$$
$$= -\frac{10^5}{2\times40}x^2 + 475\,x + 150$$
$$= -1250\,x^2 + 475\,x + 150$$

(*a*) Let the maximum temperature in the rod be at $x = x_m$

Then $\frac{dt}{dx} = 0$ at $x = x_m$

That is $0 = -\frac{q_g}{k}x_m + C_1$

or
$$x_m = \frac{C_1 k}{q_g} = \frac{475\times40}{10^5} = 0.19 \text{ m}$$

(*b*) Heat flow at left end, *i.e.*, at $x = 0$

$$Q_1 = -kA\left.\frac{dt}{dx}\right|_{x=0}$$
$$= -kA\,C_1$$
$$= -40\times(5\times10^{-4})\times475 = 9.5 \text{ W}$$

Heat flow at right end, *i.e.*, at $x = l = 0.5$ m

$$Q_2 = -kA\left.\frac{dt}{dx}\right|_{x=l}$$
$$= -kA\left[-\frac{q_g}{k}l + C_1\right]$$
$$= -40\times(5\times10^{-4})\left[-\frac{10^5}{40}\times0.5 + 475\right]$$
$$= -0.02\times(-775) = 15.5 \text{ W}$$

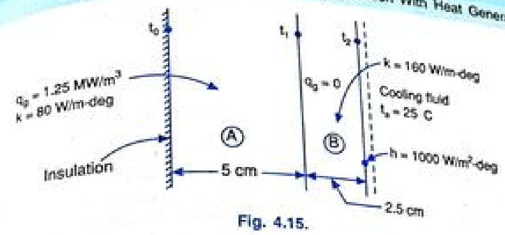


Fig. 4.15.

EXAMPLE 4.16

The plane wall A of thickness 5 cm and thermal conductivity 80 W/m-deg and having volumetric heat generation of 1.25 MW/m³ is insulated on one side while the other side is in contact with surface of another wall B. The wall B has no heat generation, is made of a material of thermal conductivity 160 W/m-deg and has a thickness of 2.5 cm. The non-contact surface of wall B is exposed to a cooling fluid at 25°C. If the convective heat transfer coefficient between wall B and the fluid is 1000 W/m²-deg, calculate temperature at the insulated surface and that at the cooled surface of this composite wall.

Solution : Considering unit surface area, the heat generated in layer *A* of the composite wall is,

$$Q_g = (1.25\times10^6)\times(0.05\times1)$$
$$= 62500 \text{ W per unit area of wall}$$

As the face at $x = 0$ is insulated, there is no flow of any heat at this surface. All the heat generated would pass through wall B and finally dissipated to the surrounding cooling heat fluid. That is

$$62500 = hA\,(t_2 - t_a)$$
$$= 1000\times1\times(t_2 - t_a)$$
$$= 1000\,(t_2 - 25)$$

$\therefore$ Temperature at the cooled surface,

$$t_2 = \frac{62500}{1000} + 25 = 87.5°C$$

Invoking Fourier law for heat conduction through wall B,

$$Q = \frac{kA(t_1 - t_2)}{\delta}$$
$$62500 = \frac{160\times1\times(t_1 - 87.5)}{0.025}$$
$$= 6400\,(t_1 - 87.5)$$

$\therefore$ Temperature at contact surface between the two walls,

$$t_1 = \frac{62500}{6400} + 87.5 = 97.26°C$$

The wall *A* represents an insulated slab of thickness 5 cm with one end maintained at a fixed temperature t_1. The temperature t_0 at the insulated surface is then given by

$$t_0 = \frac{q_g}{2k}\delta^2 + t_1$$
$$= \frac{1.25\times10^6}{2\times80}(0.05)^2 + 97.26$$
$$= 116.79°C$$

EXAMPLE 4.17

A rectangular copper conductor, 7 mm × 5 mm in cross-section, rests in an insulation trough so that the heat transfer from one face and both edges is negligible. Measurements indicate that when a current of 6500 ampere flows through the conductor, a steady state temperature of 52°C prevails at its bare face. Proceed from basic principles to determine the maximum temperature experienced by the bar and its location. Also find the temperature at the centre of the bar. For copper :

Thermal conductivity k = 375 W/mK

Resistivity $\rho = 2\times10^{-8}\ \Omega$ m

Solution : *Under specified conditions of one-dimensional and steady state, the conduction equation may be written as*

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0$$

where q_g is the rate of heat generation per unit volume.

Upon integration,

$$\frac{dt}{dx} = -\frac{q_g}{k}x + C_1 \qquad ...(i)$$

$$t = -\frac{q_g}{k}\frac{x^2}{2} + C_1x + C_2 \qquad ...(ii)$$

The constants of integration C_1 and C_2 can be evaluated by applying the following boundary conditions :

(i) At any distance x from the insulated face

$$Q = -kA\left(\frac{dt}{dx}\right)_x \text{ (Fourier Equation)}$$

At $x = 0$, $Q = 0$ and accordingly $dt/dx = 0$

That gives $C_1 = 0$

(ii) At $x = 0.005$ m, $t = 52°C$

$$Q_g = I^2R = I^2\frac{\rho l}{A}$$

$$q_g = I^2\frac{\rho l}{A} \times \frac{1}{AL} = \left(\frac{I}{A}\right)^2 \times \rho$$

$$= \left(\frac{6500}{0.075 \times 0.005}\right)^2 \times (2 \times 10^{-8})$$

$$= 6.10^6 \text{ W}$$

From expression (ii)

$$C_2 = t + \frac{q_g}{k}\frac{x^2}{2} \quad \text{...(because } C_1 = 0)$$

$$= 52 + \frac{6 \times 10^6}{375} \times \frac{(0.005)^2}{2} = 52.2$$

The expression for temperature distribution then takes the form :

$$t = -\frac{q_g}{k}\frac{x^2}{2} + 52.2$$

Maximum temperature occurs at the insulated face ($x = 0$) and its value is

$$t = 52.2°C$$

At the midpoint $x = 0.0025$ m

$$\therefore \quad t_{mid} = -\frac{6 \times 10^6}{375} \times \frac{(0.0025)^2}{2} + 52.2$$

$$= 52.15°C$$

EXAMPLE 4.18

Two large steel plates at temperature of 90°C and 70°C are separated by a steel rod 2.5 cm diameter and 0.25 m long. The rod is welded to each plate. The space between the plates is filled with insulation which also insulates the circumference of the rod. Because of voltage difference between the two plates current flows through the rod and the electrical energy is dissipated at a rate of 10 W. Determine the maximum temperature in the rod and the heat flux at each end. Proceed to compare the net heat flow rate of the two ends with the total rate of heat generation.

Thermal conductivity for the rod material is 42.5 W/m-deg.

Solution : Heat generated per unit volume

$$q_g = \frac{10}{\frac{\pi}{4}(0.025)^2 \times 0.25}$$

$$= 81528 \text{ W/m}^3$$

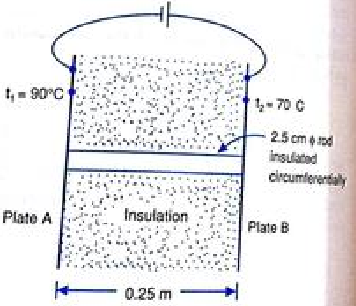


Fig. 4.16.

The appropriate heat conduction equation for the rod is

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0$$

Integrating twice :

$$\frac{dt}{dx} + \frac{q_g}{k}x = C_1$$

$$t + \frac{q_g}{2k}x^2 = C_1x + C_2$$

The appropriate boundary conditions are:

(i) At $x = 0$, $t = t_1$

That gives $C_2 = t_1$

(ii) At $x = \delta$, $t = t_2$

$$t_2 + \frac{q_g}{2k}\delta^2 = C_1\delta + t_1$$

$$C_1 = \frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta$$

With these integration constants, the temperature distribution may be written as :

$$t = -\frac{q_g}{2k}x^2 + \frac{t_2 - t_1}{\delta}x + \frac{q_g\delta}{2k}x + t_1 \qquad ...(i)$$

The maximum temperature occurs where the derivative $dt/dx = 0$

$$\frac{dt}{dx} = -\frac{q_g}{k}x + \frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta = 0$$

$$\therefore \quad x = \frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}$$

Substituting this value of x in expression (i), we get

$$t_{max} = -\frac{q_g}{2k}\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right]^2 + \frac{t_2 - t_1}{\delta}\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right] + \frac{q_g}{2k}\delta\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right] + t_1$$

$$t_{max} = t_1 - \frac{q_g}{2k}\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right]^2 + \left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right]\left[\frac{q_g}{2k}\delta + \frac{t_2 - t_1}{\delta}\right]$$

$$= t_1 - \frac{q_g}{2k}\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right]^2 + \frac{q_g}{k}\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right]^2$$

$$= t_1 + \frac{q_g}{2k}\left[\frac{k(t_2 - t_1)}{q_g\delta} + \frac{\delta}{2}\right]^2$$

Substituting the relevant values,

$$t_{max} = 90 + \frac{81528}{2 \times 42.5} \times \left[\frac{42.5(70 - 90)}{81528 \times 0.25} + \frac{0.25}{2}\right]^2$$

$$= 90 + 959.15\ (0.006938)$$

$$= 96.65°C$$

Heat flow from the rod at $x = 0$ is

$$Q_1 = -kA\left.\frac{dt}{dx}\right|_{x=0}$$

$$= -kA\left[-\frac{q_g}{k}x + \frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta\right]_{x=0}$$

$$= -kA\left[\frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta\right]$$

$$= -42.5 \times \frac{\pi}{4}(0.025)^2 \times \left[\frac{70 - 90}{0.25} + \frac{81528 \times 0.25}{2 \times 42.5}\right]$$

$$= -3.332 \text{ W}$$

Heat flow from the rod of $x = \delta$ is

$$Q_2 = -kA\left.\frac{dt}{dx}\right|_{x=\delta}$$

$$= -kA\left[-\frac{q_g}{k}\delta + \frac{t_2 - t_1}{\delta} + \frac{q_g}{2k}\delta\right]$$

$$= -kA\left[\frac{t_2 - t_1}{\delta} - \frac{q_g}{2k}\delta\right]$$

$$= -42.5 \times \frac{\pi}{4}(0.025)^2 \times \left[\frac{70 - 90}{0.25} - \frac{81528 \times 0.25}{2 \times 42.5}\right]$$

$= 6.668 \text{ W}$

The sum of heat losses from the rod is

$= (3.332 + 6.668)$

$= 10 \text{ W}$

and that is the same as electrical power generated.

4.2. DIELECTRIC HEATING

Quite often, insulating material like wool-rubber-plastics and textiles etc., are quickly heated by applying high frequency, high voltage alternating current to the plates of a condenser with the insulating material lying between the plates. Heat generated is directly proportional to voltage, frequency, area of condenser plates and inversely proportional to distance between the plates. The heat generated per unit volume of the material is constant, and the situation then corresponds to conduction in a plane wall with uniform internal heat source.

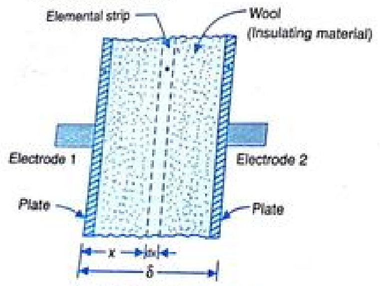


Fig. 4.17. Dielectric heating

Consider the insulating material to be packed in the form of slab between the plates of condenser. Let,

θ_1 = temperature of electrode 1 above the atmospheric temperature, $\theta_1 = (t_{w1} - t_a)$

θ_2 = temperature of electrode 2 above the atmospheric temperature, $\theta_2 = (t_{w2} - t_a)$

q_g = heat generated per unit volume of the dielectric medium (wool)

h_1, h_2 = heat transfer coefficients on the surface of condenser plates

The appropriate differential equation describing the temperature distribution can be obtained by making an energy balance on an elemental strip of thickness dx and at a distance x from the left hand electrode. For this strip θ is the difference between the material temperature and the atmospheric temperature.

Q_x (heat conducted in at distance x)

$$= -kA \frac{d\theta}{dx}$$

Q_g (heat generated due to dielectric heating)

$$= q_g\, A dx$$

Q_{x+dx} (heat conducted out at distance $x + dx$)

$$= Q_x + \frac{d}{dx}(Q_x)\, dx$$

For steady conduction of heat flow

$$Q_x + Q_g = Q_{x+dx}$$

$$= Q_x + \frac{d}{dx}(Q_x)\, dx$$

$$\therefore \quad Q_g = \frac{d}{dx}(Q_x)\, dx$$

$$q_g\, A dx = \frac{d}{dx}\left(-kA\frac{d\theta}{dx}\right) dx$$

$$= -kA\frac{d^2\theta}{dx^2} dx$$

This yields the appropriate form of heat equation as

$$\frac{d^2\theta}{dx^2} + \frac{q_g}{k} = 0 \qquad ...(i)$$

Integrating twice, we obtain the general solution for temperature distribution :

$$\frac{d\theta}{dx} + \frac{q_g}{k} x = C_1 \qquad ...(ii)$$

$$\theta + \frac{q_g}{2k} x^2 = C_1 x + C_2 \qquad ...(iii)$$

The constants of integration can be determined from the known boundary conditions.

Heat conducted at $x = 0$ equals the heat convected to the surrounding at ambient temperature t_a. That is

$$-kA\left|\frac{d\theta}{dx}\right|_{x=0} = h_1 A\,(t_{w1} - t_a)$$

$$kAC_1 = h_1 A\theta_1$$

$$C_1 = \frac{h_1\theta_1}{k}$$

$$\therefore \theta + \frac{q_g}{2k} x^2 = \frac{h_1\theta_1}{k} x + C_2$$

Also $\theta = \theta_1$, at $x = 0$ and therefore $C_2 = \theta_1$

The temperature distribution may thus be specified as

$$\theta = -\frac{q_g}{2k} x^2 + \frac{h_1\theta_1}{k} x + \theta_1 \qquad ...(iv)$$

Corresponding to electrode 2, $x = \delta$ and $\theta = \theta_2$

$$\therefore \quad \theta_2 = -\frac{q_g}{2k}\delta^2 + \frac{h_1\theta_1}{k}\delta + \theta_1 \qquad ...(v)$$

For steady state heat transfer, the total heat generated within the insulating material equals the surface heat loss from both the electrodes. That is

$$q_g A\delta = h_1 A\,(t_{w1} - t_a) + h_2 A\,(t_{w2} - t_a)$$

$$\text{or} \quad q_g\, \delta = h_1\,\theta_1 + h_2\,\theta_2 \qquad ...(vi)$$

The expressions (v) and (vi) can be solved to obtain the electrode temperature t_{w1} and t_{w2}.

With dielectric heating, there is uniform rise in temperature and the internal heat generated is of equal intensity on the surface and core.

EXAMPLE 4.19

A slab of insulation material of thickness 6 cm and thermal conductivity 1.4 kJ/m-hr-deg is placed between and is in contact with two parallel electrodes, and is then subjected to high frequency dielectric heating at a uniform rate of 140,000 kJ/m³-hr. When steady state conditions are attained the coefficients of combined radiation and convection for exposed electrode surfaces are 42 and 48 kJ/m²-hr-deg. If the atmospheric temperature is 25°C, make calculations for the (a) surface temperatures, and (b) location and magnitude of maximum temperature in the system.

Assume unidirectional heat flow and each electrode to be at a uniform temperature equal to that of the face of slab with which it is in contact.

Solution : With uniform dielectric heating, the general solution for temperature distribution is

$$\theta = -\frac{q_g}{2k}x^2 + \frac{h_1\theta_1}{k}x + \theta_1$$

$$= -\frac{140000}{2 \times 1.4}x^2 + \frac{42\theta_1}{1.4}x + \theta_1$$

$$= -50000\, x^2 + 30\,\theta_1 x + \theta_1 \qquad ...(i)$$

At $\quad x = 0.06$ m, $\theta = \theta_2$

$$\therefore \quad \theta_2 = -50000 \times 0.06^2 + 30\,\theta_1 \times 0.06 + \theta_1$$

$$= -180 + 1.8\,\theta_1 + \theta_1$$

$$= -180 + 2.8\,\theta_1 \qquad ...(ii)$$

For steady state conditions, the heat generated due to dielectric heating equals the convective heat loss from the electrode surfaces. That is

$$q_g\, A\delta = h_1\, A\theta_1 + h_2\, A\theta_2$$

$$q_g\,\delta = h_1\theta_1 + h_2\theta_2$$

$$140000 \times 0.06 = 42\,\theta_1 + 48\,\theta_2$$

$$\text{or} \quad 175 = 0.875\,\theta_1 + \theta_2 \qquad ...(iii)$$

From expressions (ii) and (iii)

$$175 = 0.875\,\theta_1 + (-180 + 2.8\,\theta_1)$$

$$= 3.675\,\theta_1 - 180$$

$$\therefore \quad \theta_1 = \frac{180 + 175}{3.675} = 96.60°\text{C}$$

$$\theta_2 = -180 + 96.60 \times 2.8 = 90.45°\text{C}$$

Therefore the electrode temperature are :

$$t_{w1} = 96.66 + 25 = 121.60°\text{C}$$

$$t_{w2} = 90.45 + 25 = 115.45°\text{C}$$

The location of maximum temperature can be worked out by differentiating expression (i) with respect to x and setting the derivative to zero

$$\frac{d\theta}{dx} = -50000 \times 2x + 30\,\theta_1 = 0$$

$$\therefore \quad x = \frac{30 \times 96.60}{50000 \times 2} = 0.02898$$
$$= 2.898 \text{ cm}$$

from the left hand electrode

The maximum temperature in the system is then obtained by susbtituting $x = 2.898$ cm in expression (*i*)

$$\theta_{max} = -50000 \times (0.02898)^2 + 30 \times 96.60 \times 0.02898 + 96.60$$
$$= -41.99 + 83.98 + 96.60$$
$$= 138.59°C$$
$$\therefore \quad t_{max} = 138.59 + 25 = 163.59°C$$

4.3. CYLINDER WITH UNIFORM HEAT GENERATION

Consider heat conduction through a long and cylindrical rod of radius R and length L.

Assumptions :

1. Steady state conditions
2. One-dimensional radial conduction
3. Constant thermal conductivity k
4. Uniform volumetric heat generation (q_g per unit volume) within the solid.

The situation corresponds to a current carrying wire or a fuel element in a nuclear reactor.

The appropriate differential equation describing the temperature distribution can be obtained by making an energy balance on a cylindrical shell of thickness dr and located at radius r.

Q_r (heat conducted in at radius r)

$$= -k\, 2\pi r l \frac{dt}{dr}$$

Q_g (heat generated in the element)

$$= (2\pi r\, dr\, l) q_g$$

Q_{r+dr} (heat conducted out at radius $r + dr$)

$$= Q_r + \frac{d}{dr}(Q_r)\, dr$$

For steady state conduction of heat flow,

$$Q_r + Q_g = Q_{r+dr}$$
$$= Q_r + \frac{d}{dr}(Q_r)\, dr$$

$$\therefore \quad Q_g = \frac{d}{dr}(Q_r)\, dr$$

$$(2\pi r\, dr\, l) q_g = \frac{d}{dr}\left(-k\, 2\pi r l \frac{dt}{dr}\right) dr$$

This yields the appropriate form of heat equation as

$$\frac{d}{dr}\left(r \frac{dt}{dr}\right) = -\frac{q_g}{k}\, r \qquad \text{...(4.15)}$$

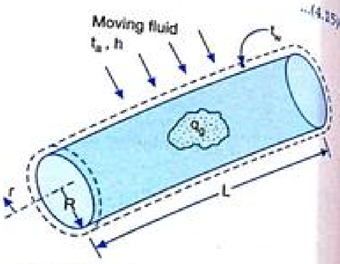


Fig. 4.18. Conduction a solid cylinder with uniform heat generation

Upon integration

$$r\frac{dt}{dr} = -\frac{q_g}{k}\frac{r^2}{2} + C_1$$

$$\frac{dt}{dr} = -\frac{q_g}{2k} r + \frac{C_1}{r} \qquad \text{...(4.16)}$$

Another integration gives the general solution for temperature distribution

$$t = -\frac{q_g}{4k} r^2 + C_1 \log_e r + C_2 \qquad \text{...(4.17)}$$

4.3.1. Solid Cylinder

The constants of integration are to be determined from the relevant boundary conditions which are:

(*i*) $t = t_w$ at $r = R$

(*ii*) Heat generated equals the heat lost by conduction at the surface

$$q_g (\pi R^2 L) = -k\, (2\pi RL) \left(\frac{dt}{dr}\right)_{r=R}$$

Another condition stems from the fact that for a solid cylinder the centre line is the line of symmetry for the temperature distribution and there the temperature gradient must be zero. This condition of $dt/dr = 0$ at $r = 0$ is satisfied automatically when the two boundary conditions are satisfied.

The temperature gradient at the surface is

$$\left(\frac{dt}{dr}\right)_{r=R} = -\frac{q_g}{2k} R + \frac{C_1}{R} \qquad \text{(Eqn. 4.16)}$$

Also

$$\left(\frac{dt}{dr}\right)_{r=R} = -\frac{q_g}{2k} R \qquad \text{(second boundary condition)}$$

$$\therefore \quad -\frac{q_g}{2k} R + \frac{C_1}{R} = -\frac{q_g}{2k} R$$

or $\quad C_1 = 0$

Applying the boundary condition $t = t_w$ at $r = R$ to equation 4.17

$$t_w = -\frac{q_g}{4k} R^2 + C_2 \quad \text{(because } C_1 = 0\text{)}$$

or $\quad C_2 = t_w + \frac{q_g}{4k} R^2$

With these values of integration constants, the general solution for the temperature distribution becomes

$$t = t_w + \frac{q_g}{4k}(R^2 - r^2) \qquad \text{...(4.18)}$$

Undoubtedly the temperature distribution is parabolic and the maximum temperature t_{max} occuring at the centre ($r = 0$) of the rod is given by

$$t_{max} = t_w + \frac{q_g}{4k} R^2 \qquad \text{...(4.19)}$$

Combining equation 4.18 and 4.19, we obtain the temperature distribution in the dimensionless form.

$$\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2 \qquad \text{...(4.20)}$$

From overall energy balance, the rate at which energy is generated within the solid cylinder must be balanced by the rate at which energy leaves by convection at the cylinder boundary,

$$q_g (\pi R^2 L) = h\, (2\pi RL)\, (t_w - t_a)$$

or $\quad t_w = t_a + \frac{q_g}{2h} R \qquad \text{...(4.21)}$

Substituting this value of surface temperature t_w in equation 4.18, one gets the temperature distribution in terms of temperature t_a of the surrounding atmosphere,

$$t = t_a + \frac{q_g}{2h} R + \frac{q_g}{4k}(R^2 - r^2) \qquad \text{...(4.22)}$$

and temperature t_{max} at $r = 0$

$$t_{max} = t_a + \frac{q_g}{2h} R + \frac{q_g}{4k} R^2 \qquad \text{...(4.23)}$$

EXAMPLE 4.20

A 25 mm diameter meat roll ($k = 1$ W/m-deg) is roasted with the help of microwave heating. For good quality roasting, it is desired that temperature at the centre of roll is maintained at 100°C when the surrounding temperature is 25°C. What should be the heating capacity in W/m³ of the microwave if the heat transfer coefficient on the surface of meat roll is 20 W/m²-deg ? Also calculate surface temperature of the roll.

Solution : Maximum temperature occurs at the centre of the roll and it is prescribed by the relation,

$$t_{max} = t_a + \frac{q_g R}{2h} + \frac{q_g R^2}{4k}$$
$$= t_a + q_g \left[\frac{R}{2h} + \frac{R^2}{4k}\right]$$

Inserting the appropriate values,

$$100 = 25 + q_g \left[\frac{0.0125}{2 \times 20} + \frac{0.025^2}{4 \times 1}\right]$$
$$= 25 + q_g (0.0003125 + 0.0000391)$$
$$= 25 + 0.0003516\, q_g$$

$\therefore$ Heating capacity of microwave,

$$q_g = \frac{100 - 25}{0.0003516}$$

$= 213310 \text{ W/m}^3$

$= 213.31 \text{ kW/m}^3$

(b) The rate at which energy is generated within the cylindrical roll is balanced by the rate at which energy leaves by convection at the roll boundary. That is

$$q_g\ (\pi R^2 l) = h\ (2\pi R l)\ (t_w - t_a)$$

or $$t_w = t_a + \frac{q_g}{2h} R$$

$$= 25 + \frac{213310}{2 \times 20} \times 0.0125 = 66.66°\text{C}$$

EXAMPLE 4.21

A stainless steel wire (conductivity = 20 W/m-deg and resistivity = 70 micro ohm-cm) of length 2 m and diameter 2.5 mm is submerged in a fluid at 50°C and an electric current of intensity 300 amps passes through it. If conductance at the wire surface is 4 kW/m²-deg, workout the steady state temperature at the centre and at the surface of the wire.

Solution : Electrical resistance of wire,

$$R_e = \frac{\rho l}{A}$$

$$= \frac{70 \times 10^{-6} \times 200}{\frac{\pi}{4} \times (0.25)^2} = 0.285\ \Omega$$

Heat generated, Q_g

$$= I^2 R_e = 300^2 \times 0.285 \text{ Watt}$$

Volume of wire, V

$$= \frac{\pi}{4} d^2 l = \frac{\pi}{4}\left(\frac{2.5}{1000}\right)^2 \times 2$$

$$= 9.81 \times 10^{-6} \text{ m}^3$$

Heat generated per unit volume, q_g

$$= \frac{300 \times 0.285}{9.81 \times 10^{-6}}$$

$$= 2.615 \times 10^9 \text{ W/m}^3$$

Radius of wire, R

$$= \frac{2.5/2}{1000} = 0.00125 \text{ m}$$

The wire surface temperature is given by,

$$t_w = t_a + \frac{q_g}{2h} R$$

$$t_w = 50 + \frac{2.615 \times 10^9}{2 \times (4 \times 1000)} \times 0.00125$$

$$= 50 + 408.59 = 458.59°\text{C}$$

Maximum temperature in the wire occurs at its geometric centre line, and can be computed from the relation,

$$t_{max} = t_a + \frac{q_g}{2h} R + \frac{q_g}{4k} R^2$$

$$= t_w + \frac{q_g}{4k} R^2$$

$$= 458.59 + \frac{2.615 \times 10^9}{4 \times 20} \times (0.00125)^2$$

$$= 458.59 + 51.07 = 510.66°\text{C}$$

EXAMPLE 4.22

A concrete column used in bridge construction is cylindrical in shape with a diameter of 1 metre. The column is completely poured in a short interval of time and the hydration of concrete results in the equivalent of a uniform source strength of 0.7 W/kg. Determine the temperature at the centre of the cylinder at a time when the outside surface temperature is 75°C. The column is sufficiently long so that temperature variation along its length may be neglected. For concrete :

Average thermal conductivity = 0.95 W/mK

Average density = 2300 kg/m³

Solution : Since the hydration of concrete results in uniform internal heat generation, the maximum temperature occurs at the centre of the cylindrical column and is described by the equation,

$$t_{max} = t_w + \frac{q_g}{4k} R^2$$

$$q_g = 0.7 \text{ W/kg} = 0.7 \times 2300 \text{ W/m}^3$$

Substituting the given values

$$t_{max} = 75 + \frac{0.7 \times 2300}{4 \times 0.95} (0.5)^2$$

$$= 180.92°\text{C}$$

EXAMPLE 4.23

Show that in a wire carrying electrical current the maximum temperature at the centre of wire is given by

$$t_{max} = t_w + \frac{i^2}{4 k k_e} R^2$$

where i is the current density, t_w is the surface temperature corresponding to outer radius R of the wire, k_e and k are the electrical and thermal conductivities of the wire material.

(b) A copper wire, 2 mm in diameter and 8 m long, is being employed for the transmission of electric current. Estimate the voltage drop if the temperature rise at the wire axis is limited to 10°C. The wire has a surface temperature of 20°C and, the electrical and thermal conductivities of the copper are stated to be 5 × 10⁷/ohm-m and 380 W/mK respectively.

Solution : (a) For a cylindrical wire conductor, the maximum temperature occurs at the centre and is prescribed by the relation,

$$t_{max} = t_w + \frac{q_g}{4k} R^2 \quad \ldots(i)$$

Total volumetric heat generation is equal to

$$I^2 R_e = I^2 \frac{\rho L}{A} = \left(\frac{I^2 L}{A}\right)\frac{1}{k_e}$$

The electrical conductivity k_e is the reciprocal of electrical resistivity ρ.

∴ Heat generation per unit volume,

$$q_g = \left(\frac{I^2 L}{A}\frac{1}{k_e}\right) \div Al$$

$$= \left(\frac{I}{A}\right)^2 \frac{1}{k_e} = \frac{i^2}{k_e} \quad \ldots(ii)$$

The term $i = I/A$ defines the current density,

From (i) and (ii)

$$t_{max} = t_w + \frac{i^2}{4 k k_e} R^2 \quad \ldots(iii)$$

which is the required expression

(b) Inserting the appropriate values in the expression (iii) derived above

$$i^2 = \frac{(t_{max} - t_w) 4 k k_e}{R^2}$$

$$= \frac{10 \times 4 \times 380 \times 5 \times 10^7}{(0.001)^2}$$

$$= 76 \times 10^{16}$$

∴ Current density i

$$= 8.72 \times 10^8 \text{ ampere/m}^2$$

Voltage drop

$$= IR_e = (iA)\ \frac{\rho l}{A}$$

$$= i l \rho = \frac{il}{k_e}$$

$$= \frac{8.72 \times 10^8 \times 8}{5 \times 10^7}$$

$$= 139.52 \text{ Volts}$$

EXAMPLE 4.24

A 66 kV transmission line carrying a current of 850 ampere is 20 mm in diameter and electrical resistance of the copper conductor is 0.075 ohm/km. Assuming that the surroundings are at 38°C and that the combined convection and radiation coefficient for heat transfer from the wire surface to the surroundings is 14.2 W/m²K, make calculations for :

(i) surface temperature of the transmission line

(ii) rate of heat generation per unit volume of the wire

(iii) maximum temperature in the line

The thermal conductivity of copper is 380 W/mK.

Solution : (i) Heat generated in the transmission line due to flow of current

$$= I^2 R$$

$$= 850^2 \times 0.075 \text{ W/km}$$

$$= \frac{850^2 \times 0.075}{1000}$$

$$= 54.187 \text{ W per metre length}$$

Heat dissipated to surroundings by combined convection and radiation

$$= h\ A\ \Delta t$$

$$= 14.2\ (\pi \times 0.02 \times 1)\ (t_w - 38) \text{ W per metre length}$$

Under steady state conditions

$$54.187 = 14.2\,(\pi \times 0.02 \times 1)\,(t_w - 38)$$

Solving, t_w (wire surface temperature)

$$= 98.76°C$$

(ii) Let q_g be the volumetric heat generated at uniform rate over the wire cross-section

$$54.187 = q_g \times \left\{\pi \times \left(\frac{0.02}{2}\right)^2 \times 1\right\}$$

$$= 3.14 \times 10^{-4}\, q_g$$

$$\therefore \quad q_g = \frac{54.187}{3.14 \times 10^{-4}}$$

$$= 1.726 \times 10^5 \text{ W/m}^3$$

(iii) Maximum temperature in the wire will occur at the geometric centre line of the wire and may be computed from the relation,

$$t_{max} = t_w + \frac{q_g}{4k} R^2$$

$$= 98.76 + \frac{1.726 \times 10^5}{4 \times 380} (0.01)^2$$

$$= 98.76 + 0.01135 = 98.771°C$$

The small difference between surface and centre temperature results from the relatively small heat generation rate and the high thermal conductivity of copper.

EXAMPLE 4.25

A copper wire 0.8 mm in diameter is insulated with plastic to an outer diameter of 2.5 mm. The wire is expected to withstand an environment temperature of 35°C. Calculate (i) maximum steady current in amperes that this wire can carry without heating any part of the plastic above 90°C (ii) maximum temperature of the wire.

The pertinent thermal and electrical properties of the wire material are :

Thermal conductivity of copper

= 380 W/m K

Thermal conductivity of plastic

= 0.4 W/m K

Electrical conductivity of copper

= 5 × 10⁵ ohm-cm and

Heat transfer coefficient from the outer surface of the plastic to the surroundings

= 10 W/m²K

Solution : Let r_1 and r_2 denote the inner and outer radii of the plastic insulation. Then the heat flow rate per metre length of the wire is given by

$$Q = \frac{t_1 - t_2}{\frac{1}{2\pi kl}\log_e \frac{r_2}{r_1} + \frac{1}{2\pi r_2 lh}}$$

$$= \frac{90 - 35}{\frac{1}{2\pi \times 0.4 \times 1} \times \log_e \frac{1.25}{0.4} + \frac{1}{2\pi \times 0.00125 \times 1 \times 10}}$$

(unit length)

$$= \frac{55}{0.4536 + 12.739}$$

$$= 4.169 \text{ W per metre length}$$

Under steady state all the heat produced on account of current flow must be transferred to the surroundings. Hence the rate of heat generation is 4.169 W per metre length of wire.

∴ Heat generated per unit volume,

$$q_g = \frac{4.169}{\pi (0.0004)^2 \times 1}$$

$$= 8.298 \times 10^6 \text{ W/m}^3$$

In terms of electrical quantities, current density i and electrical conductivity k_e, the heat generated per unit volume equals i^2/k_e

$$\therefore \quad \frac{i^2}{k_e} = 8.298 \times 10^6$$

$$i^2 = (8.298 \times 10^6) \times (5 \times 10^5 \times 100)$$

Current density i

$$= 20.36 \times 10^6 \text{ ampere/m}^2$$

∴ Current flow, I

$$= i \times A$$

$$= (20.36 \times 10^6) \times \pi \times (0.0004)^2$$

$$= \mathbf{10.22 \text{ ampere}}$$

(b) The maximum temperature occurs at the centre and is prescribed by the relation,

$$t_{max} = t_w + \frac{q_g}{4k} R^2$$

$$= 90 + \frac{8.298 \times 10^6}{4 \times 380} (0.004)^2$$

$$= 90 + 8.73 + 10^{-4} = \mathbf{90°C}$$

The insignificant difference between surface and centre temperature may be attributed to the fact that the heat generation rate is small, and the thermal conductivity of copper is high.

4.3.2. Hollow Cylinder with Outside Surface Insulated (Adiabatic)

The general solution for temperature distribution is

$$t = -\frac{q_g}{4k} r^2 + C_1 \log_e r + C_2 \quad \text{...(i)}$$

$$\frac{dt}{dr} = -\frac{q_g}{2k} r + \frac{C_1}{r} \quad \text{...(ii)}$$

The constants of integration are determined from the relevant boundary conditions which are

(i) $t = t_1$ at $r = r_1$

(ii) At $r = r_2$, the conduction region is perfectly insulated and hence heat flow is zero.

From Fourier's law, $Q = -kA\frac{dt}{dr}$ and accordingly the temperature derivative must be zero at $r = r_2$. Hence using expression (ii), we get

$$0 = -\frac{q_g}{2k} r_2 + \frac{C_1}{r_2} \quad \text{or} \quad C_1 = \frac{q_g}{2k} r_2^2$$

Applying the boundary condition $t = t_1$ at $r = r_1$ to expression (i) and

noting that $C_1 = \frac{g}{2k} r_2^2$, we get

$$t_1 = -\frac{q_g}{4k} r_1^2 + \frac{q_g}{2k} r_2^2 \log_e r_1 + C_2$$

$$\text{or} \quad C_2 = t_1 + \frac{q_g}{2k}\left[\frac{1}{2} r_1^2 - r_2^2 \log_e r_1\right]$$

With these values of integration constants, the general solution for temperature distribution becomes

$$t = -\frac{q_g}{4k} r^2 + \frac{q_g}{2k} r_2^2 \log_e r + t_1 + \frac{q_g}{2k}\left[\frac{1}{2} r_1^2 - r_2^2 \log_e r_1\right]$$

$$= t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r^2}{2} + r_2^2 \log_e\left(\frac{r}{r_1}\right)\right] \quad \text{...(4.24)}$$

Apparently the temperature distribution is parabolic.

EXAMPLE 4.26

An internally cooled copper conductor of 2 cm outer radius and 0.75 cm inner radius carries a current density of 5000 amp/cm². A constant temperature of 70°C is maintained at the inner surface and there is no heat transfer through insulation surrounding the copper. Set up an equation for temperature distribution through copper. Proceed to calculate the maximum temperature of copper and the radius at which it occurs. Also find the internal heat transfer rate and check that this equals the total energy generation in the conductor.

For copper :

thermal conductivity k = 380 W/m-deg

resistivity ρ = 2 × 10⁻⁸ Ω m

Solution : Total volumetric heat generation,

$$= I^2 R = I^2 \frac{\rho l}{A}$$

Heat generated per unit volume,

$$q_g = I^2 \frac{\rho l}{A} \div Al = \rho\left(\frac{I}{A}\right)^2$$

$$= 2 \times 10^{-8} (5000 \times 10^4)^2$$

$$= 50 \times 10^6 \text{ W/m}^3$$

For steady state conditions, the radial temperature distribution for a hollow cylinder with outside surface insulated is given by

$$t = t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r^2}{2} + r_2^2 \log_e\left(\frac{r}{r_1}\right)\right]$$

The maximum temperature occurs at the insulated surface, *i.e.*, at the outer radius and it equals

$$t_{max} = t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r_2^2}{2} + r_2^2 \log_e \frac{r_2}{r_1}\right]$$

Inserting the appropriate values,

$$t_{max} = 70 + \frac{50 \times 10^6}{2 \times 380}$$

$$\times \left[\frac{0.0075^2 - 0.02^2}{2} + 0.02^2 \log_e \frac{0.02}{0.0075}\right]$$

$$= 70 + 65789.5\ [-\ 0.0001719 + 0.0003923]$$

$$= 70 + 14.50 = \mathbf{84.5°C}$$

The internal heat transfer rate can be obtained by finding the temperature gradient at the inner radius, *i.e.*, at $r = 0.0075$ m and then invoking the Fourier's law of heat conduction.

$$t = \frac{q_g}{4k}(-2r) + \frac{q_g}{2k} r_2^2 \left(\frac{1}{r}\right)$$

$$= -\frac{q_g}{2k} r + \frac{q_g}{2k}\frac{r_2^2}{r}$$

$$\left|\frac{dt}{dr}\right|_{r=r_1} = -\frac{q_g}{2k} r_1 + \frac{q_g}{2k}\frac{r_2^2}{r_1}$$

$$= -\frac{50 \times 10^6}{2 \times 380} \times 0.0075 + \frac{50 \times 10^6}{2 \times 380} \times \frac{0.02^2}{0.0075}$$

$$= -493.42 + 3508.77 = 3015.35$$

$$\therefore \quad Q = -kA\frac{dt}{dr}$$

$$= -380 \times (2\pi \times 0.0075 \times 1) \times 3015.35$$

$$= -53968.7 \text{ W/m length of conductor}$$

The –ve sign indicates that the heat flow is radially inwards.

Check : *Since the outer surface is insulated, the entire heat generated within the conductor must be dissipated internally. Therefore the internal heat transfer must be*

= (volume per m length of conductor) × q_g

$$= \pi\,(0.002^2 - 0.0075^2) \times 50 \times 10^6$$

$$= 53968 \text{ W/m length of conductor}$$

EXAMPLE 4.27

A hollow cylindrical conductor (thermal conductivity k) with inside radius r_1, outside radius r_2 is perfectly insulated at its outside radius and is held at temperature t_1 by a coolant at the inside radius. Electrical energy is dissipated within the conductors at the constant rate of q_g per unit volume. If the steady state conditions prevail and the temperature distribution is primarily radial, establish a relation for the temperature as a function of the radial coordinate r.

A hollow conductor with $r_1 = 0.6$ cm and $r_2 = 0.75$ cm is made of a metal of thermal conductivity 17.5 W/m-deg and electrical resistance of 2.5×10^{-2} ohm per metre. Find the maximum allowable current if the temperature is not to exceed 48.5°C anywhere in the conductor. The cooling fluid at the inside is at 37.5°C.

Solution : For steady state conditions, the radial temperature distribution for a hollow cylinder having outside surface insulated is given by

$$t = t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r^2}{2} + r_2^2 \log_e \frac{r}{r_1}\right]$$

The maximum temperature occurs at the insulated surface, *i.e.*, at the outside radius and it equals

$$t_{max} = t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r_2^2}{2} + r_2^2 \log_e \frac{r_2}{r_1}\right]$$

Inserting the appropriate values

$$48.5 = 37.5 + \frac{q_g}{2 \times 17.5}$$

$$\left[\frac{0.0006^2 - 0.0075^2}{2} + 0.0075^2 \log_e \frac{0.0075}{0.006}\right]$$

$$= 37.5 + \frac{q_g}{2 \times 17.5} \times 0.243 \times 10^{-5}$$

$\therefore$ Heat generated per unit volume,

$$q_g = \frac{(48.5 - 37.5) \times 2 \times 17.5}{0.243 \times 10^{-5}}$$

$$= 1584 \times 10^5 \text{ W/m}^3$$

Total volumetric heat generation

$$= 1584 \times 10^5 \times \left\{\frac{\pi}{4}(0.0075^2 - 0.006^2) \times 1\right\}$$

$$= 2518 \text{ W per metre length of conductor}$$

In terms of electrical quantities,

Heat generated

$$= I^2 \times R$$

$$= I^2 \times (2.5 \times 10^{-2}) \text{ W per metre length}$$

$$\therefore \quad 2518 = I^2 \times (2.5 \times 10^{-2})$$

Hence the maximum allowable current,

$$I = \left[\frac{2518}{2.5 \times 10^{-2}}\right]^{\frac{1}{2}} = \mathbf{317.4\ ampere}$$

4.3.3. Hollow Cylinder with Inside Surface Insulated (Adiabatic)

The general solution for temperature distribution in a cylinder is

$$t = -\frac{q_g}{4k} r^2 + C_1 \log_e r + C_2 \qquad ...(i)$$

$$\frac{dt}{dr} = -\frac{q_g}{2k} r + \frac{C_1}{r} \qquad ...(ii)$$

The constants of integration are determined from the relevant boundary conditions which are

(*i*) At $r = r_1$, the conduction region is perfectly insulated and hence heat flow is zero.

From Fourier's law, $Q = -kA\frac{dt}{dr}$ and accordingly the temperature derivative must be zero at $r = r_1$. Hence using expression (*i*), we get

$$0 = -\frac{q_g}{2k} r_1 + \frac{C_1}{r_1} \quad \text{or} \quad C_1 = \frac{q_g}{2k} r_1^2$$

(*ii*) $t = t_2$ at $r = r_2$

Applying this boundary condition to expression (*i*) and noting that $C_1 = \frac{q_g}{2k} r_1^2$, we get

$$t_2 = -\frac{q_g}{4k} r_2^2 + \frac{q_g}{2k} r_1^2 \log_e r_2 + C_2$$

or

$$C_2 = t_2 + \frac{q_g}{2k}\left[\frac{1}{2} r_2^2 - r_1^2 \log_e r_2\right]$$

With these values of integration constants, the general solution for temperature distribution becomes

$$t = -\frac{q_g}{4k} r^2 + \frac{q_g}{2k} r_1^2 \log_e r + t_2 + \frac{q_g}{2k}\left[\frac{1}{2} r_2^2 - r_1^2 \log_e r\right]$$

$$= t_2 + \frac{q_g}{2k}\left[\frac{r_2^2 - r^2}{2} + r_1^2 \log_e \frac{r}{r_2}\right] \qquad ...(4.25)$$

Apparently the temperature distribution or is parabolic.

The maximum temperature occurs at the insulated surfac, *i.e.*, at the inner radius and it equals

$$t_{max} = t_2 + \frac{q_g}{2k}\left[\frac{r_2^2 - r_1^2}{2} + r_1^2 \log_e \frac{r_1}{r_2}\right] \qquad ...(4.26)$$

EXAMPLE 4.28

A hollow cylinder (k = 3 W/m-deg)of inner radius 7 cm and outside radius 9 cm has a heat generation rate of 5×10^6 W/m³. The inside surface is insulated and heat is removed by convection over the outside surface by a fluid at 100°C with convection coefficient 335 W/m²-deg. Make calculations for the temperatures at the outside and inside surfaces of the cylinder.

Solution : Since the inside surface is insulated, the heat generated is convected at the outside surface. That is

$$q_g \times \pi(r_2^2 - r_1^2) l = h_o 2\pi r_o l (t_2 - t_a)$$

Considering unit length of the cylinder and inserting appropriate values;

$5 \times 10^6 \times \pi\,(0.09^2 - 0.07^2) \times 1$

$= 335 \times (2\pi \times 0.09 \times 1) \times (t_2 - 100)$

or $50240 = 189.34\,(t_2 - 100)$

$$\therefore \quad t_2 = \frac{50240}{189.34} + 100 = 365.34°C$$

For steady state conditions, the radial temperature distribution for a hollow cylinder with inside surface insulated is given by

$$t = t_2 + \frac{q_g}{2k}\left[\frac{r_2^2 - r^2}{2} + r_1^2 \log_e \frac{r}{r_2}\right]$$

The temperature at the inside surface is determined using the above expression and replacing r by r_1

$$t = t_2 + \frac{q_g}{2k}\left[\frac{r_2^2 - r_1^2}{2} + r_1^2 \log_e \frac{r_1}{r_2}\right]$$

$$= 365.34 + \frac{5 \times 10^6}{2 \times 30}\left[\frac{0.09^2 - 0.07^2}{2} + 0.07^2 \log_e \frac{0.07}{0.09}\right]$$

$$= 365.34 + \frac{5 \times 10^6}{60}\left[\frac{0.0081 - 0.0049}{2} + 0.0049 \times (-0.2513)\right]$$

$$= 365.34 + \frac{5 \times 10^6}{60} \times [0.0016 - 0.001231]$$

$$= 397.59°C$$

4.3.4. Hollow Cylinder with Temperatures Specified at the Inside and Outside Surfaces

The general solution for temperature distribution is,

$$t = -\frac{q_g}{4k} r^2 + C_1 \log_e r + C_2$$

The constants of integration are determined from the relevant boundary conditions which are

$t = t_1$ at $r = r_1$

and $t = t_2$ at $r = r_2$

That gives,

$$t_1 = -\frac{q_g}{4k} r_1^2 + C_1 \log_e r_1 + C_2 \quad \text{...(i)}$$

$$t_2 = -\frac{q_g}{4k} r_2^2 + C_1 \log_e r_2 + C_2 \quad \text{...(ii)}$$

From identities (i) and (ii), the integration constants workout as

$$C_1 = \frac{(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)}{\log_e \frac{r_1}{r_2}}$$

and

$$C_2 = t_1 + \frac{q_g}{4k} r_1^2 - \frac{(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)}{\log_e \frac{r_1}{r_2}} \log_e r_1$$

With these values of constants, the general solution for temperature distribution becomes

$$t = -\frac{q_g}{4k} r^2 + \frac{(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1)}{\log_e \frac{r_1}{r_2}} \log_e r + t_1 + \frac{q_g}{4k} r_1^2 - \frac{(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)}{\log_e \frac{r_1}{r_2}} \log_e r_1$$

or

$$t - t_1 = \frac{q_g}{4k}(r_1^2 - r^2) + \frac{(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)}{\log_e \frac{r_1}{r_2}} \times (\log_e r - \log_e r_1)$$

$$= \frac{q_g}{4k}(r_1^2 - r^2) + \left[(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)\right] \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_1}{r_2}}$$

EXAMPLE 4.29

A hollow cylinder of 3 cm inner radius and 4.5 cm outer radius has a heat generation rate of 5×10^6 W/m³. The inner and outer surfaces are maintained at temperatures of 380°C and 360°C respectively and thermal conductivity of the cylinder material is 30 W/m-deg. Make calculations for the temperature at mid radius.

Solution : $r_1 = 0.03$ m ;

$r_2 = 0.045$ m ;

and $r_3 = 0.0375$ m at mid radius

For the specified boundary conditions, the temperature distribution is given by

$$t - t_1 = \frac{q_g}{4k}(r_1^2 - r^2) + \left[(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)\right] \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_1}{r_2}}$$

Inserting the appropriate data,

$$t - 380 = \frac{5 \times 10^6}{4 \times 30}(0.03^2 - 0.0375^2) + \left[(380 - 360) - \frac{5 \times 10^6}{4 \times 30} \times (0.045^2 - 0.03^2)\right] \frac{\log_e \frac{0.0375}{0.03}}{\log_e \frac{0.03}{0.045}}$$

$$= -21.09 + (20 - 46.875) \times \frac{0.223}{(-0.4055)}$$

$$= -6.31$$

$\therefore$ Temperature at mid radius, t

$= 380 - 6.31 = 373.69°C$

EXAMPLE 4.30

A long hollow cylinder (k = 0.5 W/m-deg) of 5 cm inner radius and 15 cm outer radius has a heat generation rate of 1000 W/m³. The outer surface is maintained at a temperature of 50°C and thermal conductivity of the cylinder material is 0.5 W/m-deg. If maximum temperature occurs at radius of 10 cm, determine the temperature at the inner surface, and the value of maximum temperature in the cylinder. Proceed from the basic heat equation.

Solution : With uniform heat generation and assuming steady state uni-directional heat flow in the radial direction, the appropriate differential equation describing the temperature distribution through a cylinder surface is

$$\frac{d}{dr}\left(r \frac{dt}{dr}\right) = -\frac{q_g r}{k}$$

Upon integration

$$r \frac{dt}{dr} = -\frac{q_g}{k}\frac{r^2}{2} + C_1$$

or

$$\frac{dt}{dr} = -\frac{q_g}{k}\frac{r}{2} + \frac{C_1}{r} \quad \text{...(i)}$$

Another integration gives the general solution for temperature distribution

$$t = -\frac{q_g}{4k} r^2 + C_1 \log_e r + C_2 \quad \text{...(ii)}$$

The appropriate boundary conditions are

(i) At $r = 10$ cm, the temperature is stated to be maximum and accordingly

$\frac{dt}{dr} = 0$ that gives

$$C_1 = \frac{q_g}{k}\frac{r^2}{2} = \frac{1000}{0.5} \times \frac{(0.1)^2}{2} = 10$$

(ii) $t = 50°C$ at $r = 15$ cm, then from expression (ii) and noting that $C_1 = 10$, we get

$$50 = -\frac{1000}{4 \times 0.5}(0.15)^2 + 10 \log_e 0.15 + C_2$$

$= -11.25 - 18.97 + C_2$;

$C_2 = 80.22$

The temperature distribution through cylinder may then be written as

$$t = -\frac{q_g}{4k}r^2 + 10\log_e r + 181.47$$

Then temperature at the inner surface where $r = 5$ cm,

$$t = -\frac{1000}{4\times0.5}(0.05)^2 + 10\log_e 0.05 + 80.22$$
$$= -1.25 - 29.95 + 80.22 = 49.02°C$$

Maximum temperature is stated to occur at $r = 10$ cm and therefore

$$t_{max} = -\frac{1000}{4\times0.5}(0.1)^2 + 10\log_e 0.1 + 80.22$$
$$= -5 - 23.02 + 80.22 = 52.2°C$$

4.4. HEAT TRANSFER THROUGH THE PISTON CROWN

With reference to Fig. 4.19, let

R = outer radius of piston,
b = thickness of piston crown
t_o = outer surface temperature of piston material,
k = thermal conductivity of piston material,
q_g = convection and radiation heat flux (heat transfer per unit area) from the gases to the piston

The appropriate differential equation describing the temperature distribution can be obtained by making an energy balance on a cylindrical element of thickness dr and radius r.

Q_r (heat conducted in at radius r)

$$= -k\,(2\pi r b)\frac{dt}{dr}$$

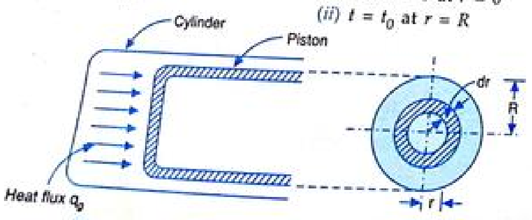


Fig. 4.19. Heat transfer through a piston crown

dr) Q_g (heat given by the gases to the element

$$= q_g \times 2\pi r\,dr$$

Q_{r+dr} (heat conducted out at radius $r + dr$)

$$= Q_r + \frac{d}{dr}(Q_r)\,dr$$

For steady conduction of heat flow,

$$Q_r + Q_g = Q_{r+dr}$$
$$= Q_r + \frac{d}{dr}(Q_r)\,dr$$

$$\therefore \quad Q_g = \frac{d}{dr}(Q_r)\,dr$$

$$q_g \times 2\pi r\,dr = \frac{d}{dr}\left(-k2\pi r b\frac{dt}{dr}\right)dr$$

This yields the appropriate form of heat equation as

$$\frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q_g}{kb}r = 0$$

Upon integration ...(i)

$$r\frac{dt}{dr} + \frac{q_g}{kb}\frac{r^2}{2} = C_1$$

$$\text{or} \quad \frac{dt}{dr} + \frac{q_g}{kb}\frac{r}{2} = \frac{C_1}{r} \qquad ...(ii)$$

Another integration gives the general solution for temperature distribution

$$t + \frac{q_g}{kb}\frac{r^2}{4} = C_1\log_e r + C_2 \qquad ...(iii)$$

The constants of integration are determined from the relevant boundary conditions which are :

(i) $dt/dr = 0$ at $r = 0$
(ii) $t = t_0$ at $r = R$

Using expression (ii) $C_1 = 0$

Applying the boundary condition $t = t_o$ at $r = R$ to expression (iii) and noting that $C_1 = 0$, we get

$$t_o + \frac{q_g}{kb}\frac{R^2}{4} = C_2$$

With these values of integration constants, the expression (iii) for the temperature takes the form

$$t + \frac{q_g}{kb}\frac{r^2}{4} = t_o + \frac{q_g}{kb}\frac{R^2}{4}$$

$$\text{or} \quad t = t_o + \frac{q_g}{4kb}(R^2 - r^2) \qquad ...(iv)$$

Undoubtedly the temperature distribution is parabolic and the maximum temperature occuring at the centre of piston ($r = 0$) is given by

$$t_{max} = t_0 + \frac{q_g}{4kb}R^2 \qquad ...(v)$$

If Q denotes the total heat given by the gases to the piston crown, then

$$Q = \pi R^2 q_g \; ; \quad q_g = \frac{Q}{\pi R^2}$$

$$\therefore \quad t_{max} = t_o + \frac{Q}{\pi R^2}\times\frac{1}{4kb}R^2$$

$$= t_o + \frac{Q}{4\pi k b} \qquad ...(vi)$$

The above expression is used for calculating the desired thickness of piston crown

$$b = \frac{Q}{4\pi k(t_{max} - t_0)} \qquad ...(vii)$$

4.5. NUCLEAR FUEL ELEMENTS WITH AND WITHOUT CLADDING

The heat generation due to fission within a nuclear fuel element is not uniform, and for a cylinderical fuel rod the internal heat generation is generally given by

$$q_g = q_0\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad ...(i)$$

where q_0 is the heat generation rate per unit volume at the centre ($r = 0$) and R is the outer radius of the solid fuel rod. Evidently q_g is a function of position r, *i.e.*, the radial distance from the axis of the rod.

For steady state one-dimensional heat conduction in the radial direction, we have

$$\frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q_g}{k}r = 0$$

$$\text{or} \quad \frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q_0}{k}\left[1 - \left(\frac{r}{R}\right)^2\right]r = 0$$

Upon integration

$$r\frac{dt}{dr} + \frac{q_0}{k}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right] = C_1 \qquad ...(ii)$$

$$\frac{dt}{dr} + \frac{q_0}{k}\left[\frac{r}{2} - \frac{r^3}{4R^2}\right] = \frac{C_1}{r}$$

Integrating again,

$$t + \frac{q_0}{k}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] = C_1\log_e r + C_2 \qquad ...(iii)$$

Invoking the boundary conditions,

$$\frac{dt}{dr} = 0 \quad \text{at} \quad r = 0$$

and $t = t_{max}$ at $r = 0$ we get,

$C_1 = 0$ (from expression ii)
$C_2 = t_{max}$ (from expression iii)

With these values of integration constants, the expression (iii) for the temperature distribution takes the form:

$$t + \frac{q_0}{k}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] = t_{max}$$

$$\text{or} \quad t - t_{max} = -\frac{q_0}{k}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] \qquad ...(iv)$$

If t_w is the temperature at the outer surface (wall) of the rod, *i.e.*, at $r = R$, then

$$t_w - t_{max} = -\frac{q_0}{k}\left[\frac{R^2}{4} - \frac{R^4}{16R^2}\right]$$

$$= -\frac{3q_0R^2}{16k} \qquad ...(v)$$

The heat flow at the surface of the rod is,

$$Q = -kA\left.\frac{dt}{dr}\right|_{r=R}$$

$$= -kA\left[-\frac{q_0}{k}\left\{\frac{r}{2}-\frac{r^3}{4R^2}\right\}\right]_{r=R}$$

$$= -kA\left[-\frac{q_0}{k}\left\{\frac{R}{2}-\frac{R^3}{4R^2}\right\}\right]$$

$$= \frac{q_0 AR}{4} \qquad \text{...(vi)}$$

Under steady state conditions, this heat would be convected from the outside surface of the rod.

$$\frac{q_0 AR}{4} = hA\,(t_w - t_a)$$

or $$t_w = t_a + \frac{q_0}{4h}R \qquad \text{...(vii)}$$

where h is the convective heat transfer coefficient and t_a is the ambient temperature.

Substituting this value of temperature at the outer surface of the rod in expression (v), the temperature distribution becomes :

$$t_a + \frac{q_0}{4h}R - t_{max} = -\frac{3q_0}{16k}R^2$$

or $$t_{max} - t_a = \frac{q_0}{4}R\left[\frac{3}{4k}R - \frac{1}{h}\right] \qquad \text{...(viii)}$$

The fuel elements of a nuclear reactor are likely to get damaged due to oxidation if these elements come in direct contact with the cooling medium. To prevent this damage, the fuel elements are usually lagged on the outside with a protective material called *cladding*.

The heat generated rate per unit volume in the fissionable (fuel rod) material is given by

$$q_g = q_0\left[1-\left(\frac{r}{R}\right)^2\right]$$

where q_g is the generating rate at any radius r, q_0 is the heat generating rate at the centre of fissionable element.

Let R_f and R_c be the outside radii of fuel rod and cladding respectively.

The temperature distribution within the fissionable element and the cladding can be worked out from the following steady state one-dimensional heat conduction equation :

$$\frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q_g}{k}r = 0$$

The heat flux q (heat flow per unit area, Q/A) is

$$q = -k\frac{dt}{dr};$$

$$\frac{dt}{dr} = -\frac{q}{k}$$

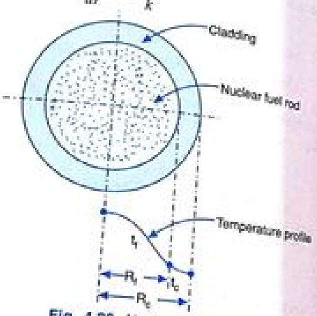


Fig. 4.20. Nuclear heat source

Assuming thermal conductivity k to be constant and with substitution of

$$dt/dr = -q/k$$

the basic governing equation becomes

$$\frac{d}{dr}\left(-\frac{rq}{k}\right) + \frac{q_g}{k}r = 0$$

or $$\frac{d}{dr}(r\,q) = q_g\,r \qquad \text{...(ix)}$$

Let the fuel rod and the cladding be denoted by f and c respectively. Then,

$$\frac{d}{dr}(r\,q_f) = q_g\,r = q_0\left[1-\left(\frac{r}{R_f}\right)^2\right]r \qquad \text{...(x)}$$

and as there is no internal generation in cladding

$$\frac{d}{dr}(r\,q_c) = 0 \qquad \text{...(xi)}$$

Integrating expression (x),

$$r\,q_f = q_0\left[\frac{r^2}{2}-\frac{r^4}{4R_f^2}\right] + C_1$$

or $$q_f = q_0\left[\frac{r}{2}-\frac{r^3}{4R_f^2}\right] + \frac{C_1}{r} \qquad \text{...(xii)}$$

and upon integration of expression (xi), we obtain

$$r\,q_c = C_2$$

or $$q_c = \frac{C_2}{r} \qquad \text{...(xiii)}$$

Applying the boundary conditions

q_f = finite at $r = 0$

$q_f = q_c$ at $r = R_f$

the integration constants workout as

$$C_1 = 0$$

and $$\frac{C_2}{R_f} = q_f = q_0\left[\frac{R_f}{2}-\frac{R_f^3}{4R_f^2}\right] = q_0\frac{R_f}{4}$$

or $$C_2 = \frac{q_0}{4}R_f^2$$

Accordingly the heat flux through fuel rod and cladding may be rewritten as

$$q_f = -k_f\frac{dt_f}{dr} = q_0\left[\frac{r}{2}-\frac{r^3}{4R_f^2}\right] \qquad \text{...(xiv)}$$

and $$q_c = -k_c\frac{dt_f}{dr} = \frac{q_0}{4r}R_f^2 \qquad \text{...(xv)}$$

The temperature t_f and t_c can be obtained by integration of the above identities

$$t_f = \frac{q_0}{k_f}\left[\frac{r^4}{16R_f^2}-\frac{r^2}{4}\right] + C_3 \qquad \text{...(xvi)}$$

and $$t_c = -\frac{q_0}{4k_f}R_f^2\log_e r + C_4$$

Applying the boundary condition $t_c = t_w$ (outside surface temperature of the system) at $r = R_c$:

$$C_4 = t_w + \frac{q_0}{4k_c}R_f^2\log_e R_c$$

$$\therefore\ t_c = -\frac{q_0}{4k_c}R_f^2\log_e r + t_w + \frac{q_0}{4k_c}R_f^2\log_e R_c$$

or $$t_c - t_w = \frac{q_0}{4k_c}R_f^2\log_e\left(\frac{R_c}{r}\right) \qquad \text{...(xviii)}$$

Applying the boundary condition $t_c = t_f$ at $r = R_f$ to expression (xvi) and substituting there in the value of t_c as evaluated above, we get

$$\frac{q_0}{k_f}\left[\frac{R_f^4}{16R_f^2}-\frac{R_f^2}{4}\right] + C_3 = -\frac{q_0}{4k_c}R_f^2\log_e R_f + t_w + \frac{q_0}{4k_c}R_f^2\log_e R_c$$

$$= t_w + \frac{q_0}{4k_c}R_f^2\log_e\frac{R_c}{R_f}$$

or $$C_3 = t_w + \frac{q_0}{4}R_f^2\left[\frac{3}{4k_f}+\frac{1}{k_c}\log_e\frac{R_c}{R_f}\right]$$

$$\therefore\ t_f = \frac{q_0}{k_f}\left[\frac{r^4}{16R_f^2}-\frac{r^2}{4}\right] + t_w + \frac{q_0}{4}R_f^2\left[\frac{3}{4k_f}+\frac{1}{k_c}\log_e\frac{R_c}{R_f}\right] \qquad \text{...(xix)}$$

The maximum value of t_f occurs at the centre of fissionable material, *i.e.*, at $r = 0$ and would be given by

$$t_{max} = t_w + \frac{q_0}{4}R_f^2\left[\frac{3}{4k_f}+\frac{1}{k_c}\log_e\frac{R_c}{R_f}\right] \qquad \text{...(xx)}$$

EXAMPLE 4.31

The internal heat generation in a cylindrical fuel rod of nuclear reactor has been prescribed by the relation

$$q_g = q_0\left[1-\left(\frac{r}{R}\right)^2\right]$$

where q_g is the heat generating rate at any radius r and q_0 is the heat generating rate at the centre

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and R is the outside radius of the fuel element. Establish a relation for the temperature drop from the centre line to the outside surface of the rod. Work out this temperature drop for a 3 cm outside diameter cylindrical rod of thermal conductivity 25 W/m-deg if heat is removed from the outside surface at the rate of 2.25 × 10⁶ W/m². Also determine the surface heat transfer coefficient if the maximum wall surface temperature is limited to 200°C and the fluid surrounding the rod is at 75°C.

Solution : The heat flow at the outside surface of the rod is

$$Q = \frac{q_0 AR}{4}\ ; \quad \frac{Q}{A} = \frac{q_0 R}{4}$$

$$\therefore \quad q_0 = \left(\frac{Q}{A}\right)\times\frac{4}{R} = 2.25\times 10^6 \times \frac{4}{0.015}$$
$$= 6\times 10^8\ \mathrm{W/m^3}$$

Temperature drop from the centre line ($t = t_{max}$) to the outside surface ($t = t_w$) of the rod is

$$t_{max} - t_w = \frac{3q_0R^2}{16R} = \frac{3\times6\times10^8\times(0.015)^2}{16\times25}$$
$$= \mathbf{1102.5°C}$$

(*b*) Invoking relation for convective heat flux (heat flow per unit area),

$$Q = hA\,(t_w - t_a)$$

∴ Surface heat transfer coefficient

$$h = \left(\frac{Q}{A}\right)\times\frac{1}{t_w - t_a}$$

$$= (2.25\times10^6)\times\frac{1}{(200-75)}$$
$$= \mathbf{18000\ W/m^2\text{-}deg}$$

4.6. SPHERE WITH UNIFORM HEAT GENERATION

Consider heat conduction through a solid sphere of radius *R*.

Assumptions :

1. Steady state conditions
2. One-dimensional radial conduction
3. Constant thermal conductivity *k*
4. Uniform volumetric heat generation (q_g per unit volume) within the solid.

The differential equation describing the temperature distribution can be set up by making an energy balance on a spherical shell of thickness *dr*, at radius *r*

Q_r (heat conducted in at radius *r*)

$$= -k4\pi r^2\frac{dt}{dr}$$

Q_g (heat generated in the element)

$$= q_g\times 4\pi r^2 dr$$

Q_{r+dr} (heat conducted out at radius *r* + *dr*)

$$= Q_r + \frac{d}{dr}(Q_r)\,dr$$

∴ For steady state conduction of heat flow,

$$Q_r + Q_g = Q_{r+dr}$$
$$= Q_r + \frac{d}{dr}(Q_r)\,dr$$

or $$Q_g = \frac{d}{dr}(Q_r)\,dr$$

or $$4\pi r^2\,dr\,q_g = \frac{d}{dr}\left(-4\pi k r^2\frac{dt}{dr}\right)dr$$
$$= -4\pi k\frac{d}{dr}\left(r^2\frac{dt}{dr}\right)dr$$

That gives the appropriate heat flow equation as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dt}{dr}\right)+\frac{q_g}{k} = 0$$

or $$\frac{1}{r^2}\left(r^2\frac{d^2t}{dr^2}+2r\frac{dt}{dr}\right)+\frac{q_g}{k} = 0$$

or $$r\frac{d^2t}{dr^2}+2\frac{dt}{dr}+\frac{q_g}{k}r = 0$$

or $$r\frac{d^2t}{dr^2}+\frac{dt}{dr}+\frac{dt}{dr}+\frac{q_g}{k}r = 0$$

or $$\frac{d}{dr}\left(r\frac{dt}{dr}\right)+\frac{dt}{dr}+\frac{q_g}{k}r = 0 \qquad ...(i)$$

Upon integration,

$$r\frac{dt}{dr}+t+\frac{q_g}{k}\frac{r^2}{2} = C_1$$

or $$\frac{d}{dr}(rt)+\frac{q_g}{k}\frac{r^2}{2} = C_1$$

Another integration gives the general solution for the steady-state radial temperature distribution,

$$rt+\frac{q_g}{k}\frac{r^3}{6} = C_1 r + C_2 \qquad ...(ii)$$

Corresponding to centre of the sphere $r = 0$ and that gives $C_2 = 0$. Applying the boundary condition $t = t_w$ at $r = R$ to expression (ii) and noting that $C_2 = 0$, we get

$$Rt_w+\frac{q_g}{k}\frac{R^3}{6} = C_1R$$

or $$C_1 = t_w + \frac{q_g}{6k}R^2$$

With these values of integration constants, the general solution for temperature distribution becomes:

$$rt+\frac{q_g}{k}\frac{r^3}{6} = \left(t_w+\frac{q_g}{6k}R^2\right)r$$

or $$t+\frac{q_g}{6k}r^2 = t_w + \frac{q_g}{6k}R^2$$

or $$t = t_w + \frac{q_g}{6k}(R^2-r^2) \qquad ...(iii)$$

Undoubtedly the temperature distribution is parabolic and the maximum temperature occuring at the centre ($r = 0$) of the sphere is given by

$$t_{max} = t_w + \frac{q_g}{6k}R^2 \qquad ...(iv)$$

Combining expressions (*iii*) and (*iv*), we obtain the temperature distribution in dimensionless form

$$\frac{t-t_w}{t_{max}-t_w} = \frac{R^2-r^2}{R^2} = 1-\left(\frac{r}{R}\right)^2 \qquad ...(v)$$

The heat flow can be evaluated by using the Fourier's relation,

$$Q = -kA\left(\frac{dt}{dr}\right)_{r=R}$$

$$= -k4\pi R^2\frac{d}{dr}\left[t_w+\frac{q_g}{6k}(R^2-r^2)\right]_{r=R}$$

$$= -k4\pi R^2\left[\frac{q_g}{6k}(-2r)\right]_{r=R}$$

$$= k4\pi R^2\times\frac{q_g}{3k}R$$

$$= \frac{4}{3}\pi R^3\times q_g$$

= volume of sphere × heat generation capacity ...(vi)

Undoubtedly, the heat conducted is equal to heat generated. But for steady state condition, the heat conducted (or generated) must be equal to heat convected from the outer surface of the sphere.

$$\frac{4}{3}\pi R^3 q_g = h\times 4\pi R^2\,(t_w - t_a)$$

or $$t_w = t_a + \frac{q_g R}{3h} \qquad ...(vii)$$

Substituting this value of surface temperature t_w in expression (iii), one gets the temperature distribution in terms of temperature t_a of the surrounding atmosphere

$$t = t_a + \frac{q_g}{3h}R + \frac{q_g}{6k}(R^2-r^2) \qquad ...(viii)$$

and temperature t_{max} at $r = 0$

$$t_{max} = t_a + \frac{q_g}{3h}R + \frac{q_g}{6k}R^2 \qquad ...(ix)$$

EXAMPLE 4.32

An 8 cm diameter orange, approximately spherical in shape, undergoes ripening process and generates 18000 kJ/m³-hr of energy. If external surface of the orange is at 6.5°C, make calculations for temperature at the centre of the orange. Also determine the heat flow from the outer surface of the orange. Take thermal conductivity

$$k = 0.8\ \mathrm{kJ/m\text{-}hr\text{-}deg}$$

for the orange material.

Solution : $q_g = 18000$ kJ/m³-hr

$$= \frac{18000 \times 10^3}{3600} = 5000 \text{ J/m}^3\text{-s} = 5000 \text{ W/m}^3$$

$$k = 0.8 \text{ kJ/m-hr-deg} = \frac{0.8 \times 10^3}{3600} = 0.222 \text{ J/m-s-deg} = 0.222 \text{ W/m-deg}$$

At the centre of the orange, temperature is maximum and its value is prescribed by the relation :

$$t_{max} = t_w + \frac{q_g}{6k} R^2$$

$$= 6.5 + \frac{5000}{6 \times 0.222}(0.04)^2 = 12.5°\text{C}$$

The heat conducted equals the heat generated

$$Q = \frac{4}{3} \pi R^3 \times q_g$$

$$= \frac{4}{3} \pi (0.04)^3 \times 5000$$

$$= 1.34 \text{ J/s} = \mathbf{4.82 \text{ kJ/hr}}$$

EXAMPLE 4.33

A solid sphere of 8 cm radius has a uniform heat generation of 4×10^6 W/m³. The outside surface is exposed to a fluid at 150°C with convective heat transfer coefficient of 750 W/m²-deg. If thermal conductivity of the solid material is 30 W/m-deg, determine; (a) maximum temperature and its location, (b) temperature at 5 cm radius.

Solution : Heat generated in the sphere is convected over the surface. That is

$$q_g \times \frac{4}{3} \pi R^3 = h_o \times 4\pi R^2 \times (t_w - t_a)$$

$$\text{or } 4 \times 10^6 \times \frac{4}{3} \pi (0.08)^3 = 750 \times 4\pi (0.08)^2 \times (t_w - 150)$$

$$\text{or } t_w - 150 = \frac{4 \times 10^6 \times 0.08}{3 \times 750} = 142.22$$

$\therefore$ Temperature at the surface of sphere,

$$t_w = 150 + 142.22 = 292.22°\text{C}$$

Maximum temperature occurs at the centre and its value is prescribed by the relation

$$t_{max} = t_w + \frac{q_g}{6k} R^2$$

$$= 292.22 + \frac{4 \times 10^6}{6 \times 30}(0.08)^2$$

$$= \mathbf{434.45°C}$$

Temperature at any radius r can be worked out from the relation,

$$\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2$$

$$\therefore \frac{t - 292.22}{434.45 - 292.22} = 1 - \left(\frac{0.05}{0.08}\right)^2 = 0.6094$$

$$t = 0.6094\,(434.45 - 292.22) + 292.22$$

$$= \mathbf{378.9°C}$$

4.6.1. Hollow Sphere with Inside Surface Insulated

The general solution for temperature distribution in a sphere is

$$rt + \frac{q_g}{k}\frac{r^3}{6} = C_1 r + C_2$$

$$t + \frac{q_g}{6k} r^2 = C_1 + \frac{C_2}{r} \quad ...(i)$$

$$\frac{dt}{dr} + \frac{q_g}{3k} r = -\frac{C_2}{r^2} \quad ...(ii)$$

The constants of integration are determined from the relevant boundary conditions which are

(*i*) At $r = r_1$, the conduction region is perfectly insulated and hence the heat flow is zero. From Fourier's law, $Q = -kA\frac{dt}{dr}$ and accordingly the temperature derivative must be zero at $r = r_1$. Hence using expression (*ii*), we get

$$0 + \frac{q_g}{3k} r_1 = -\frac{C_2}{r_1^2} \quad \text{or} \quad C_2 = \frac{q_g}{3k} r_1^3$$

(*ii*) $t = t_2$ and $r = r_2$

Applying this boundary condition to expression (*i*) and noting that $C_2 = \frac{q_g}{3k} r_1^3$, we get

$$t_2 + \frac{q_g r_2^2}{6k} = C_1 + \frac{q_g}{3k}\frac{r_1^3}{r}$$

or

$$C_1 = t_2 + \frac{q_g r_2^2}{6k} - \frac{q_g}{3k}\frac{r_1^3}{r}$$

Substituting these values of integration constants in expression (*i*) and upon rearrangement,

$$t = t_2 + \frac{q_g}{6k}(r_2^2 - r^2) - \frac{q_g}{3k} r_1^3 \left(\frac{1}{r} - \frac{1}{r_2}\right)$$

The maximum temperature occurs at the insulated surface, *i.e.*, at the inner radius and it equals

$$t = t_2 + \frac{q_g}{6k}(r_2^2 - r_1^2) - \frac{q_g}{3k} r_1^3 \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

EXAMPLE 4.34

A hollow sphere (k = 30 W/m-deg) of inner radius 6 cm and outside radius 8 cm has a heat generation rate of 4×10^6 W/m³. The inside surface is insulated and heat is removed by convection over the outside surface by a fluid at 100°C with surface conductance 300 W/m²-deg. Make calculations for the temperature at the outside and inside surfaces of the sphere.

Solution : Since the inside surface is insulated, the heat generated is convected at the outside surface. That is

$$q_g \frac{4}{3}\pi(r_2^3 - r_1^3) = h_o \times 4\pi\, r_2^2 (t_2 - t_a)$$

$$4 \times 10^6 \times \frac{4}{3} \pi (0.08^3 - 0.06^3) = 300 \times 4\pi\,(0.08)^2 \times (t_2 - 100)$$

$$\text{or } 4957.01 = 24.11\,(t_2 - 100)\,;$$

$$t_2 = \frac{4957.01 + 100}{24.11} = 305.6°\text{C}$$

Temperature at any radius r can be worked out from the relation

$$t = t_2 + \frac{q_g}{6k}(r_2^2 - r^2) - \frac{q_g}{3k} r_1^3 \left(\frac{1}{r} - \frac{1}{r_2}\right)$$

$$= 305.6 + \frac{4 \times 10^6}{6 \times 30}(0.08^2 - 0.06^2) - \frac{4 \times 10^6}{3 \times 30}(0.06)^3 \left(\frac{1}{0.06} - \frac{1}{0.08}\right)$$

$$= 305.6 + 62.22 - 40 = 327.8°\text{C}$$

SALIENT POINTS

1. There occurs heat conduction and internal heat generation at uniform rate within the conducting medium itself in the following cases:
   (*i*) resistance heating in electrical appliances
   (*ii*) chemical and combustion processes
   (*iii*) fuel elements in a nuclear reaction
   (*iv*) drying and setting of concrete
2. For a large homogenous flat plate with both surfaces maintained at a common temperature, the temperature distribution is given by

$$t = \frac{q_g}{2k}(\delta - x)\,x + t_w$$

where q_g is rate of heat generation per unit volume of the plate, t_w is the wall surface temperature and δ is the thickness of plate.

The temperature distribution is parabolic and symmetrical about the mid plane. Maximum value of temperature occurs at $x = \frac{\delta}{2}$ and it equals

$$t_{max} = \frac{q_g}{8k}\delta^2 + t_w$$

Heat conducted to the wall surface is finally dissipated to the surrounding atmosphere at temperature t_a. Then for each surface

$$\frac{A\delta q_g}{2} = hA\,(t_w - t_a)$$

That gives:

$$t_w = t_a + \frac{q_g}{2g}\delta$$

Then considering h and t_a

$$t = t_a + \frac{q_g}{h}\delta + \frac{q_g}{2k}(\delta - x)\,x$$

and $$t_{max} = t_a + q_g\left[\frac{\delta}{2h} + \frac{\delta^2}{8k}\right]$$

When both surfaces the wall have different temperatures, then

$$\frac{t - t_{w2}}{t_{w1} - t_{w2}} = \left[1 - \frac{x}{\delta}\right]\left[\frac{Bx}{\delta} + 1\right]$$

and $$\frac{t_{max} - t_{w2}}{t_{w1} - t_{w2}} = \frac{(B+1)^2}{4B}$$

where $$B = \frac{q_g}{2k} \times \frac{\delta^2}{(t_{w1} - t_{w2})}$$

3. When heat generated in a material is due to passage of electric current, then

$$q_g = i^2\rho = \frac{i^2}{k_c}$$

where i is the current density $\left(\frac{I}{A}\right)$ and electrical conductivity k_c is the resistivity ρ of the medium.

4. For a long solid cylinder of radius R

$$t = t_w + \frac{q_g}{4k}(R^2 - r^2)$$

The temperature distribution in dimensionless form works out as

$$\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2$$

In terms of temperature t_a of the surroundings and outside connective film coefficient h

$$q_g \times \pi r^2 l = h \times 2\pi R l \times (t_w - t_a)$$

That gives :

$$t_w = t_a + \frac{q_g}{2h}R$$

cordingly,

$$t = t_a + \frac{q_g}{2h}R + \frac{q_g}{4k}(R^2 - r^2)$$

emperature t_{max} occurring at $r = 0$ is

$$t_{max} = t_a + \frac{q_g}{2h}R + \frac{q_g}{4k}R^2$$

5. For a nuclear cylindrical rod

(i) Without cladding

$$t_w - t_{max} = -\frac{3q_0R_f^2}{16k}$$

and $$t_{max} - t_a = \frac{q_0}{4}R\left[\frac{3}{4k}R_f + \frac{1}{h}\right]$$

(ii) With cladding

$$t_{max} = t_w + \frac{q_0}{4}R_f^2\left[\frac{3}{4k_f} + \frac{1}{k_c}\log_e\frac{R_c}{R_f}\right]$$

where q_0 is heat generation at centre of the rod, R_f is outer radius of fuel rod, R_c is outer radius of cladding, k_f is thermal conductivity of fuel rod material and k_c is thermal conductivity of cladding rod.

6. For a sphere with uniform heat generation

$$t = t_w + \frac{q_g}{6k}(R^2 - r^2)$$

The temperature distribution is parabolic and the maximum temperature occurring at the centre ($r = 0$) of the sphere is given by

$$t_{max} = t_w + \frac{q_g}{6k}R^2$$

In dimensionless form

$$\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2$$

In terms of temperature t_a of the surroundings and outside convective film coefficient,

$$\frac{4}{3}\pi R^3 q_g = h \times 4\pi R^2 (t_w - t_a)$$

That gives:

$$t_w = t_a + \frac{q_g}{3h}R$$

Accordingly

$$t = t_a + \frac{q_g}{3h}R + \frac{q_g}{6k}(R^2 - r^2)$$

and temperature t_{max} occuring at $r = 0$ is

$$t_{max} = t_a + \frac{q_g}{3h}R + \frac{q_g}{6k}R^2$$

REVIEW QUESTIONS

Conceptual and conventional questions:

1. Mention some of the cases where heat is generated internally at uniform rate in the conducting medium itself.
2. Derive suitable expressions to show that temperature profile for heat conduction through a wall of constant thermal conductivity is a straight line and in the presence of a uniform heat source it takes the shape of a parabola.
3. A flat plate with uniform heat generation has its surfaces maintained at the same constant temperature. Sketch the temperature distribution and temperature. Sketch the temperature distribution and indicate which point of the plate will attain maximum temperature.
4. Consider a long solid cylinder with uniform heat generation and with its outer surface maintained at a constant temperature. Set up an expression which describe the temperature distribution and specify the location of maximum temperature.
5. Set up expressions for temperature distribution during steady state heat conduction in a solid sphere with internal heat generation.
6. Show that for a plane wall of thickness $2l$ with a uniformly distributed heat generation q_g per unit volume, the temperature t_0 at the mid plane is prescribed by the relation,

$$t_0 = \frac{q_g l^2}{2k} + t_w$$

where t_w is the temperature on either side of the wall and k is the thermal conductivity of wall material.

7. Consider a plane wall of thickness $2l$ with temperature t_1 and t_2 on the bounding surfaces. The wall has uniformly distributed heat generation q_g per unit volume. Determine an analytical expression for the dimensionless temperature $(t - t_2)/(t_c - t_2)$ where t_c is the temperature at the centre line of the wall. Further show that the temperature at the centre is the maximum temperature when q_g is positive and the minimum temperature when q_g is negative.
8. A plane wall of thickness 10 cm and thermal conductivity 25 W/m-deg has a volumetric heat generation of 0.3×10^6 W/m^3. The wall is insulated on one side and the other side is exposed to fluid at 90°C temperature. Determine the maximum temperature in the wall if the convective heat transfer coefficient between the wall and fluid is 500 W/m^2 K.
9. A copper bus bar of rectangular cross-section 5 mm × 125 mm experiences uniform heat generation q_g W/m^3 given by

$$q_g = 0.015\, I^2\ \text{W/m}^3\ A^2$$

where I is the electric current in ampere passing through the conductor and A is its area in metre square.

The bar is exposed to ambient air with heat transfer coefficient $h = 5$ W/m^2-deg. Determine the allowable current carrying capacity if the maximum temperature must not exceed that of air by more than 25°C.

10. An electric current of 34000 ampere flows along a flat steel plate (conductivity $k = 55$ W/m-deg and resistivity $\rho = 12 \times 10^{-6}$ ohm/cm) of the thickness 12 mm and width 100 mm. The temperature at one surface of the plate is 100°C and that at the other is 120°C. Workout the temperature distribution across the plate, value and position of the maximum temperature and flow of heat from each surface of the plate. Neglect the end effects and presume that ohmic heating is generated uniformly across the section.
11. Show that in a long-cylinder of radius R with uniformly distributed heat sources, the temperature distribution is prescribed by the relation :

$$\frac{t - t_w}{t_{max} - t_w} = 1 - \left(\frac{r}{R}\right)^2$$

where t_w is the temperature at the outer surface of the cylinder and t_{max} is the temperature along the cylinder axis.

12. An electric current of 150 ampere passes through the 20 mm diameter wire of an electrical resistance heater. Determine the centre line temperature rise above the surface temperature. The wire has electrical resistivity $\rho = 8 \times 10^{-7}$ ohm-cm and thermal conductivity $k = 19$ W/m-deg.

(Ans. 23.75°C)

13. An electric heater made from nichrome wire, measuring 2 mm diameter and 10 m long, carries a current of 25 ampere. Make calculations for the rate of heat flow from 1m length of the heater, and also the temperature at the surface and centre line of the wire. Given that the specific resistance of nichrome

$$\rho = 101 \text{ ohm-mm}^2/\text{m};$$

thermal conductivity of nichrome

$$k = 17.5 \text{ W/m-deg.}$$

and the cold air is blown at the heater at a temperature $t_a = 20°C$ giving a local coefficient of heat transfer from the surface of heater to the air $h = 46.5$ W/m^2-deg.

(Ans. 218.5 W/m; 769 and 77°C)

14. A nuclear element in the form of a hollow cylinder has inner and outer radii 5 cm and 10 cm respectively. The cylinder is insulated at the inner surface and the heat generated is convected over the outside surface of a fluid at 50°C and surface conductance

$$100 \text{ W/m}^2\text{-deg}$$

If thermal conductivity of the cylinder material is 50 W/m-deg, make calculations for the rate of heat generated so that maximum temperature in the system will not exceed 200°C.

(Ans. 380 kW/m^3)

B. Multiple choice questions :

1. Notable examples of uniform generation of heat within the conducting medium are
   1. energy generated in the fuel element of a nuclear reactor
   2. liberation of energy due to some exothermic chemical reactions
   3. resistance heating in electrical appliances

   Which of the statements made above are correct?

   (a) 1, 2 and 3 (b) 1 and 2
   (c) 1 and 3 (d) only 2

2. For a plane wall of thickness l with uniformly distributed heat generation q_g per unit volume, the temperature t_0 at the mid-plane is given by

   (a) $t_0 = \dfrac{q_g l^2}{2k} + t_w$ (b) $t_0 = \dfrac{q_g l^2}{4k} + t_w$

   (c) $t_0 = \dfrac{q_g l^2}{8k} + t_w$ (d) $t_0 = \dfrac{q_g l^2}{16k} + t_w$

   where t_w is the temperature on either side of the wall and k is the thermal conductivity of the wall material

3. The temperature drop in a plane wall with uniformly distributed heat generation can be decreased by reducing

   (a) wall thickness
   (b) heat generation rate
   (c) thermal conductivity of wall material
   (d) convection coefficient at the surface

4. Consider a slab of thickness δ with one side ($x = 0$) insulated and other side ($x = \delta$) maintained at constant temperature T_0 as shown below

Fig. 4.21.

   The rate of uniform heat generation within the slab is q_g W/m^3. Presuming that the heat conduction is in steady state and one-dimensional along x-direction, the maximum temperature in the slab would occur at x equal to

   (a) zero (b) $\delta/4$
   (c) $\delta/2$ (d) δ

5. For a long cylinder of radius R with uniformly distributed heat sources, the temperature distribution in the dimensionless form is

   (a) $\dfrac{t - t_w}{t_{max} - t_w} = 1 - \dfrac{r}{R}$

   (b) $\dfrac{t - t_w}{t_{max} - t_w} = 1 - \left(\dfrac{r}{R}\right)^2$

   (c) $\dfrac{t - t_w}{t_{max} - t_w} = 1 - \left(\dfrac{r}{R}\right)^3$

   (d) $\dfrac{t - t_w}{t_{max} - t_w} = 1 - \left(\dfrac{r}{R}\right)^4$

   where t_w is the temperature at the outer surface of the cylinder and t_{max} is the temperature along the cylinder axis.

6. Consider heat conduction through a long solid cylindrical rod of radius R which has uniformly distributed heat generation q_g per unit volume. Maximum temperature occurring at the centre of rod is

   (a) $t_{max} = t_w + \dfrac{q_g}{k} R^2$

   (b) $t_{max} = t_w + \dfrac{q_g}{2k} R^2$

   (c) $t_{max} = t_w + \dfrac{q_g}{3k} R^2$

   (d) $t_{max} = t_w + \dfrac{q_g}{4k} R^2$

   where t_w is the temperature at the outer surface of cylinder and k is thermal conductivity of the cylinder material.

7. For a cylindrical rod with uniformly distributed heat sources, the thermal gradient dt/dr at half the radius location will be

   (a) one-fourth of that at the surface
   (b) one-half of that at the surface
   (c) twice of that at the surface
   (d) four times of that at the surface

8. Consider convection heat flow to water at 75°C from a cylindrical nuclear reactor fuel rod of 50 mm diameter. Under steady state condition, the rate of heat generation within the fuel element is 5×10^7 W/m^3 and the convective heat transfer coefficient is 1 kW/m^{2}K. The outer surface temperature of the fuel element would be

   (a) 700°C (b) 625°C
   (c) 550°C (d) 400°C

Answers :

1. (a)	2. (c)	3. (a)	4. (a)	5. (b)
6. (d)	7. (b)	8. (a)		

HINTS AND COMMENTS

2(c):

For a plane with uniformly distributed heat generation, the temperature distribution is

$$t = \frac{q_g}{2k}(l - x)\, x + t_w$$

At the mid plane, *i.e.*, at $x = \dfrac{l}{2}$

$$t = \frac{q_g}{2k}\left(l - \frac{l}{2}\right)\frac{l}{2} + t_w$$

$$= \frac{q_g}{8k} l^2 + t_w$$

4(a):

The maximum temperature occurs at the insulated face of the wall where $x = 0$.

8(a):

For heat conduction through a cylindrical wall with uniform heat generation

$$t_w = t_a + \frac{q_g}{2h} R$$

$$= 75 + \frac{5 \times 10^7}{2 \times 1000} \times 0.025 = 700°C$$

7(b):

The temperature distribution through a cylindrical rod with uniformly distributed heat sources is parabolic and prescribed as

$$t = t_w + \frac{q_g}{4k}(R^2 - r^2)$$

$$\frac{dt}{dr} = \frac{q_g}{4k}(-2r)$$

$$\left(\frac{dt}{dr}\right)_{r=R} = \frac{q_g}{4k} \times (-2R)$$

$$\left(\frac{dt}{dr}\right)_{r=\frac{R}{2}} = \frac{q_g}{4k} \times \left(-2 \times \frac{R}{2}\right)$$

Apparently

$$\left(\frac{dt}{dr}\right)_{r=\frac{R}{2}} = \frac{1}{2}\left(\frac{dt}{dr}\right)_{r=R}$$

Chapter

5 Heat Transfer From Extended Surfaces

Learning objectives : A study of the subject matter included in this chapter will enable the reader to have an understanding of

- *finned surfaces, their configuration and common applications*
- *steady flow of heat along a rod*
- *heat dissipation from (i) an infinitely long fin (ii) a fin losing heat at the tip (iii) a fin insulated at the tip*
- *performance parameters : efficiency and effectiveness of fin*
- *thermometric well and its utility*

In many engineering situations, means are often sought to improve heat dissipation from a surface to its surroundings. The Newton-Rikhman relation $Q = hA\,(t - t_a)$ reveals that the convective heat flow can be enhanced by increasing the film coefficient h, the surface area A and the temperature difference $(t - t_a)$. The convective coefficient is a function of the geometry, fluid properties and the flow rate. Control of h through these parameters helps to obtain its optimum value. With regard to the effect of temperature excess $(t - t_a)$, difficulties are encountered when the ambient temperature t_a is too high particularly in hot weather conditions. The surface area exposed

to the surroundings is frequently increased by the attachment of protrusions to the surfaces, and the arrangement provides a means by which heat transfer rate can be substantially improved. The protrusions are called *fins* or *spines*, and these extensions can take a variety of forms; the most common types are illustrated in Fig. 5.1.

A straight fin is an extended surface attached to a plane wall; the cross-sectional area of the fin may be uniform or it may vary with distance from the wall (Fig. 5.1 *a, b*). Annular fins are attached circumferentially to a cylindrical surface and their cross-sectional area varies with radius from the centre line of the cylinder (Fig. 5.1*c*). However, both the straight and annular fins are of rectangular cross-section, whose area can be expressed as a product of the fin thickness δ and width *l* for straight fins or the circumference 2πr for annular fins. In contrast a pin fin or a spine is an extended surface of circular cross-section (Fig. 5.1*d*) which may be uniform or non-uniform. Thus a spine represents a thin cylindrical or conical rod protruding from a wall.

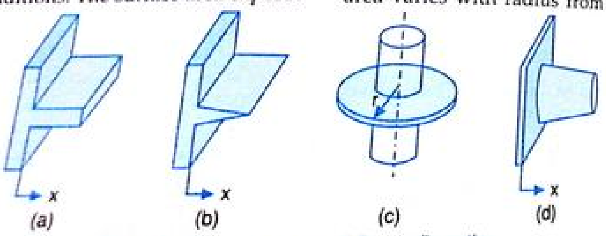


Fig. 5.1. Common types of fin configuration

Common applications of finned surfaces are with

- Economisers for steam power plants ;
- Convectors for steam and hot water heating of systems ;
- Air cooled cylinders of aircraft engines, I.C. engines and air compressors ;
- Electrical transformers and motors ;
- Cooling coils and condenser coils in refrigerators and air conditioners ;
- Electronic equipment etc.

This chapter concentrates on determination of temperature distribution and heat flow from different types of fins. The knowledge of temperature distribution is necessary for their optimum design with regard to size and weight.

5.1. STEADY FLOW OF HEAT ALONG A ROD

(*Governing differential equation*)

Consider a straight rectangular fin or a pin fin (spine) protruding from a wall surface (Fig. 5.2). The characteristic dimensions of the fin are its length l, constant cross-sectional area A_c and the circumferential parameter P. Thus for a rectangular fin

$$A_c = b\,\delta \;;\quad P = 2\,(b + \delta)$$

and for the spine

$$A_c = \frac{\pi}{4} d^2 \;;\quad P = \pi d$$

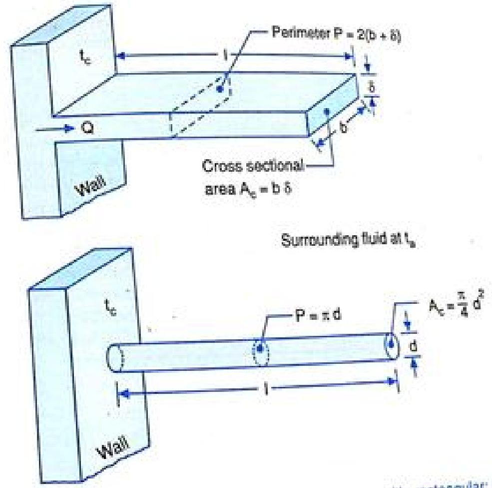


Fig. 5.2. Extended surfaces with uniform cross-section(a) Fin with rectangular; (b) Fin with circular profile (spine)

The temperature at the base of the fin is t_0 and the temperature of the ambient fluid into which the rod extends is considered to be constant at temperature t_a. The base temperature t_0 is highest and the temperature along the fin length goes on diminishing (Fig. 5.3)

Analysis of heat flow form the finned surface is made with the following assumptions :

- *thickness of the fin is small compared with the length and width; temperature gradients over the cross-section are neglected and heat conduction treated one dimensional*
- *homogeneous and isotropic fin material; the thermal conductivity k of the fin material is constant*
- *uniform heat transfer coefficient h over the entire fin surface*
- *no heat generation within the fin itself*
- *joint between the fin and the heated wall offers no bond resistance; temperature at root or base of the fin is uniform and equal to temperature t_0 of the wall*

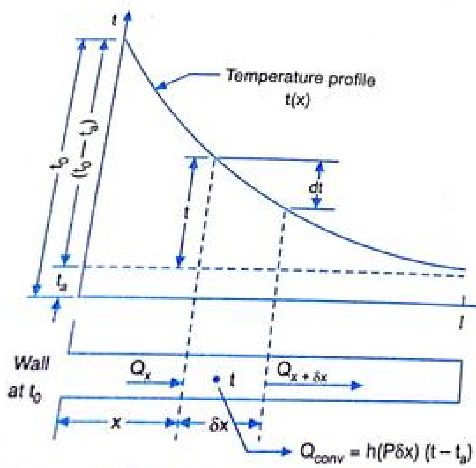


Fig. 5.3. Heat transfer through a fin

- negligible radiation exchange with the surroundings; radiation effects, if any, are considered as included in the convection coefficient h
- steady state heat dissipation

Heat from the heated wall is conducted through the fin and convected from the sides of the fin to the surroundings. Let attention be focused on an infinitesimal element of the fin; the element has thickness δx and is located at a distance x from base wall (Fig. 5.3)

(*i*) Heat conducted into the element at plane x

$$Q_x = -kA_c\left(\frac{dt}{dx}\right)_x \qquad ...(5.1)$$

(*ii*) Heat conducted out of the element at plane $(x + \delta x)$

$$Q_{x+\delta x} = -kA_c\left(\frac{dt}{dx}\right)_{x+\delta x}$$

$$= -kA_c\frac{d}{dx}\left(t + \frac{dt}{dx}\delta x\right) \qquad ...(5.2)$$

(*iii*) Heat convected out of the element between the planes x and $(x + \delta x)$

$$Q_{conv} = h\,(P\,\delta x)\,(t - t_a) \qquad ...(5.3)$$

Here temperature t of the fin has been assumed to be uniform and non-variant for the infinitesimal element.

A heat balance on the element gives:

$$Q_x = Q_{x+\delta x} + Q_{conv}$$

$$-kA_c\frac{dt}{dx} = -kA_c\frac{d}{dx}\left(t + \frac{dt}{dx}\delta x\right) + h\,P\,\delta x\,(t - t_a)$$

Upon rearrangement and simplification

$$\frac{d^2t}{dx^2} - \frac{hP}{kA_c}(t - t_a) = 0 \qquad ...(5.4)$$

Equation 5.4 is further simplified by transforming the dependent variable by defining the temperature excess θ as,

$$\theta(x) = t(x) - t_a$$

Since the ambient temperature t_a is constant, we get by differentiation

$$\frac{d\theta}{dx} = \frac{dt}{dx}\ ;\quad \frac{d^2\theta}{dx^2} = \frac{d^2t}{dx^2}$$

Thus,
$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad ...(5.5)$$

where
$$m = \sqrt{\frac{hP}{kA_c}}$$

Equations 5.4 and 5.5 provide a general form of the energy equation for one-dimensional heat dissipation from an extended surface. For a given fin, the parameter m is constant provided the convective film coefficient h is constant over the entire surface and the thermal conductivity k is constant within the considered temperature range. Then the general solution of this linear, homogeneous second order differential equation is of the form

$$\theta = C_1\,e^{mx} + C_2\,e^{-mx} \qquad ...(5.6)$$

The constant C_1 and C_2 are to be determined with the aid of relevant boundary conditions.

The following relations will be found useful while making further analysis of heat dissipation through a finned surface.

(*i*) $\sinh mx = \dfrac{e^{mx} - e^{-mx}}{2}$

$\sinh(0) = 0$

$\dfrac{d}{dx}\sinh mx = m\cosh mx$

(*ii*) $\cosh mx = \dfrac{e^{mx} + e^{-mx}}{2}$

$\cosh(0) = 1$

$\dfrac{d}{dx}\cosh mx = m\sinh mx$

5.2. HEAT DISSIPATION FROM AN INFINITELY LONG FIN ($l \to \infty$)

The relevant boundary conditions are

(*i*) Temperature at the base of fin equals the temperature of the surface to which the fin is attached.

$$t = t_0 \quad \text{at} \quad x = 0$$

In terms of excess temperature

$$t - t_a = t_0 - t_a$$

or $\quad \theta = \theta_0 \quad \text{at} \quad x = 0$

(*ii*) Temperature at the end of an infinitely long fin equals that of the surroundings.

$$t = t_a \quad \text{at} \quad x = \infty$$

$$\theta = 0 \quad \text{at} \quad x = \infty$$

Substitution of these boundary conditions in equation 5.6 gives :

$$C_1 + C_2 = \theta_0 \qquad ...(a)$$

$$C_1\,e^{m\infty} + C_2\,e^{-m(\infty)} = 0 \qquad ...(b)$$

Since the term $C_2\,e^{-m(\infty)}$ is zero, the equality is valid only if $C_1 = 0$. Then it follows from relation (a) that $C_2 = \theta_0$. Substituting these values of constants C_1 and C_2 in equation 5.6, one obtains the following expression for temperature distribution along the length of the fin

$$\theta = \theta_0\,e^{-mx}\ ;$$

$$(t - t_a) = (t_0 - t_a)\,e^{-mx} \qquad ...(5.7)$$

Fig. 5.4 shows the dependence of dimensionless temperature $(t - t_a)/(t_0 - t_a)$ along the length of fin for different values of parameter m ($m_1 < m_2 < m_3$). The plot indicates that the dimensionless temperature falls more

with increase in factor m. With the fin length extending to infinity, $x \to \infty$, all the curves approach $(t - t_a)/(t_0 - t_a) = 0$ asymptotically.

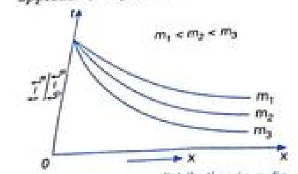


Fig. 5.4. Temperature distribution in a fin

The most important design variable for a fin is the amount of heat that it can remove from the heated wall and dissipate it to the surroundings.

An estimate of the heat flow rate can be made by writing the Fourier rate equation corresponding to root section of the fin

$$Q_{fin} = -kA_c\left(\frac{dt}{dx}\right)_{x=0}$$

From the expression for temperature distribution

$$t = t_a + (t_0 - t_a)\, e^{-mx}$$

$$\left(\frac{dt}{dx}\right)_{x=0} = [-m\,(t_0 - t_a)\, e^{-mx}]_{x=0}$$

$$= -m\,(t_0 - t_a)$$

$$\therefore \quad Q_{fin} = kA_c\, m\,(t_0 - t_a) \qquad \text{...(5.8)}$$

Recalling that

$$m = \sqrt{\frac{Ph}{kA_c}}$$

$$Q_{fin} = \sqrt{P\,h\,k\,A_c}\;(t_0 - t_a) \qquad \text{...(5.8a)}$$

Alternatively the heat flow rate can be worked out by integrating the expression for convective heat transport from the infinitesimal element of the fin surface to the surroundings.

$$Q_{fin} = \int_0^\infty hP\,dx\,(t - t_a)$$

$$= \int_0^\infty hP\,(t_0 - t_a)\,e^{-mx}\,dx$$

$$= hP\,(t_0 - t_a)\int_0^\infty e^{-mx}\,dx$$

$$= hP(t_0 - t_a)\frac{1}{m}$$

$$= hP(t_0 - t_a)\sqrt{\frac{kA_c}{Ph}}$$

$$= \sqrt{PhkA_c}\;(t_0 - t_a)$$

which is the same as evaluated above (Equation 5.8)

Equations 5.7 and 5.8 are reasonable approximations of temperature distribution and heat flow rate in a finite fin if its length is very large compared to its cross-sectional area.

The temperature distribution (Equation 5.7) would suggest that the temperature drops towards the tip of the fin. Hence area near the fin tip is not utilized to the extent as the lateral area near the base. Obviously an increase in the fin length beyond a certain point does not pay much regarding an increase in the heat dissipation. A tapered fin is then considered to be a better design as it has more lateral area near the base where the difference in temperature is high.

Heat flow rates through solids can be compared by having an arrangement consisting essentially of a box to which rods of different materials are attached (Ingen-Hausz experiment). The rods are of same length and area of cross-section (same size and shape); their outer surfaces are electroplated with the same material and are equally polished. This is to ensure that for each rod, the surface heat transfer will be same. The procedure would involve coating the rods with wax and filling the box with be same. Heat flow from the box along the rod would melt the wax for a distance which would depend upon the rod material. Let

θ_0 = excess of temperature of the hot bath above the ambient temperature

θ_m = excess of temperature of melting point of wax above the ambient temperature

$l_1, l_2, l_3, \ldots$ = lengths upto which wax melts.

Then for different rods (treating each as fin of infinite length),

$$\theta_m = \theta_0\, e^{-m_1 l_1}$$
$$= \theta_0\, e^{-m_2 l_2}$$
$$= \theta_0\, e^{-m_3 l_3}$$

Obviously then

$$m_1 l_1 = m_2 l_2 = m_3 l_3$$

or $$\sqrt{\frac{hP}{k_1 A}}\,l_1 = \sqrt{\frac{hP}{k_2 A}}\,l_2 = \sqrt{\frac{hP}{k_3 A}}\,l_3$$

or $$\frac{l_1}{\sqrt{k_1}} = \frac{l_2}{\sqrt{k_2}} = \frac{l_3}{\sqrt{k_3}} = \text{const}$$

or $$\frac{k_1}{l_1^2} = \frac{k_2}{l_2^2} = \frac{k_3}{l_3^2} = \text{const}$$

Thus, the thermal conductivity of the material of the rod is directly proportional to the square of the length upto which the wax melts on the rod.

EXAMPLE 5.1

Two long rods of the same diameter, one made of brass (k = 85 W/m-deg) and the other of copper (k = 375 W/m-deg), have one of their ends inserted into a furnace. At a section 10.5 cm away from the furnace, the temperature of the brass rod is 120°C. At what distance from the furnace end, the same temperature would be reached in the copper rod. Both rods are exposed to the same environment.

Solution : Treating the rods as infinitely long fins, the temperature distribution is prescribed by the relation,

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = e^{-mx}$$

or $$t = t_a + (t_0 - t_a)\, e^{-mx}$$

For brass rod :

$$120 = t_a + (t_0 - t_a)\, e^{-m_1 l_1} \qquad \text{...(i)}$$

For copper rod :

$$120 = t_a + (t_0 - t_a)\, e^{-m_2 l_2} \qquad \text{...(ii)}$$

where l_1 and l_2 are the lengths upto which same temperature is reached in both the rods. Obviously from expressions (i) and (ii), we get

$$m_1 l_1 = m_2 l_2$$

or $$\sqrt{\frac{h_1 P_1}{k_1 A_1}}\,l_1 = \sqrt{\frac{h_2 P_2}{k_2 A_2}}\,l_2$$

Since the rods are exposed to the same environment ($h_1 = h_2$) and are of the same diameter ($P_1 = P_2$ and $A_1 = A_2$) we get

$$\frac{l_1}{\sqrt{k_1}} = \frac{l_2}{\sqrt{k_2}}$$

or $$l_2 = l_1\sqrt{\frac{k_2}{k_1}} = 10.5\sqrt{\frac{375}{85}} = 22.05 \text{ cm}$$

EXAMPLE 5.2

Three rods, one made of silver (k = 420 W/m-deg), second made of aluminium (k = 210 W/m-deg) and the third made of wrought iron (k = 70 W/m-deg) are coated with a uniform layer of wax all around. The rods are placed vertically in a boiling water bath with 250 mm length of each rod projecting outside. If all rods are 15 mm diameter, 300 mm length and have identical surface coefficient 12.5 W/m²-deg, work out the ratio of lengths upto which wax will melt on each rod.

Solution : Let l_1, l_2, and l_3 be the lengths upto which wax will melt on each rod. Then

$$\frac{k_1}{l_1^2} = \frac{k_2}{l_2^2} = \frac{k_3}{l_3^2}$$
(i) (ii) (iii)

From (i) and (ii) we get,

$$\frac{l_1}{l_2} = \left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} = \left(\frac{420}{210}\right)^{\frac{1}{2}} = 1.414$$

From (ii) and (iii) we get,

$$\frac{l_2}{l_3} = \left(\frac{k_2}{k_3}\right)^{\frac{1}{2}} = \left(\frac{210}{70}\right)^{\frac{1}{2}} = 1.732$$

Therefore,

$$l_1 : l_2 : l_3 = (1.414 \times 1.732) : 1.732 : 1$$
$$= 2.45 : 1.732 : 1$$

EXAMPLE 5.3

Estimate the energy input required to solder together two very long pieces of bare copper wire 0.1625 cm in diameter with a solder that melts at 195°C. The wires are positioned vertically in air at 24°C and the heat transfer coefficient on the wire surface is 17 W/m²-deg. For the wire alloy, take the thermal conductivity 335 W/m-deg.

Solution : The physical situation approximates as two infinite fins with a base temperature of 195°C in an environment at 24°C with the given value of surface coefficient.

Cross-sectional area,

$$A_c = \frac{\pi}{4} d^2 = \frac{\pi}{4}(0.001625)^2 = 2.073 \times 10^{-6}\ \text{m}^2$$

Perimeter, P

$$= \pi d = \pi \times 0.001625 = 0.0051\ \text{m}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{17 \times 0.0051}{335 \times 2.073 \times 10^{-6}}} = 11.17\ \text{m}^{-1}$$

Heat dissipation from an infinitely long fin is

$$Q_{fin} = k A_c\, m\, (t_0 - t_a)$$
$$= 335 \times (2.073 \times 10^{-6}) \times 11.17\,(195 - 24)$$
$$= 1.326\ \text{W}$$

Therefore, the energy input required for two wires is **2.652 W**

EXAMPLE 5.4

A rod of 10 mm square section and 160 mm length with thermal conductivity of 50 W/m-deg protrudes from a furnace wall at 200°C, and is exposed to air at 30°C with convection coefficient

20 W/m²-deg

Make calculations for the heat convected upto 80 mm and 158 mm lengths and comment on the result. Adopt a long fin model for the arrangement.

Solution : Heat dissipation for an infinitely long fin is

$$Q = k A_c\, m\, (t_0 - t_a)$$

where

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{20 \times (4 \times 0.01)}{50 \times (0.01 \times 0.01)}} = 12.649\ \text{m}^{-1}$$

$$\therefore \quad Q = 50 \times (0.01 \times 0.01) \times 12.649 \times (200 - 30)$$
$$= \mathbf{10.75\ W}$$

For the long fin model, the temperature distribution is

$$\frac{\theta}{\theta_0} = \frac{T - T_a}{T_0 - T_a} = e^{-mx}$$

At $x = 80$ mm $= 0.08$ m

$$mx = 12.649 \times 0.08 = 1.01192$$

$$\frac{T - 30}{200 - 30} = e^{-1.01192} = 0.3635$$

$$T = 0.3635 \times (200 - 30) + 30 = 91.8°\text{C}$$

At $x = 158$ m $= 0.158$ m ;

$$mx = 12.649 \times 0.158 = 1.9985$$

$$\frac{T - 30}{200 - 30} = e^{-1.9985} = 0.1355$$

$$T = 0.1355 \times (200 - 30) + 30 = 53.04°\text{C}$$

Heat conducted upto any length is worked out by taking the difference of total heat and heat conducted at that section.

∴ Heat convected upto 0.08 m length

$$= 10.75 - k A_c\, m\, (t_{0.08} - t_a)$$
$$= 10.75 - 50 \times (0.01 \times 0.01) \times 12.649\,(91.8 - 30)$$
$$= 6.84\ \text{W}$$

which is $\frac{6.84}{10.75} \times 100 = 63.63\%$ of total heat dissipation.

Fig. 5.5.

Heat convected upto 0.158 m length

$$= 10.75 - k A_c\, m\, (t_{0.158} - t_a)$$
$$= 10.75 - 50 \times (0.1 \times 0.1) \times 12.649\,(53.04 - 30)$$
$$= 9.293\ \text{W}$$

which is $\frac{9.293}{10.75} \times 100 = 86.4\%$ of total heat dissipation.

Comments: Most of heat is dissipated in a short length of the fin. Accordingly it is uneconomical to extend the fin length beyond a certain value.

5.3. HEAT DISSIPATION FROM A FIN INSULATED AT THE TIP

The fin is of any finite length with the end insulated and so no heat is transferred from the tip. Therefore, the relevant boundary conditions are :

(*i*) $\theta = \theta_0$ at $x = 0$

(*ii*) $\frac{dt}{dx} = 0$ at $x = l$

Applying these boundary conditions to equation 5.6

$$C_1 + C_2 = \theta_0 \qquad \text{...(a)}$$

Further

$$t - t_a = C_1 e^{mx} + C_2 e^{-mx}$$

$$\frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}$$

$$\left(\frac{dt}{dx}\right)_{x=l} = mC_1 e^{ml} - mC_2 e^{-ml}$$

$$\therefore C_1 e^{ml} - C_2 e^{-ml} = 0 \qquad \text{...(b)}$$

Solving expressions (*a*) and (*b*), the constants are determined as follows :

$$C_1 = \theta_0 \left[\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right]$$

$$C_2 = \theta_0 \left[\frac{e^{ml}}{e^{ml} + e^{-ml}}\right]$$

Substituting these values of constant C_1 and C_2 in equation 5.6, one obtains the following expression for temperature distribution along the length of fin

$$\frac{\theta}{\theta_0} = \frac{e^{m(l-x)} + e^{-m(l-x)}}{e^{ml} + e^{-ml}}$$

Expressing in terms of hyperbolic functions,

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh m(l - x)}{\cosh ml} \qquad \text{...(5.9)}$$

The rate of heat flow from the fin is given by :

$$Q_{fin} = -kA_c \left(\frac{dt}{dx}\right)_{x=0}$$

From the expression for temperature distribution

$$t - t_a = (t_0 - t_a)\frac{\cosh m(l - x)}{\cosh ml}$$

$$\frac{dt}{dx} = (t_0 - t_a)\frac{\sinh m(l - x)}{\cosh ml}(-m)$$

$$\left(\frac{dt}{dx}\right)_{x=0} = -m\,(t_0 - t_a)\tanh ml$$

$$\therefore \quad Q_{fin} = kA_c m (t_0 - t_a)\tanh ml$$
$$= \sqrt{PhkA_c}\,(t_0 - t_a)\tanh ml \qquad \text{...(5.10)}$$

EXAMPLE 5.5

A carbon steel rod (k = 55 W/m-deg) has been attached to a plane wall which is maintained at a temperature of 350°C. The rod is 8 cm long and has the cross-section of an equilateral triangle with each side 5 mm. Determine the heat dissipation from the rod if it is exposed to a convection environment at 25°C with unit surface conductance 100 W/m²-deg. Consider end surface loss to be negligible.

Solution : For a fin of triangular cross-section

$$P = 3a$$

Fig. 5.6.

$$A_c = \frac{1}{2}a\left(\sqrt{3}\frac{a}{2}\right) = \frac{\sqrt{3}}{4}a^2$$

$$m = \sqrt{\frac{hP}{kA_c}}$$

$$= \left[\frac{100 \times (3 \times 0.005)}{55 \times \frac{\sqrt{3}}{4} \times (0.005)^2}\right]^{½}$$

$= 50.19 \text{ m}^{-1}$

For a fin with end loss negligible (tip insulated)

$$Q = k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

$$= 55 \times \frac{\sqrt{3}}{4}(0.005)^2 \times 50.19$$

$$\times (350 - 40)\tanh(50.19 \times 0.08)$$

$= 9.26$ **W**

EXAMPLE 5.6

Which of the following arrangement of pin fins will give higher heat transfer rate from a hot surface ?

6 fins of 10 cm length

12 fins of 5 cm length

The base temperature of the fin is maintained at 200°C and the fin is exposed to a convection environment at 15°C with convection coefficient 25 W/m²-deg. Each fin has cross-sectional area 2.5 cm², perimeter 5 cm and is made of a material having thermal conductivity 250 W/m-deg.

Neglect the heat loss from the tip of fin.

Solution : The heat loss from n-fins is given by

$$Q = n\,k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

where

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{25 \times 0.05}{250 \times 2.5 \times 10^{-4}}} = 4.472 \text{ m}^{-1}$$

Case I $n = 6$ and $l = 10$ cm $= 0.1$ m

$$ml = 4.4772 \times 0.1 = 0.4472$$

$$Q_1 = 6\left[250 \times 2.5 \times 10^{-4} \times 4.472 \times (200 - 15)\tan h\,(0.4472)\right]$$

$$= 130.18 \text{ W}$$

Case II $n = 12$ and $l = 5$ cm $= 0.05$ m

$$ml = 4.472 \times 0.05 = 0.2236$$

$$Q_2 = 12\left[250 \times 2.5 \times 10^{-4} \times 4.472 \times (200 - 15)\tan h\,(0.2236)\right]$$

$$= 138.63 \text{ W}$$

The arrangement II is to be preferred as it gives higher rate of heat transfer.

EXAMPLE 5.7

An array of 10 fins of anodised aluminium (k = 180 W/m-deg) is used to cool a transistor operating at a location where the ambient conditions correspond to temperature 35°C and convective coefficient 12 W/m²-deg. Each fin measures 3 mm wide × 0.4 mm thick × 5 cm length and has its base at 60°C. Determine the power dissipated by the fin array.

Solution : Refer Fig 5.7 for the fin array. The length of the fin is represented by projection perpendicular to the plane of the pipe.

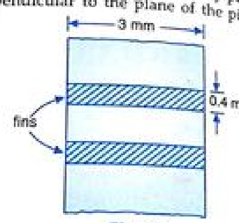


Fig. 5.7.

For a fin of rectangular cross-section,

$$P = 2(b + \delta)$$

$$= 2(3 + 0.4) = 6.8 \text{ mm}$$

$$A_c = b \times \delta = 3 \times 0.4 = 1.2 \text{ mm}^2$$

$$m = \sqrt{\frac{hP}{kA_c}}$$

$$= \sqrt{\frac{12 \times (6.8 \times 10^{-3})}{180 \times (1.2 \times 10^{-6})}}$$

$$= 19.44 \text{ m}^{-1}$$

The arrangement corresponds to a fin with tip insulated and for that

$$Q = k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

$$= 180 \times (1.2 \times 10^{-6}) \times 19.44 \times (60 - 35)\tanh(19.44 \times 0.05)$$

$$= 0.0786 \text{ W}$$

∴ Heat loss from the array of 10 fins,

$= 0.0786 \times 10 =$ **0.786 W**

EXAMPLE 5.8

A heating unit is made in the form of a vertical tube of 50 mm outside diameter and 1.2 m height. The tube is fitted with 20 steel fins of rectangular section with height 40 mm and thickness 2.5 mm. The temperature at the base of fin is 75°C, the surrounding air temperature is 20°C and the heat transfer coefficient between the fin as well as the tube surface and the surrounding air is 9.5 W/m²K. If thermal conductivity of the fin material is 55 W/mK, make calculations for the amount of heat transferred from the tube with and without fin.

Solution : Heat flow rate from the tube surface without fin

$$Q_1 = h\,A\,\Delta t \quad = h \times \pi\,doH \times (t_o - t_\infty)$$

$$= 9.5 \times (\pi \times 0.05 \times 1.2) \times (75 - 20)$$

$$= 98.44 \text{ W}$$

(ii) Heat flow rate convected from the base

$$Q_b = h\,A_b\,(t_o - t_\infty)$$

where

$$A_b = (\pi \times 0.05 \times 1.2) - 20\,(1.2 \times .0025)$$

$$= 0.1284 \text{ m}^2$$

$$\therefore \quad Q_b = 9.5 \times 0.1284 \times (75 - 20)$$

$$= 67.09 \text{ W}$$

Heat flow rate convected from the fins,

$$Q_f = n\,k\,A_c\,m\,(t_o - t_a)\tanh ml$$

where A_c = cross-sectional area of fin

$$= 1.2 \times 0.0025 = 0.003 \text{ m}^2$$

P = perimeter of fin

$$= 2(1.2 + 0.0025) = 2.405 \text{ m}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{9.5}{55} \times \frac{2.405}{0.003}} = 11.77$$

Then, $Q_f = 20 \times 55 \times 0.003 \times 11.77 \times (75 - 20)\tan h\,(11.77 \times 0.04)$

$$= 937.75 \text{ W}$$

∴ Heat flow rate from the tube surface when fins are fitted,

$$Q_2 = Q_b + Q_f$$

$$= 67.09 + 937.75 = 1004.84 \text{ W}$$

EXAMPLE 5.9

An electronic semiconductor device generates 0.16 kJ/hr of heat. To keep the surface temperature at the upper safe limit of 75°C, it is desired that the generated heat should be dissipated to the surrounding environment which is at 30°C. The task is accomplished by attaching aluminium fins, 0.5 mm² square and 10 mm to the surface. Work out the number of fins if thermal conductivity of fin material is 690 kJ/m-hr-deg and the heat transfer coefficient is 45 kJ/m²-hr-deg. Neglect the heat loss from the tip of the fin.

Solution : For a fin of rectangular cross-section,

$$P = 2(b + \delta) = 2(0.5 + 0.5)$$

$$= 2 \text{ mm} = 2 \times 10^{-3} \text{ m}$$

$$A_c = b \times \delta = 0.5 \times 0.5$$

$$= 0.25 \text{ mm}^2$$

$$= 0.25 \times 10^{-6} \text{ m}^2$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{45 \times 2 \times 10^{-3}}{690 \times 0.25 \times 10^{-6}}} = 22.84 \text{ m}^{-1}$$

For a fin with insulated tip,

$$Q_{fin} = k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

$$= 690 \times (0.25 \times 10^{-6}) \times 22.84 \times (75 - 30)\tanh(22.84 \times 0.01)$$

$$= 39.77 \times 10^{-3} \text{ kJ/hr per fin}$$

∴ Number of fins

$$= \frac{0.16}{39.77 \times 10^{-3}} = 4.02$$

Thus, 4 fins are needed to dissipate the required amount of heat.

EXAMPLE 5.10

A steel strap is serving as a support for the steam pipe shown in the adjoining figure. The strap is welded to the pipe and bolted to the ceiling. The junction between the support strut and the ceiling is adiabatic, and the outside temperature of the steam pipe is 105°C. The strut is 60 cm high, 12.5 cm wide and 0.3 cm thick. Work out the rate at which heat is lost to the surrounding air by the support strut. It may be assumed that thermal

conductivity for steel is 45 W/m-deg, the total outside surface coefficient is 17 W/m²-deg, and the surrounding air is at 32°C.

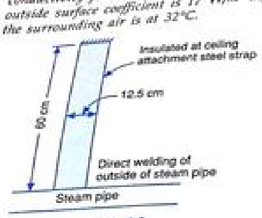


Fig. 5.8.

Solution : For a fin of rectangular cross-section

$$P = 2\,(b + \delta)$$
$$= 2\,(0.125 + 0.003) = 0.256 \text{ m}$$
$$A_c = b \times \delta = 0.125 \times 0.003$$
$$= 3.75 \times 10^{-4} \text{ m}^2$$
$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{17 \times 0.256}{45 \times 3.75 \times 10^{-4}}}$$
$$= 16.06 \text{ m}^{-1}$$

For the support strut with its tip insulated

$$Q = k\,A_c\,m\,(t_0 - t_a) \tanh ml$$
$$= 45 \times (3.75 \times 10^{-4}) \times 16.06 \times (105 - 32) \tanh (16.06 \times 0.6)$$
$$= \mathbf{178\ W}$$

EXAMPLE 5.11

A rod of 10 mm diameter and 80 mm length with thermal conductivity 16 W/m-deg protrudes from a surface at 160°C. The rod is exposed to air at 30°C with a convection coefficient of 25 W/m²-deg. How does the heat flow from this rod get affected if the same material volume is used for two fins of the same length ? Assume short fin with end insulated.

Solution : For a short fin with insulated tip,

$$Q = k\,A_c\,m\,(t_0 - t_a) \tanh ml$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h \times \pi d}{k\frac{\pi}{4}d^2}} = \sqrt{\frac{4h}{kd}}$$

Case I :

$$m_1 = \sqrt{\frac{4 \times 25}{16 \times 0.01}} = 25 \text{ m}^{-1}$$
$$m_1\,l = 25 \times 0.08 = 2$$
$$\therefore \quad Q_1 = 16 \times \frac{\pi}{4}\,(0.01)^2 \times 25 \times (160 - 30) \tanh (2)$$
$$= 3.935 \text{ W}$$

Case II : Let d be the new diameter. Then for the same material volume

$$2 \times \frac{\pi}{4}\,d^2 \times 0.08 = \frac{\pi}{4}\,(0.01)^2 \times 0.08\ ;$$
$$d = 0.00707 \text{ m}$$

Then :

$$m_2 = \sqrt{\frac{4 \times 25}{16 \times 0.00707}} = 29.73$$
$$m_2\,l = 29.73 \times 0.08 = 2.378$$
$$Q_2 = 16\frac{\pi}{4}\,(0.00707)^2 \times 29.73 \times (160 - 30) \tanh (2.378)$$
$$= 2.385 \text{ W}$$

For two fins

$$Q_2 = 2 \times 2.385 = \mathbf{4.770\ W}$$

Percentage increase in heat flow

$$= \frac{4.77 - 3.935}{3.935} \times 100 = \mathbf{21.2\%}$$

Comments : A thinner or a low sectional area fin is a better choice.

EXAMPLE 5.12

A plate fin of 10 mm thickness and 80 mm length is dissipating heat from a surface at 190°C. The fin is exposed to air at 25°C with a convection coefficient of 22 W/m²-deg. If thermal conductivity of the fin material is 200 W/m-deg, determine the heat dissipation. Consider 1 m width of fin.

(b) To increase the heat dissipation, the following two alternatives have been suggested with the same material volume.

(i) Split the fin into two fins of 5 mm thickness each

Single fin 5 mm thick and 160 mm long. Which will be the better choice ? The fins may be considered short with tip insulated.

Solution : For a short fin with tip insulated

$$Q = k\,A_c\,m\,(t_0 - t_a) \tanh ml$$

Here $P = 2\,(b + \delta)$

$$= 2\,(1 + 0.01) = 2.02 \text{ m}$$
$$A_c = b \times \delta = 1 \times 0.01 = 0.01 \text{ m}^2$$
$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{22 \times 2.02}{200 \times 0.01}}\ 4.714 \text{ m}^{-1}$$
$$ml = 4.714 \times 0.08 = 0.377$$

Heat dissipation,

$$Q = 200 \times 0.01 \times 4.714\,(190 - 25) \tanh (0.377)$$
$$= \mathbf{560.18\ W\ per\ m\ width}$$

(i) **Case I :** Two fins 5 mm thick and 80 mm length

$$P = 2\,(b + \delta) = 2\,(1 + 0.005)$$
$$= 2.01 \text{ m}$$
$$A_c = b \times \delta = 1 \times 0.005 = 0.005 \text{ m}^2$$
$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{22 \times 2.01}{200 \times 0.005}}$$
$$= 6.65 \text{ m}^{-1}$$
$$ml = 6.65 \times 0.08 = 0.532$$
$$Q = 200 \times 0.005 \times 6.65\,(190 - 25) \tanh (0.532)$$
$$= 534.26 \text{ W for one fin}$$

Heat dissipation from two fins

$$= \mathbf{1068.52\ W\ per\ m\ width}$$

Percentage increase in heat dissipation

$$= \frac{1068.52 - 560.18}{560.18} \times 100$$
$$= \mathbf{90.65\%}$$

Case II : One fin 5 mm thick and 160 mm length

$$P = 2\,(b + \delta)$$
$$= 2\,(1 + 0.005) = 2.01 \text{ m}$$
$$A_c = b\delta = 1 \times 0.005 = 0.005 \text{ m}^2$$
$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{22 \times 2.01}{200 \times 0.005}}$$
$$= 6.65 \text{ m}^{-1}$$
$$ml = 0.65 \times 0.16 = 1.064$$
$$Q = 200 \times 0.005 \times 6.65\,(190 - 25) \times \tanh (1.064)$$
$$= 863.744 \text{ W}$$

Percentage increase in heat dissipation

$$= \frac{863.744 - 560.18}{560.18} \times 100$$
$$= 54.19\%$$

There is an improvement of only 54.19% against 90.65% in case of two fins of the original length. Obviously increase in length is not effective.

EXAMPLE 5.13.

A transistor, idealized as a long cylinder of 1 mm radius and 2 cm length, is provided 1 mm thick sheath covering of thermal conductivity 0.125 W/mK.

The transistor is 200 mW rating and it loses heat from its surface to the surroundings at 25°C with surface heat transfer coefficient of 15 W/m²K. For proper functioning, the surface temperature (at radius 1 mm) of the transistor is not to exceed 75°C. Calculate the heat that needs to be dissipated to the atmosphere by three brass leads (k = 125 W/mK). Proceed to find the length of these leads if they are circular with 0.3 mm diameter. Consider the lead as fin with insulted end.

Solution : The heat transfer from the transistor to the atmosphere is

$$Q = \frac{\Delta t}{R_1 + R_2} = \frac{t_1 - t_a}{R_1 + R_2}$$

where R_1 = resistance due to conduction through insulation

$$= \frac{1}{2\pi kl}\log_e \frac{r_2}{r_1}$$
$$= \frac{1}{2\pi \times 0.125 \times 0.02}\log_e\left(\frac{0.002}{0.001}\right)$$
$$= 44.14 \text{ K/W}$$

and R_2 = resistance due to convection from the surface

$$= \frac{1}{h\,A_2} = \frac{1}{h \times 2\pi r_2 l}$$

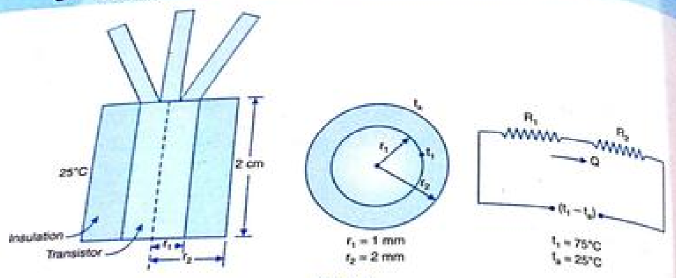


Fig. 5.9.

$$= \frac{1}{15 \times (2 \times \pi \times 0.002 \times 0.02)}$$

$= 265.39$ K/W

∴ *Radial heat flow from the transistor*

$$= \frac{75-25}{44.14+265.39} = 0.1615 \text{ W}$$

Rating of transistor = 200 mW = 0.2 W

Then, heat to be dissipated by the three leads

$= 0.2 - 0.1615 = 0.0385$ W

Heat to dissipated by one lead

$$= \frac{0.0385}{3} = 0.0128 \text{ W}$$

Now, for a fin with insulated end,

$$Q = \sqrt{hP\,kA}\;\theta_0 \tanh ml$$

$$\text{where } m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{15 \times (\pi \times 0.3 \times 10^{-3})}{125 \times \frac{\pi}{4}(0.3 \times 10^{-3})^2}}$$

$= 40$

$$\therefore 0.0128 = \sqrt{15 \times (\pi \times 0.3 \times 10^{-3}) \times 125 \times \frac{\pi}{4}(0.3 \times 10^{-3})^2} \times (75 - 25) \times \tanh (40\, l)$$

$= 3.5325 \times 10^{-4} \times 50 \tanh (40\, l)$

That gives:

$$\tanh (40\, l) = \frac{0.0128}{3.5325 \times 10^{-4} \times 50} = 0.7247$$

or $\quad 40\, l = 0.9174$

∴ Length of the lead,

$$l = \frac{0.9174}{40} = 0.0229 \text{ m}$$

$= 22.9$ mm

EXAMPLE 5.14

Two rods A and B of the same length and diameter protrude from a surface at 120°C and are exposed to air at 25°C. The temperatures measured at the end of the rods are 50°C and 75°C. If thermal conductivity of material A is 20 W/m-deg, calculate the thermal conductivity of material B. Adopt the condition of a fin insulated at the tip.

Solution : The temperature distribution for a short fin with tip insulated is given by

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{1}{\cosh ml}$$

For rod A :

$$\frac{50-25}{120-25} = \frac{1}{\cosh m_1 l}$$

$$\cosh m_1 l = \frac{95}{25} = 3.8 \;;\; m_1 l = 2.01$$

For rod B :

$$\frac{75-25}{120-25} = \frac{1}{\cosh m_2 l}$$

$$\cosh m_2 l = \frac{95}{50} = 1.9 \;;\; m_2 l = 1.257$$

$$\therefore \quad \frac{m_1}{m_2} = \frac{2.01}{1.257} = 1.599$$

$$\text{or} \quad \sqrt{\frac{h_1 P_1}{k_1 A_1} \times \frac{k_2 A_2}{h_2 P_2}} = 1.599$$

Since the rods are of same length and diameter, and are exposed to the same environment

$A_1 = A_2$, $P_1 = P_2$ and $h_1 = h_2$

That gives :

$$\sqrt{\frac{k_2}{k_1}} = 1.599$$

∴ Thermal conductivity of rod B,

$k_2 = k_1\,(1.599)^2$

$= 20 \times (1.599)^2 = \mathbf{51.13}$ **W/m-deg**

EXAMPLE 5.15

A centrifugal pump which circulates a hot liquid metal at 500°C is driven by a 3600 rpm electric motor. The motor is coupled to the pump impeller by a horizontal steel shaft 25 mm in diameter. If the temperature of the motor is limited to a maximum value of 60°C with the ambient air at 25°C, what length of the shaft should be specified between the motor and the pump. It may be presumed that the thermal conductivity of the shaft material is 35 W/mK, and that the convective film coefficient between the steel shaft and the ambient air is 15.7 W/m²K.

Solutio.. : The shaft conducts heat from the pump (t_0 = 500°C) towards the motor (t_1 = 60°C) and also loses energy from its surface to the surroundings (t_a = 25°C) by convection. Treating the shaft as a fin insulated at the tip, the temperature distribution would be prescribed by the relation :

$$\frac{\theta_x}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(l-x)}{\cosh ml}$$

$t_x = t_1$ at $x = l$

(length of the shaft)

$$\therefore \frac{60-25}{500-25} = \frac{\cosh m(l-l)}{\cosh ml} = \frac{1}{\cosh ml}$$

$$\cosh ml = \frac{500-25}{35} = 13.57 \;;\; ml = 3.3$$

For a circular shaft of diameter d,

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kd}}$$

$$= \sqrt{\frac{4 \times 15.7}{35 \times 0.025}} = 8.47 \text{ m}^{-1}$$

∴ $8.47\, l = 3.3$

$$l = \frac{3.3}{8.47} = 0.3896 \text{ m} = 38.96 \text{ cm}$$

EXAMPLE 5.16

The temperature of a gas stream is to be measured by using two thermocouples attached to a tube of perimeter 50 mm and wall cross-sectional area 15 mm². The tube is 250 mm long and is mounted normal to the duct wall. If thermocouples are attached to the tube at 125 mm and 250 mm from the duct wall and indicate tube wall temperatures of 350°C and 390°C respectively, estimate the gas temperature and the duct wall temperature. The heat transfer coefficient between the tube and the gas stream is 1.2 W/m²K and the thermal conductivity of the tube material is 25 W/mK. Neglect any heat transfer into the exposed end of the tube.

Solution : The temperature distribution for a fin with insulated end (no heat transfer at the exposed end) is given by relation:

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh m(l-x)}{\cosh ml}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{1.2 \times 50 \times 10^{-3}}{25 \times 15 \times 10^{-6}}}$$
$$= 12.65 \text{ m}^{-1}$$

Inserting data appropriate to the two locations of the thermocouples :

$$\frac{350 - t_a}{t_0 - t_a} = \frac{\cosh 12.65\,(0.25 - 0.125)}{\cosh\,(12.65 \times 0.25)}$$
$$= 0.214 \quad \ldots(i)$$

and
$$\frac{390 - t_a}{t_0 - t_a} = \frac{\cosh 12.65\,(0.25 - 0.25)}{\cosh\,(12.65 \times 0.25)}$$
$$= 0.0845 \quad \ldots(ii)$$

Simultaneous solution of expressions *(i)* and *(ii)* would give :

Temperature of the gas stream t_a = **417.8°C**

Temperature of the duct wall t_0 = **114.83°C**

EXAMPLE 5.17

The handle of a saucepan, 30 cm long and 2 cm in diameter, is subjected to 100°C temperature during a certain cooking operation. The average unit surface conductance over the handle surface is 7.35 W/m²-deg in the kitchen air at 24°C. The cook is likely to grasp the last 10 cm of the handle and hence the temperature in this region should not exceed 38°C. What should be the thermal conductivity of the handle material to accomplish it ? The handle may be treated as a fin insulated at the tip.

Solution : The handle conducts heat from the fire end (t_0 = 100°C) towards the hand (t_x = 38°C) and also loses heat from its surface to the surroundings (t_a = 24°C) by convection. Since the handle approximates to a fin insulated at the tip, the temperature distribution would be prescribed by the relation,

$$\frac{\theta_x}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m\,(l - x)}{\cosh ml} \quad \ldots(i)$$

The distance x corresponds to a position on the rod at which the temperature can be allowed to reach the value 38°C.

Inserting the appropriate values in expression (i) we get,

$$\frac{38 - 24}{100 - 24} = \frac{\cosh m\,(0.3 - 0.2)}{\cosh\,(m \times 0.3)}$$

$$0.184 = \frac{\cosh\,(0.1\,m)}{\cosh\,(0.3\,m)}$$

By trial and error : $m = 9.2 \text{ m}^{-1}$

Now, for a circular handle of diameter *d*,

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kd}}$$

Therefore, the thermal conductivity of the handle material should be

$$k = \frac{4h}{m^2 d} = \frac{4 \times 7.35}{9.2^2 \times 0.02}$$
$$= \mathbf{17.36 \text{ W/m-deg}}$$

EXAMPLE 5.18

A very long copper rod 20 mm in diameter extends horizontally from a plane heated wall maintained at 100°C. The surface of the rod is exposed to an air environment at 20°C with convective heat transfer coefficient of 8.5 W/m²-deg. Workout the heat loss if the thermal conductivity of copper is 400 W/m-deg. Further estimate how long the rod be in order to be considered infinite.

Solution : For a circular shaft of diameter *d*

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kd}}$$

$$= \sqrt{\frac{4 \times 8.5}{400 \times 0.02}} = 2.061 \text{ m}^{-1}$$

Heat dissipation from an infinitely long fin is prescribed by the relation,

$$Q_{fin} = k\,A_c\,m\,(t_0 - t_a)$$

$$= 400 \times \frac{\pi}{4}\,(0.02)^2 \times 2.061\,(100 - 20)$$

$$= \mathbf{20.71 \text{ W}}$$

There is no heat loss from the tip of an infinitely long rod and such it behaves as if the tip were insulated. Therefore, an estimate of the validity of this approximation can be made by comparing the following two expressions for the fin.

$$Q_{fin} = k\,A_c\,m\,(t_0 - t_a) \qquad \text{infinitely long fin}$$

$$Q_{fin} = k\,A_c\,m\,(t_0 - t_a)\tanh ml \qquad \text{fin with tip insulated}$$

These expressions provide equivalent results if $\tanh ml \geq 0.99$ or $ml \geq 2.65$

Hence the rod can be considered infinite if

$$l \geq \frac{2.65}{m}$$
$$\geq \frac{2.65}{2.061} = \mathbf{1.285 \text{ m}}$$

EXAMPLE 5.19

It is desired that heat dissipation for a metal tank containing cooling oil should be increased by 60% by attaching straight rectangular fins to its surface area which is at 100°C temperature. The fins are to be 6 mm thick and spaced 100 mm between centres. Make calculations for the height (distance between root and tip) of each fin on the assumption that natural convection coefficient remains unchanged at 35 W/m²-K, that the surface temperature of the tank is expected to drop to 95°C when the fins are fitted, and that the heat transfer from the tips of fins is neglected. Thermal conductivity for the fin and tank material is 280 W/mK and the surroundings are at 20°C.

Solution : Let the tank dimensions be 1 m × 1 m; that gives surface area without fins as 1 m².

Heat transfer rate without fins

$$= h\,A\,(t_0 - t_a)$$
$$= 35 \times 1 \times (100 - 20) = 2800 \text{ W}$$

Desired heat flow rate

$$= 2800 \times 1.6 = 4480 \text{ W}$$

With 0.1 m spacing between centres, 10 fins can be fitted per metre length of the tank surface.

Area occupied by 10 fins

$$= (1 \times 0.006) \times 10 = 0.06 \text{ m}^2$$

Thus 1 m² of primary surface is reduced to (1 − 0.06) = 0.94 m². Heat transfer from this unfinned area

$$= 35 \times 0.94\,(95 - 20)$$
$$= 2467.5 \text{ W}$$

∴ Heat dissipated from 10 fins

$$= 4480 - 2467.5 = 2012.5 \text{ W}$$

Thus each fin dissipates 201.25 W/m length.

For a straight rectangular fin,

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h}{k} \times \frac{2}{\delta}}$$

$$= \sqrt{\frac{35 \times 2}{280 \times 0.006}} = 6.45 \text{ m}^{-1}$$

For a fin with insulated tip

$$Q_{fin} = k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

$$201.25 = 280 \times (0.006 \times 1) \times 6.45 \times (95 - 20)\tanh\,(6.45\,l)$$

$$\tanh\,(6.45\,l) = 0.2476\;;\; 6.45\,l = 0.252$$

Therefore, height of each fin is,

$$l = \frac{0.252}{6.45} = 0.039 \text{ m} = 3.9 \text{ cm}$$

5.4. HEAT DISSIPATION FROM A FIN LOSING HEAT AT THE TIP

The fin is of any finite length with the tip exposed for heat dissipation. The relevant boundary conditions are :

(*i*) $\theta = \theta_0$ at $x = 0$

(*ii*) The fin is losing heat at the tip, *i.e.*, the heat conducted to the fin at $x = l$ equals the heat convected from the end to the surroundings

$$-k\,A_c\left(\frac{dt}{dx}\right)_{x=l} = h\,A_s\,(t - t_a)$$

At the tip of fin, the cross-sectional area for heat conduction A_c equals the surface area A_s from which the convective heat transport occurs. Thus

$$\frac{dt}{dx} = -\frac{h\theta}{k} \quad \text{at} \quad x = l$$

Applying these boundary conditions to equation 5.6,

$$C_1 + C_2 = \theta_0 \qquad \text{...(a)}$$

Further,

$$t - t_a = C_1 e^{mx} + C_2 e^{-mx}$$

$$\left(\frac{dt}{dx}\right)_{x=l} = m\, C_1 e^{ml} - m\, C_2 e^{-ml} = -\frac{h\theta}{k}$$

$$\text{or } C_1 e^{ml} - C_2 e^{-ml} = -\frac{h\theta}{km}$$

$$= -\frac{h}{km}\left[C_1 e^{ml} + C_2 e^{-ml}\right] \qquad \text{...(b)}$$

This is because θ at $x = l$ equals $(C_1 e^{ml} + C_2 e^{-ml})$.

Solving expressions (a) and (b), the constants are determined as follows

$$C_1 e^{ml} - (\theta_0 - C_1) e^{-ml}$$

$$= -\frac{h}{km}\left[C_1 e^{ml} + (\theta_0 - C_2) e^{-ml}\right]$$

$$\text{or } C_1 e^{ml} - \theta_0 e^{-ml} + C_1 e^{-ml}$$

$$= -\frac{h}{km} C_1 e^{ml} - \frac{h}{km}\theta_0 e^{-ml} + \frac{h}{km} C_1 e^{-ml}$$

$$\text{or } C_1\left[\left(e^{ml} + e^{-ml}\right) + \frac{h}{km} e^{ml} - \frac{h}{km} e^{-ml}\right]$$

$$= \theta_0 e^{-ml} - \frac{h}{km}\theta_0 e^{-ml}$$

$$\text{or } C_1\left[\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)\right]$$

$$= \theta_0 e^{-ml}\left[1 - \frac{h}{km}\right]$$

$$\therefore \quad C_1 = \frac{\theta_0\left(1 - \frac{h}{km}\right)e^{-ml}}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)}$$

$$\text{and } C_2 = \theta_0 - C_1$$

$$= \theta_0 - \frac{\theta_0\left(1 - \frac{h}{km}\right)e^{-ml}}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)}$$

$$= \theta_0\left[1 - \frac{\left(1 - \frac{h}{km}\right)e^{-ml}}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)}\right]$$

$$= \theta_0\left[\frac{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}e^{ml} - \frac{h}{km}e^{-ml} - e^{-ml} + \frac{h}{km}e^{-ml}}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)}\right]$$

$$= \theta_0\left[\frac{e^{ml} + \frac{h}{km}e^{ml}}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)}\right]$$

$$= \frac{\theta_0\left(1 + \frac{h}{km}\right)e^{ml}}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{km}\left(e^{ml} - e^{-ml}\right)}$$

Substituting these values of constants C_1 and C_2 in equation. 5.6, one obtains the following expression for temperature distribution along the length of the fin.

$$\frac{\theta}{\theta_0} = \frac{e^{m(l-x)} + e^{-m(l-x)} + \frac{h}{km}\left\{e^{m(l-x)} - e^{-m(l-x)}\right\}}{e^{ml} + e^{-ml} + \frac{h}{km}\left\{e^{ml} - e^{-ml}\right\}}$$

Expressing in terms of hyperbolic functions

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a}$$

$$= \frac{\cosh m(l-x) + \frac{h}{km}\sinh m(l-x)}{\cosh ml + \frac{h}{km}\sinh ml} \qquad \text{...(5.11)}$$

The rate of heat flow from the fin is given by

$$Q_{fin} = -k\, A_c \left(\frac{dt}{dx}\right)_{x=0}$$

From the expression for temperature distribution,

$$t - t_a = (t_0 - t_a)\, \frac{\cosh m\,(l - x) + \frac{h}{km}\sinh m(l-x)}{\cosh ml + \frac{h}{km}\sinh ml}$$

$$\frac{dt}{dx} = (t_0 - t_a)\, \frac{m \sinh m\,(l + x) - m\left[\frac{h}{km}\cosh m(l-x)\right]}{\cosh ml + \frac{h}{km}\sinh ml}$$

$$\left(\frac{dt}{dx}\right)_{x=0} = -(t_0 - t_a)\, m\, \frac{\sinh ml + \frac{h}{km}\cosh ml}{\cosh ml + \frac{h}{km}\sinh ml}$$

$$\therefore \quad Q_{fin} = k\, A_c\, m\, (t_0 - t_a)\, \frac{\sinh ml + \frac{h}{km}\cosh ml}{\cosh ml + \frac{h}{km}\sinh ml}$$

$$= \sqrt{Ph\,k\,A_c}\,(t_0 - t_a)\frac{\sinh ml + \frac{h}{km}\cosh ml}{\cosh ml + \frac{h}{km}\sinh ml}$$

$$= \sqrt{Ph\,k\,A_c}\,(t_0 - t_a)\frac{\tanh ml + \frac{h}{km}}{1 + \frac{h}{km}\tanh ml} \qquad \text{...(5.12)}$$

EXAMPLE 5.20

A horizontal steel shaft, 30 mm diameter and 600 mm long, has its first bearing located 100 mm from the end connected to the impeller of a centrifugal pump. If the impeller is immersed in a hot liquid metal at 500°C, work out the temperature at the bearings under the conditions: (a) the shaft is very long (b) the heat flow through the end of the shaft is negligible and (c) the heat is transferred to the surroundings from the end.

The temperature and convection coefficient associated with the fluid adjoining the shaft are 35°C and 68 kJ/m²-hr-deg. For steel shaft, thermal conductivity k = 72 kJ/m-hr-deg.

Solution : For the circular shaft

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4}d^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kd}} = \sqrt{\frac{4 \times 68}{72 \times 0.03}} = 11.22 \text{ m}^{-1}$$

(a) For an infinitely long fin

$$\frac{\theta_x}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = e^{-mx}$$

Therefore, temperature t_x at the bearing ($x = 100$ mm $= 0.1$ m) is

$$t_x = t_a + (t_0 - t_a)\, e^{-mx}$$
$$= 35 + (500 - 35)\, e^{-11.22 \times 0.10}$$
$$= 35 + 465 \times 0.3256 = \mathbf{186.42°C}$$

(b) For a fin with no heat loss from the tip end

$$\frac{\theta_x}{\theta_c} = \frac{(t_x - t_a)}{(t_0 - t_a)} = \frac{\cosh m(l-x)}{\cosh ml}$$

Therefore, temperature t_x at the bearing ($x = 100$ mm $= 0.1$ m) is

$$t_x = t_a + (t_0 - t_a)\frac{\cosh m(l-x)}{\cosh ml}$$

$$= 35 + (500 - 35)\frac{\cosh 11.22\,(0.6 - 0.1)}{\cosh(11.22 \times 0.6)}$$

$$= 35 + 465 \times \frac{135.57}{419.41} = \mathbf{186°C}$$

(c) For a fin dissipating heat to the surroundings from its tip end

$$\frac{\theta_x}{\theta_0} = \frac{(t_x - t_a)}{(t_0 - t_a)}$$

$$= \frac{\cosh m(l-x) + \frac{h}{mk}\sinh m(l-x)}{\cosh ml + \frac{h}{mk}\sinh ml}$$

Therefore, temperature t_x at the bearing (x = 100 mm = 0.1 m) is

$$t_x = t_a + (t_0 - t_a)$$

$$\times \frac{\cosh m(l-x) + \frac{h}{mk}\sinh m(l-x)}{\cosh ml + \frac{h}{mk}\sinh ml}$$

Now,

$$t_0 - t_a = 500 - 35 = 465$$

$$\cosh m(l-x) = \cosh 11.22\,(0.6 - 0.2) = 136.57$$

$$\frac{h}{mk} = \frac{68}{11.22 \times 7.2} = 0.084$$

$$\sinh m(l-x) = \sinh 11.22\,(0.6 - 0.1) = 136.57$$

$$\cosh ml = \cosh (11.22 \times 0.6) = 419.41$$

$$\therefore \quad t_x = 35 + 465\left(\frac{136.57 + 0.084 \times 136.57}{419.41 + 0.084 \times 419.41}\right)$$

$$= 35 + 465 \times 0.32 = \mathbf{183.8°C}$$

EXAMPLE 5.21

The figure shows a 5 cm diameter rod, 90 cm long, which is having its lower face grinded smooth. The remainder of the rod is exposed to the 32°C room air and a surface coefficient heat transfer equal to 6.8 W/m²-deg exists between the rod surface and the room air. The grinder dissipates mechanical energy at the rate of 35 W. If thermal conductivity of rod material is 41.5 W/m-deg, find the temperature of the rod at the point where the grinding is taking place.

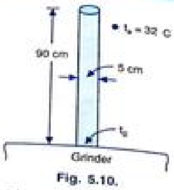


Fig. 5.10.

Solution : The entire dissipated energy goes into the rod at its lower face and is then transferred by convection from the outside surface area of the rod to the surrounding air. Treating the rod as fin losing heat at the tip, the heat flow through the rod is given by

$$Q = k A_c m (t_0 - t_a) \frac{\tanh ml + \frac{h}{km}}{1 + \frac{h}{km}\tanh ml}$$

For a circular rod of diameter d

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4}d^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kd}}$$

$$= \sqrt{\frac{4 \times 6.8}{41.5 \times 0.05}} = 3.62 \text{ m}^{-1}$$

$$ml = 3.62 \times 0.9 = 3.26$$

$$\tanh ml = \tanh 3.26 = 0.997$$

$$\frac{h}{km} = \frac{6.8}{41.5 \times 3.62} = 0.045$$

Then

$$35 = 41.5 \times \frac{\pi}{4}(0.05)^2 \times 3.62\,(t_0 - 32) \times \left(\frac{0.997 + 0.045}{1 + 0.045 \times 0.997}\right)$$

$$= 0.293\,(t_0 - 32)$$

∴ Temperature of the rod at the grinding surface,

$$t_0 = 32 + \frac{35}{0.293} = \mathbf{151.45°C}$$

EXAMPLE 5.22

A turbine blade 5 cm long, 4.5 cm² cross-sectional area and 10 cm perimeter is made of stainless steel of thermal conductivity 100 kJ/m-hr-deg. The temperature at the root of blade is 500°C and it is exposed to products of combustion passing through the turbine at 850°C. Determine the temperature at the middle of blade and the rate of heat flow from it. The film coefficient between the blade and the combustion gases is

1100 kJ/m²-hr-deg

The blade may be treated as a fin losing heat at the tip.

Solution : For a fin losing heat at the tip, the temperature distribution is prescribed by the relation :

$$\frac{\theta_x}{\theta_0} = \frac{(t_x - t_a)}{(t_0 - t_a)}$$

$$= \frac{\cosh m(l-x) + \frac{h}{km}\sinh m(l-x)}{\cosh ml + \frac{h}{km}\sinh ml}$$

where

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{1100 \times 0.10}{100 \times 4.5 \times 10^{-4}}} = 49.44 \text{ m}^{-1}$$

$$ml = 49.44 \times 0.05 = 2.472$$

$$\frac{h}{km} = \frac{1100}{100 \times 49.44} = 0.222$$

$$\sinh ml = 5.881\,; \quad \cosh ml = 5.965$$

$$\tanh ml = 0.986$$

$$\therefore \frac{t_x - 850}{500 - 850} = \frac{\cosh m(l-x) + 0.222\sinh m(l-x)}{5.965 + 0.222 \times 5.881}$$

From this expression, temperature at any distance x from the root of the blade can be worked out. At the mid of turbine blade x = 0.025 m.

$$\cosh m(l-x) = \cosh 49.44\,(0.05 - 0.025) = 1.236$$

$$\sinh m(l-x) = \sinh 49.44\,(0.05 - 0.025) = 1.576$$

$$\therefore \frac{t_x - 850}{500 - 850} = \frac{1.236 + 0.222 \times 1.576}{5.965 + 0.222 \times 5.881} = 0.218$$

$$t_x = 850 + 0.218\,(500 - 850) = 773.7°C$$

(b) The heat flow through the blade is given by

$$Q = k A_c m (t_0 - t_a) \frac{\tanh ml + \frac{h}{km}}{1 + \frac{h}{km}\tanh ml}$$

$$= 100 \times (4.5 \times 10^{-4})\,(49.44)\,(500 - 850)\,\frac{0.986 + 0.222}{1 + 0.222 \times 0.986}$$

$$= \mathbf{-771.64 \text{ kJ}}$$

The -ve sign indicates that the heat flows from the combustion gases to the blade.

EXAMPLE 5.23

A cylinder 5 cm in diameter and 1 m long is provided with 10 longitudinal straight fins of material having thermal conductivity 120 W/m-deg. The fins are 0.75 mm thick and protrude 12.5 mm from the cylinder surface. The system is placed in an atmosphere at 40°C and the heat transfer coefficient from the cylinder and fins to the ambient air is 20 W/m²-deg. If the surface temperature of the cylinder is 150°C, calculate the rate of heat transfer and the temperature at the end of fins.

Consider the fin to be of finite length

Solution : For a fin of rectangular cross-section

$$A_c = b \times \delta = 1 \times (0.75 \times 10^{-3}) = 0.75 \times 10^{-3} \text{ m}^2$$

$$P = 2(b + \delta) = 2b = 2 \times 1 = 2 \text{ m}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{20 \times 2}{120 \times 0.75 \times 10^{-3}}} = 21.08 \text{ m}^{-1}$$

For a fin of finite length, the heat conducted to the end is convected away to the ambient air and is prescribed by the relation

$$Q = k\,A_c\,m\,(t_0 - t_a) \times \frac{\tanh ml + \dfrac{h}{km}}{1 + \dfrac{h}{km} \times \tanh ml}$$

$$ml = 21.08 \times (12.5 \times 10^{-3}) = 0.2635$$

$$\frac{h}{km} = \frac{20}{120 \times 21.08} = 0.007906$$

$$\tanh ml = \tanh\,(0.2635) = 0.2576$$

$$\therefore\ Q = 120 \times 0.75 \times 10^{-3} \times 21.08 \times (150 - 40) \times \frac{0.2576 + 0.007906}{1 + 0.007906 \times 0.2576}$$

$$= 55.29 \text{ W per fin}$$

Heat transfer from unfinned (base surface)

$$= h\,[\pi\,d - n\,A_c]\,l \times (t_0 - t_a)$$

$$= 20\,[\pi \times 0.05 - 10 \times (0.75 \times 10^{-3})] \times 1 \times (150 - 40)$$

$$= 328.9 \text{ W}$$

$$\therefore\ Q_{total} \text{ for 10 fins} = 328.9 + 10 \times 55.29$$

$$= \mathbf{881.8\ W}$$

(*b*) For a fin dissipating heat to the surroundings from its tip end

$$\frac{\theta_x}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m\,(l - x) + \dfrac{h}{km} \times \sinh m\,(l - x)}{\cosh ml + \dfrac{h}{km} \times \sinh ml}$$

At the tip ($x = l$), the above identity reduces to

$$\frac{t_l - t_a}{t_0 - t_a} = \frac{1}{\cosh ml + \dfrac{h}{km} \times \sinh ml}$$

or

$$\frac{t_l - 40}{150 - 40} = \frac{1}{\cosh 0.2635 + 0.007906 \times \sinh 0.2635}$$

$$= 0.964$$

$$\therefore\ t_l = 40 + (150 - 40) \times 0.964$$

$$= \mathbf{146.04°C}$$

5.5. FIN PERFORMANCE

The utility of a fin in dissipating a given quantity of heat is generally assessed on the basis of the following parameters :

• ***Efficiency of fin*** relates the performance of an actual fin to that of an ideal or fully effective fin. A fin will be most effective, *i.e.*, it would dissipate heat at maximum rate if the entire fin surface area is maintained as the base temperature.

$$\eta_f = \frac{\text{actual heat transfer rate from the fin}}{\text{heat that would be dissipated if the whole surface of the fin were maintained at the base temperature}}$$

Thus for a fin insulated at tip

$$\eta_f = \frac{\sqrt{phkA_c}\,(t_0 - t_a)\tanh ml}{h\,(Pl)\,(t_0 - t_a)} \qquad ...(5.13)$$

The parameter Pl represents the total surface area exposed for convective heat flow. Upon simplification,

$$\eta_f = \frac{\tanh ml}{\sqrt{Ph/kA_c}\;l} = \frac{\tanh ml}{ml} \qquad ...(5.14)$$

An estimate of the fin efficiency can thus be made by substituting the value of m in the above relation. An insight into this expression shows that,

(*i*) For a very long fin

$$\frac{\tanh ml}{ml} \rightarrow \frac{1}{\text{large number}}$$

Obviously the fin efficiency drops with an increase in its length.

(*ii*) For small values of ml, the fin efficiency increases. When the length is reduced to zero, then,

$$\frac{\tanh ml}{ml} \rightarrow \frac{ml}{ml} = 1$$

Thus the fin efficiency reaches its maximum value of 100% for a trivial value of $l = 0$, *i.e.*, no fin at all. Naturally maximization of fin performance with respect to its length does not constitute the design criterion for a fin. The efficiency of fin,

however, forms a criterion for judging the relative merits of fins of different geometries or materials.

• ***Effectiveness of fin*** (ϵ_f) represents the ratio of the fin heat transfer rate (heat dissipation with a fin) to the heat transfer rate that would exist without a fin.

Fig. 5.11 shows the base heat transfer surface before and after the fin has been attached. The heat transfer through the root area A_c before fin attachment is :

$$Q = h\,A_c\,(t_0 - t_a)$$

After the attachment of an infinitely long fin, the heat transfer rate through the root area becomes:

$$Q_{fin} = \sqrt{PhkA_c}\,(t_0 - t_a)$$

$\therefore$ Fin effectiveness, ϵ_f

$$= \frac{\sqrt{PhkA_c}\,(t_0 - t_a)}{hA_c\,(t_0 - t_a)}$$

$$= \sqrt{\frac{Pk}{hA_c}} \qquad ...(5.15)$$

In case of a straight rectangular fin of thickness δ and width b,

$$\frac{P}{A_c} = \frac{2\,(b + \delta)}{b\delta} = \frac{2}{\delta}$$

$$\therefore\ \epsilon_f = \sqrt{\frac{2k}{h\delta}} \qquad ...(5.16)$$

A close examination of the relations for fin effectiveness helps us to infer the following important results :

(*i*) If the fin is to improve heat dissipation from the primary surface, then the fin effectiveness must be greater than unity. That is,

$$\sqrt{\frac{Pk}{hA_c}} > 1$$

Literature suggests that use of fins on surfaces is justified only if the ratio Pk/hA_c is greater than 5.

(*ii*) An improvement in fin effectiveness occurs when the fin is made from a material of high thermal conductivity such as copper and aluminium alloys. Although copper (k = 385 W/mK) is superior to aluminium (k = 225 W/mK) regarding the thermal conductivity, yet fins are generally made of aluminium because of their additional advantage related to lower cost and weight.

(*iii*) Increase in the ratio of perimeter P to the cross-sectional area A_c also brings about improvement in the effectiveness of fin. That explains the preference for thin but closely spaced fins. The lower limit on the pitch (distance between two adjacent fins) is governed by the thickness of boundary layer that develops on the fin surface.

(*iv*) A high value of film coefficient has an adverse effect on heat dissipation. Use of fins would be better justified under conditions for which the convective coefficient is small, i.e., when the fluid is gas rather than a liquid. When fins are to be used on a surface separating a gas and a liquid, better results are achieved when fins are located on the

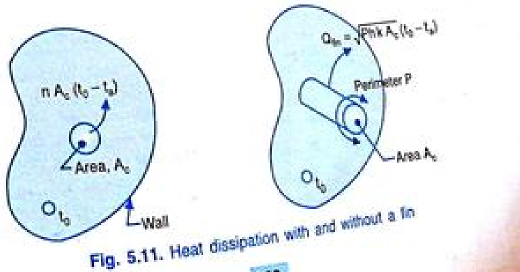


Fig. 5.11. Heat dissipation with and without a fin

gas side where the convective film coefficient has a low value. This aspect also explains why fins are not used on steam condenser tubes; values of h are quite high in case of condensation heat transfer. A marginal improvement in heat dissipation does not justify the added cost of finned surfaces.

From the definitions of fin efficiency and fin effectiveness, it is evident that these performance parameters are related to each other by the following expression (*written for a fin insulated at the tip*).

$$\eta_f = \frac{\sqrt{PhkA_c}\,(t_0 - t_a)\tanh ml}{hPl\,(t_0 - t_a)}$$

$$\epsilon_f = \frac{\sqrt{PhkA_c}\,(t_0 - t_a)\tanh ml}{hA_c\,(t_0 - t_a)}$$

$$\therefore \quad \epsilon_f = \eta_f \frac{hPl\,(t_0 - t_a)}{hA_c\,(t_0 - t_a)}$$

$$= \eta_f \frac{Pl}{A_c} \qquad \ldots(5.17)$$

$$= \eta_f \frac{\text{surface area of the fin}}{\text{cross-sectional area of the fin}}$$

An increase in the fin effectiveness can be obtained by extending the length of fin but that rapidly becomes a losing proposition in terms of efficiency.

EXAMPLE 5.24

A steel rod (k = 30 W/m-deg) 1 cm in diameter and 5 cm long protrudes from a wall which is maintained at 100°C. The rod is insulated at its tip and is exposed to an environment with h = 50 W/m²-deg and t_a = 30°C. Calculate the fin efficiency, temperature at the tip of fin and the rate of heat dissipation.

Solution : The fin efficiency is given by

$$\eta_{fin} = \frac{\tanh ml}{ml}$$

where $$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h \times \pi d}{k\frac{\pi}{4}d^2}} = \sqrt{\frac{4h}{kd}}$$

$$= \sqrt{\frac{4 \times 50}{30 \times 0.01}} = 25.82 \text{ m}^{-1}$$

$$\therefore \quad \eta_{fin} = \frac{\tanh(25.82 \times 0.05)}{(25.82 \times 0.05)}$$

$$= 0.6657 \text{ or } 66.57\%$$

(*b*) The temperature distribution for a fin with insulated tip (no heat transfer at the exposed end) is given by the relation

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh m(l - x)}{\cosh ml}$$

At tip ($x = l$) and the above equation reduces to

$$\frac{\theta_l}{\theta_0} = \frac{t_l - 30}{100 - 30} = \frac{1}{\cosh ml}$$

or $$t_l = 30 + \frac{70}{\cosh(25.82 \times 0.05)}$$

$$= 30 + 35.79 = \mathbf{65.79\ °C}$$

(*c*) The heat loss from a fin with insulated tip is

$$Q = k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

$$= 30 \times \frac{\pi}{4}(0.01)^2 \times 25.82 \times (100 - 30) \times \tanh(25.82 \times 005)$$

$$= \mathbf{3.658\ W}$$

EXAMPLE 5.25

The cylindrical head of an engine is 1 m long and has an outside diameter of 50 mm. Under typical operating conditions, the outer surface of the head is at a temperature of 150°C and is exposed to ambient air at 40°C with a convective coefficient of 80 kJ/m²-hr-deg. The head has been provided with 12 longitudinal straight fins which are 0.75 mm thick and protrude 2.5 cm from the cylindrical surface. Workout the increase in heat dissipation due to addition of fins. Also calculate the temperature at the centre of the fin.

It may be presumed that the fins have insulated tips and that the thermal conductivity of the cylinder head and fin material is

260 kJ/m-hr-deg

Solution : For a straight rectangular fin

$$\frac{P}{A_c} = \frac{2(b + \delta)}{b\delta} = \frac{2b}{b\delta}$$

$$= \frac{2}{\delta} \text{ because } \delta << b$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{2h}{k\delta}}$$

$$= \sqrt{\frac{2 \times 80}{260 \times 0.0075}} = 28.64 \text{ m}^{-1}$$

$$ml = 28.64 \times 0.025 = 0.716$$

For a fin with insulated tip,

$$Q_{fin} = k\,A_c\,m\,(t_0 - t_a)\tanh ml$$

$$= 260 \times (1 \times 0.00075) \times 28.64 \times (150 - 40)\tanh 0.716$$

$$= 377.45 \text{ kJ/hr}$$

For 12 fins, the heat dissipation will be equal to

$$12 \times 377.45 = 4529.4 \text{ kJ/hr}$$

Surface area of cylinder head not occupied by the fins is equal to

$$(\pi \times 0.05 \times 1) - (12 \times 0.00075 \times 1) = 0.148 \text{ m}^2$$

Heat dissipation from this surface is by convection

$$Q_{con} = h\,A_c\,(t_0 - t_a)$$

$$= 80 \times 0.148 \times (150 - 40)$$

$$= 1302.4 \text{ kJ/hr}$$

Hence the total heat transfer is equal to $4529.4 + 1302.4 = 5831.8$ kJ/hr

Had the cylinder been bare, then the heat dissipation would have been,

$$Q_{bare} = h\,A\,(t - t_a)$$

$$= 80 \times (\pi \times 0.05 \times 1)(150 - 40)$$

$$= 1381.6 \text{ kJ/hr}$$

$\therefore$ % increase in heat dissipation

$$= \frac{5831.8 - 1381.6}{1381.6} \times 100 = \mathbf{322\%}$$

Thus the fins significantly increase heat dissipation from the cylinder head. Considerable improvement could still be obtained by increasing the number of fins (by reducing both fin thickness and the separation between fins).

(*b*) Invoking the expression for temperature distribution along the fin length,

$$\frac{\theta_x}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(l - x)}{\cosh ml}$$

Therefore, temperature t_x at the centre of the fin (x = 0.0125 m) is,

$$t_x = t_a + (t_0 - t_a)\frac{\cosh m(l - x)}{\cosh ml}$$

$$= 40 + (150 - 40) \frac{\cosh 28.64\,(0.025 - 0.0125)}{\cosh(28.64 \times 0.025)}$$

$$= 40 + 110 \times \frac{1.064}{1.267} = \mathbf{132.37°C}$$

EXAMPLE 5.26

In a chemical process the heat transfer from a surface to distilled water is increased by providing a number of thin fins each 50 mm long and 2 mm thick. The metal fins are coated with 0.15 mm thick layer of plaster to prevent ionization of water and the ends of the fin are attached to an insulated wall. The temperature at the base of fins is 80°C, mean water temperature is 20°C and the heat transfer coefficient between the water and plastic coating is 250 W/m²K. Determine the temperature at the tip of fins, and the fin efficiency. Thermal conductivities for the fin material and the plastic are 205 W/mK and 0.55 W/mK respectively.

Solution : Since there is a layer of plastic coating on the metallic fins, its resistance to the heat transfer is also to be considered. This is achieved by replacing the convective coefficient h in the expression $\sqrt{\frac{hP}{kA_c}}$ by the overall heat transfer coefficient U given by :

$$\frac{1}{U} = \frac{1}{h} + \frac{\delta_p}{k}$$

where, δ_p is the thickness of plastic coating

$$\frac{1}{U} = \frac{1}{250} + \frac{0.15 \times 10^{-3}}{0.55}$$

$\therefore$ $U = 234\ W/m^2K$

For a rectangular fin of thickness δ and width b,

$$\frac{P}{A_c} = \frac{2(b+\delta)}{b\delta} = \frac{2b}{b\delta} = \frac{2}{\delta}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{2h}{k\delta}}$$

$$= \sqrt{\frac{2 \times 234}{205 \times 0.002}} = 33.78\ m^{-1}$$

For a fin with insulated tip

$$\frac{\theta_x}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(l-x)}{\cosh ml}$$

Therefore, temperature t_x at the tip of the fin ($x = 0.05$) is :

$$t_x = t_a + (t_0 - t_a) \times \frac{\cosh 33.78\,(0.05 - 0.05)}{\cosh (33.78 \times 0.05)}$$

$$= 20 + (80 - 20) \times \frac{1}{2.799}$$

$$= \mathbf{41.45°C}$$

(*b*) The actual heat transferred from the fin is :

$$Q_a = k A_c\, m\, (t_0 - t_a) \tanh ml$$
$$= k\,(b\,\delta)\, m\, (t_0 - t_a) \tanh ml$$

For unit width of the fin,

$$Q_a = 205\,(1 \times 0.002) \times 33.78\,(80 - 20) \tanh (33.78 \times 0.050)$$
$$= 776\ W$$

The heat that would be dissipated if the whole surface of the fin were maintained at the base temperature is:

$$Q = h\,(Pl)\,(t_0 - t_a)$$

The parameter Pl represents the total surface area exposed for the convective heat flow.

$$Pl = 2\,(b + \delta)\, l = 2bl$$

$$\therefore \quad Q = 234 \times (2 \times 1 \times 0.05) \times (80 - 20)$$
$$= 1404\ W$$

The fin efficiency is then

$$\eta_{fin} = \frac{776}{1404} = 0.5527 \text{ or } 55.27\ \%$$

5.6. DESIGN CONSIDERATIONS FOR FINS

The following factors need consideration for the optimum design of fins :

- cost
- manufacturing difficulties
- pressure drop caused by fins
- space considerations, *i.e.*, geometrical configuration of the channel where the fin is to be attached
- weight considerations in the case of automobiles and aircrafts

The design will be considered optimum when the fins require minimum cost of manufacture, offer minimum resistance to the fluid flow, are light in weight, and are easy to manufacture. There is a limit on the length of fin imposed by manufacturing difficulties, stability etc.

Space Considerations : With regard to the space considerations, the limiting condition exists when increase in length no longer brings an increase in the heat condition, *i.e.*,

$$dQ/dl = 0$$

Recalling that for a fin losing heat at the tip, the rate of heat dissipation by the fin is:

$$Q = k A_c\, m\, (t_0 - t_a) \frac{\tanh ml + \frac{h}{km}}{1 + \frac{h}{km} \tanh ml}$$

The factors k, A_c, m and $(t_0 - t_a)$ are constants and therefore for maximum heat dissipation we may write :

$$\frac{d}{dl}\left[\frac{\tanh ml + \frac{h}{km}}{1 + \frac{h}{km}(\tanh ml)}\right] = 0$$

or
$$\frac{\left[1 + \frac{h}{km}\tanh ml\right] m\,\mathrm{sech}^2\, ml - \left(\tanh ml + \frac{h}{km}\right)\left[\frac{h}{km}\mathrm{sech}^2\, ml\right]}{\left[1 + \frac{h}{km}\tanh ml\right]^2} = 0$$

or
$$\left[1 + \frac{h}{km}\tanh ml\right] m\,\mathrm{sech}^2\, ml - \left[\tanh ml + \frac{h}{km}\right]\left[\frac{h}{km}\mathrm{sech}^2\, ml\right] = 0$$

Upon simplification

$$m - \frac{h^2}{mk^2} = 0\ ; \quad m\,k = h \qquad \text{...(5.18)}$$

When this result is substituted in the expression for heat dissipation, we would get: $Q = h A_c (t_0 - t_a)$ which represents the heat transfer rate through the root area before the fin attachment. If $mk > h$, the heat dissipation will be greater than the value prescribed by this relation and the attachment of fins would be advantageous. Corresponding to a straight rectangular fin for which $m = \sqrt{2h/k\delta}$ the limiting condition becomes.

$$\frac{2k}{\delta} = h\ ; \quad \frac{1}{h} = \frac{\delta/2}{k} \qquad \text{...(5.19)}$$

The parameter $1/h$ prescribes the thermal convection resistance, and the parameter $(\delta/2)/k$ is a measure of thermal conduction resistance of a plane wall of thickness equal to one-half of the fin thickness. Thus the limit is reached when both the resistances have the same magnitude.

Fig 5.11 depicts a variation of heat dissipation Q with respect to fin length l for particular values of parameter $k/h\delta$. The plot shows that as $k/h\delta \to 1$, the value of $dQ/dl \to 0$ and the fin becomes ineffective. Use of fin on surfaces is justified only if the parameter $k/h\delta$ has values exceeding five. Further, the rate of increase of Q with increase in length is quite less as compared with the rate of increase of Q with increase in $k/h\delta$.

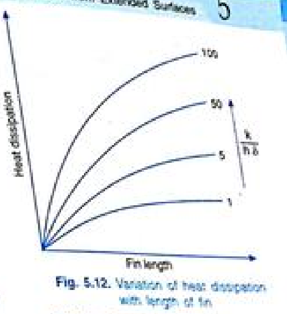


Fig. 5.12. Variation of heat dissipation with length of fin

Weight Considerations : The weight of fin is particularly important in the case of automobiles and the aircrafts; the desirable requirement being that maximum heat dissipation should be achieved with a minimum weight of the heat exchange system.

For a rectangular fin (length l, thickness δ and width b) with its end insulated, we have

$$Q = k A_c\, m\, (t_0 - t_a) \tanh ml$$

Substituting the value of $m = \sqrt{2h/k\delta}$ and expressing length l in terms of area A ($A = l \times \delta$)

$$Q = l\sqrt{2kh\delta}\,(t_0 - t_a) \tanh\left(\sqrt{2\frac{h}{k}}\frac{A_c}{\delta^{3/2}}\right)$$

Conditions for maximum heat flow rate for a given weight (area) can be worked out by differentiating the heat dissipation Q with respect to δ and setting the derivative equal to zero. Thus

$$l(t_0 - t_a)\sqrt{2kh}\left[\sqrt{\delta}\,\mathrm{sech}\left\{\sqrt{2\frac{h}{k}}\frac{A}{\delta^{3/2}}\right\} \sqrt{2\frac{h}{k}} A_c \left(-\frac{3}{2}\right)\delta^{-5/2} + \tanh\left\{\sqrt{2\frac{h}{k}}\frac{A_c}{\delta^{3/2}}\right\}\frac{1}{2}\delta^{-1/2}\right] = 0$$

Upon simplification

$$\tanh\left[\sqrt{\frac{2h}{k\delta}}\frac{A_c}{\delta}\right] = \frac{3\left[\sqrt{2h/k}\,(A_c/\delta)\right]}{\cosh^2\left[\sqrt{2h/k\delta}\,(A_c/\delta)\right]}$$

A numerical or graphical solution of this transcendental equation would yield,

$$\sqrt{\frac{2h}{k\delta}}\frac{A_c}{\delta} = 1.419 \qquad ...(5.20)$$

which prescribes the condition for maximum heat flow through a fin of given weight. Substituting $A = l \times \delta$, this relation can be recast as

$$\frac{l}{\delta/2} = 1.419\sqrt{\frac{2k}{h\delta}} \qquad ...(5.21)$$

The ratio of length to half the fin thickness thus depends upon the factor $2k/h\delta$ which may be written as

$$\frac{h\delta}{2k} = \frac{\delta/2k}{1/h}$$

$$= \frac{\text{thermal conduction resistance}}{\text{thermal convection resistance}}$$

In the heat conduction problems, the group $h\delta/2k$ is referred to as ***Biot number.***

EXAMPLE 5.27

A steel tube of 80 mm internal diameter and 6 mm wall thickness has 8 longitudinal fins of 1.5 mm thickness. Each fin extends 35 mm from the pipe wall and thermal conductivity of fin material is 50 W/m-deg. The wall temperature, the ambient temperature and surface heat transfer coefficient are stated to be 150°C, 25°C and 80 W/m²-deg. Make calculations for the percentage increase in the rate of heat transfers for the finned tube over the plain tube.

Solution : For a fin of rectangular cross-section,

$$P = 2(b + \delta) \approx 2b\ ;\ A_c = b \times \delta$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h \times 2b}{k(b+\delta)}} = \sqrt{\frac{2h}{k\delta}}$$

$$= \sqrt{\frac{2 \times 80}{50 \times 0.0015}} = 46.19 \text{ m}^{-1}$$

Efficiency of the fin, η_f

$$= \frac{\tanh ml}{ml}$$

$$= \frac{\tanh(46.19 \times 0.035)}{46.19 \times 0.035}$$

$$= \frac{0.924}{1.6166} = 0.571$$

Consider 1 m length of tube. Then length of fin = 1 m ;

breadth of fin = 0.035 m

area of single fin = 2× (1 × 0.035)

= 0.07 m² (both sides included)

total area of 8 fins = 8 × 0.07 = 0.56 m²

If temperature all over the fin is same and equal to the base temperature, then maximum rate of heat transfer from the fin surface

$$= h\,A\,\Delta t$$

$$= 80 \times 0.56 \times (150 - 25)$$

$$= 5600 \text{ W}$$

Actual rate of heat transfer from the fins,

Q_1 = fin efficiency × maximum rate of heat transfer

= 0.571 × 5600 = 3197.6 W

Area of contact of all the fins with the tube wall

$$= 8 \times (1 \times 0.0015) = 0.012 \text{ m}^2$$

Outside tube radius, r_o

$$= \frac{80}{2} + 6 = 46 \text{ mm} = 0.046 \text{ m}$$

Surface area per metre length

$$= 2\,\pi\,r_o\,l$$

$$= 2\,\pi \times 0.046 \times 1 = 0.289 \text{ m}^2$$

Free outside area of finned tube

$$= 0.289 - 0.012 = 0.277 \text{ m}^2$$

Heat transfer form free outside area of finned tube,

$$Q_2 = h\,A\,\Delta t$$

$$= 80 \times 0.277 \times (150 - 25)$$

$$= 2770 \text{ W}$$

Total heat transfer from the finned tube,

$$Q_f = Q_1 + Q_2$$

$$= 3197.6 + 2770 = 5867.6 \text{ W}$$

Heat transfer from the bare or unfinned tube

$$Q_b = h\,A\,\Delta t$$

$$= 80 \times 0.289 \times (150 - 25)$$

$$= 2890 \text{ W}$$

∴ Percentage increase in heat transfer when the fins are provided,

$$= \frac{Q_f - Q_b}{Q_b} \times 100$$

$$= \frac{5867.6 - 2890}{2890} \times 100 = 103.03\%$$

EXAMPLE 5.28

A plane wall at a uniform temperature of 225°C is exposed to fluid with temperature 25°C and convective film coefficient 800 kJ/m²-hr-deg. The wall has been provided with rectangular fins of thermal conductivity 200 kJ/m-hr-deg and profile area 1.8 cm². Make calculations for (a) the height and thickness of fin and (b) the maximum heat dissipation possible from 1 metre width of fin. Presume that the fins lose heat at the tip.

Solution : For maximum heat dissipation through a fin of given weight or profile area, we know that

$$\sqrt{\frac{2h}{k\delta}}\frac{A_c}{\delta} = 1.419$$

$$\sqrt{\frac{2 \times 800}{200}} \times \frac{1.8 \times 10^{-4}}{\delta^{3/2}} = 1.419$$

Thickness of the fin,

$$\delta = 5.049 \times 10^{-3} \text{ m} = 5.049 \text{ mm}$$

Height (length) of fin,

$$l = \frac{A}{\delta} = \frac{1.8 \times 10^{-4}}{5.049 \times 10^{-3}}$$

$$= 0.0356 \text{ m} = 35.6 \text{ mm}$$

(b) For a rectangular fin,

$$m = \sqrt{\frac{2h}{k\delta}} = \sqrt{\frac{2 \times 800}{200 \times 5.049 \times 10^{-3}}}$$

$$= 39.84 \text{ m}^{-3}$$

Now the heat dissipation from a fin losing heat at the tip,

$$Q = k\,A_c\,m\,(t_0 - t_a)\,\frac{\tanh ml + \frac{h}{km}}{1 + \frac{h}{km}\tanh ml}$$

$$A_c = b \times \delta$$

$$= (1 \times 0.005049) = 0.005049 \text{ m}^2$$

$$ml = 39.84 \times 0.0357 = 1.422$$

$$\tanh ml = \tanh 1.422 = 0.890$$

$$\frac{h}{km} = \frac{800}{200 \times 39.84} = 0.1$$

$$t_0 - t_a = 225 - 25 = 200$$

$$\therefore \quad Q = 4 \times 0.005049 \times 39.84 \times 200\left(\frac{0.890 + 0.1}{1 + 0.1 \times 0.89}\right)$$

$$= 6776 \text{ kJ/hr}$$

5.7. GENERALISED EQUATION FOR FINS

Consider a fin with an arbitrary geometry, i.e., a fin for which the cross-sectional and surface areas (A_c and A_s respectively) are varying along the heat flow direction (x-direction).

A heat balance for an elemental cross-section of thickness dx at distance x from the base wall gives

$$Q_x = Q_{x+dx} + Q_{conv}$$

$$= \left[Q_x + \frac{d}{dx}(Q_x)dx\right] + Q_{conv}$$

or

$$\frac{d}{dx}(Q_x)\,dx + Q_{conv} = 0$$

$$\frac{d}{dx}\left(-kA_c\frac{dt}{dx}\right)dx + h\,dA_s\,(t - t_a) = 0$$

Here temperature of the fin has been presumed to be uniform and non-varient for the infinitesimal element Further, taking the thermal conductivity of the fin material to be constant within the considered temperature range, we have

$$-k\,A_c\frac{d^2t}{dx^2}\,dx - k\frac{dA_c}{dx}\frac{dt}{dx}\,dx + h\,dA_s\,(t - t_a) = 0$$

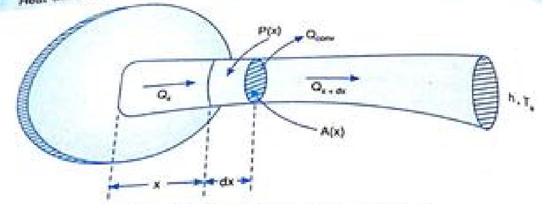


Fig. 5.13. A general fin of varying cross-section

Dividing throughout by $k\,A_c\,dx$ and upon re-arrangement

$$\frac{d^2t}{dx^2}+\frac{1}{A_c}\frac{dA_c}{dx}\frac{dt}{dx}-\frac{h}{kA_c}\frac{dA_s}{kA_c}(t-t_a)=0 \qquad ...(5.22)$$

The above equation is further simplified by defining temperature excess θ as $\theta = t - t_a$. Since ambient temperature is constant, we get by differentiation,

$$\frac{d\theta}{dx}=\frac{dt}{dx} \quad \text{and} \quad \frac{d^2\theta}{dx^2}=\frac{d^2t}{dx^2}$$

Equation 5.22 may then be written as

$$\frac{d^2\theta}{dx^2}+\frac{1}{A_c}\frac{dA_c}{dx}\frac{d\theta}{dx}-\frac{h}{kA_c}\frac{dA_s}{dx}\theta=0$$

The term dA_s/dx represents perimeter P of the surface at the section under consideration

$$\therefore \frac{d^2\theta}{dx^2}+\frac{1}{A_c}\frac{dA_c}{dx}\frac{d\theta}{dx}-\frac{hP}{kA_c}\theta=0 \qquad ...(5.23)$$

The above identity provides a general form of the energy equation for steady, one-dimensional heat dissipation for a fin of any cross-section.

If the fin is of constant cross-sectional area, the term dA_c/dx vanishes and the equation reduces to

$$\frac{d^2\theta}{dx^2}-\frac{hP}{kA_c}\theta=0$$

or $\frac{d^2\theta}{dx^2}-m^2\theta=0$ where $m=\sqrt{\frac{hP}{kA_c}}$

This is the same as equation 5.5 derived earlier.

5.8. HEAT FLOW THROUGH TRIANGULAR AND PARABOLIC FINS

It is worthwhile to mention that :

(i) the heat dissipation by every segment of the fin is not the same

(ii) part of the fin away from the base (root) is much less effective in heat dissipation; the end cross-section of the fin is very poorly utilized. Most of the heat is dissipated in a short length of the fin near the base

(iii) the section required for the same temperature gradient continuously decreases with length. Thus if a fin of a constant cross section is used, there would be wastage of material.

These aspects have led to the development of straight triangular and parabolic fins.

5.8.1. Straight triangular fins

A tapered (straight traingular fin) is of great practical importance because it yields the maximum heat flow per unit weight:

The dimensions and the co-ordinate system for analysing a triangular (tapered) fin have

been illustrated in Fig. 5.14. The fin is of length l between the base and the tip (origin), width b perpendicular to the plane of the paper and the fin thickness increases uniformly from zero at the tip to δ at the base. The temperature at the base of fin is t_0 and it goes on decreasing towards the tip.

The fin is sufficiently thin ($\delta << b$) so that heat flow pertains to one-dimensional heat conduction. For an elemental strip at distance x from the tip of fin,

thickness $= \frac{x}{l}\delta$

cross-sectonal area $A_c = \frac{x}{l}\delta$

perimeter $= 2b$

(Neglecting the effect of edges)

In terms of excess temperature, $\theta = t - t_a$, the controlling differential equation is

$$\frac{d^2\theta}{dx^2}+\frac{1}{A_c}\frac{dA_c}{dx}\frac{d\theta}{dx}-\frac{hP}{kA_c}\theta=0$$

Substituting the relevant values,

$$\frac{d^2\theta}{dx^2}+\frac{1}{\frac{x\delta b}{l}}\frac{d}{dx}\left(\frac{x\delta b}{l}\right)\frac{d\theta}{dx}-\frac{h\times 2b}{\frac{k\,x\delta b}{l}}\theta=0$$

$$\text{or} \quad \frac{d^2\theta}{dx^2}+\frac{1}{x}\frac{d\theta}{dx}-\frac{2hl}{k\delta}\frac{\theta}{x}=0 \qquad ...(5.24)$$

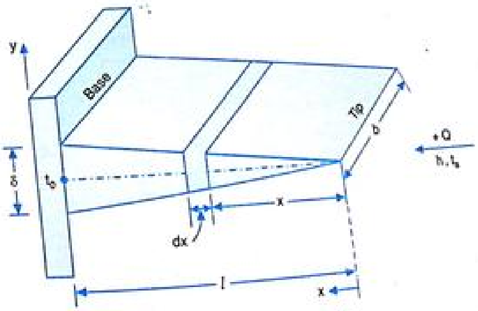


Fig. 5.14. Traingular fin

Introducing the factors $B^2 = 2\frac{hl}{k\delta}$ and multiplying throughout by x^2, we obtain

$$x^2\frac{d^2\theta}{dx^2}+x\frac{d\theta}{dx}-B^2x\theta=0 \qquad ...(5.25)$$

This equation can be recast into a form identical to the Bessel equation by carrying out the following transformation :

Let $\quad z = 2B\sqrt{x}\ ; \frac{dz}{dx} = Bx^{-1/2}$

$$\frac{d\theta}{dx}=\frac{d\theta}{dz}\times\frac{dz}{dx}=Bx^{-1/2}\frac{d\theta}{dz}$$

$$\frac{d^2\theta}{dx^2}=\frac{d}{dx}\left(\frac{d\theta}{dx}\right)=\frac{d}{dx}\left(Bx^{-1/2}\frac{d\theta}{dz}\right)$$

$$=B\times\left(-\frac{1}{2}x^{-3/2}\right)\times\frac{d\theta}{dz}+Bx^{-1/2}\frac{d^2\theta}{dz^2}\cdot\frac{dz}{dx}$$

$$=B\times\left(-\frac{1}{2}x^{-3/2}\right)\frac{d\theta}{dz}+Bx^{-1/2}\frac{d^2\theta}{dz^2}\times Bx^{-1/2}$$

$$=B\left[-\frac{1}{2}x^{-3/2}\frac{d\theta}{dz}+Bx^{-1}\frac{d^2\theta}{dz^2}\right]$$

Upon substitution of these values, the equation 5.25 takes the form

$$x^2\times B\left[-\frac{1}{2}x^{-3/2}\frac{d\theta}{dz}+Bx^{-1}\frac{d^2\theta}{dz^2}\right]+x\times\left[Bx^{-1/2}\frac{d\theta}{dz}\right]-B^2x\theta=0$$

$$\text{or} \quad B\left[-\frac{1}{2}\sqrt{x}\frac{d\theta}{dz} + Bx\frac{d^2\theta}{dz^2}\right] + B\sqrt{x}\frac{d\theta}{dz} - B^2x\,\theta = 0$$

$$\text{or} \quad -\frac{1}{2}B\sqrt{x}\frac{d\theta}{dz} + B^2x\frac{d^2\theta}{dz^2} + B\sqrt{x}\frac{d\theta}{dz} - B^2x\theta = 0$$

Replacing $B = \frac{z}{2\sqrt{x}}$, we get

$$-\frac{1}{2}\left(\frac{z}{2\sqrt{x}}\right)\sqrt{x}\frac{d\theta}{dz} + \left(\frac{z}{2\sqrt{x}}\right)^2 x\frac{d^2\theta}{dz^2} + \left(\frac{z}{2\sqrt{x}}\right)\sqrt{x}\frac{d\theta}{dz} - \left(\frac{z}{2\sqrt{x}}\right)^2 x\theta = 0$$

$$\text{or} \quad -\frac{z}{4}\frac{d\theta}{dz} + \frac{z^2}{4}\frac{d^2\theta}{dz^2} + \frac{z}{2}\frac{d\theta}{dz} - \frac{z^2}{4}\theta = 0$$

$$\text{or} \quad \frac{z^2}{4}\frac{d^2\theta}{dz^2} + \frac{z}{4}\frac{d\theta}{dz} - \frac{z^2}{4}\theta = 0$$

$$\text{or} \quad \frac{d^2\theta}{dz^2} + \frac{1}{z}\frac{d\theta}{dz} - \theta = 0 \qquad ...(5.26)$$

This is a form of Bessel's differential equation of zero order and its general solution is

$$\theta = C_1 I_0(z) + C_2 k_0(z)$$

$$= C_1 I_0(2B\sqrt{x}) + C_2 k_0(2B\sqrt{x}) \qquad ...(5.27)$$

where I_0 and k_0 are modified zero order Bessel's functions of the first and second kind repectively. Table 5.1 shows some typical values of $I_0(z)$ and $k_0(z)$.

The constants of integration C_1 and C_2 can be worked out by applying the boundary conditions at the two ends of the triangular fin.

(*i*) At the tip :

$$\theta = \text{finite at } x = 0$$

$$\theta = C_1 I_0(0) + C_2 k_0(0)$$

But $I_0(0) = 1$ and $k_0(0) = \infty$

As θ is finite, C_2 must be zero otherwise θ would take infinite value which is not possible

$$\therefore \quad \theta = C_1 I_0(2B\sqrt{x}) \qquad ...(5.28)$$

(*ii*) At the root (base) :

$$\theta = \theta_0 \quad \text{at} \quad x = l$$

$$\theta_0 = C_1 I_0(2B\sqrt{l})$$

$$\text{or} \quad C_1 = \frac{\theta_0}{I_0(2B\sqrt{l})}$$

Table 5.1. Typical Values of Bessel Function

z	$I_0(z)$	$I_1(x)$	$\frac{2}{\pi}K_0(z)$	$\frac{2}{\pi}K_1(z)$
0.0	1.000	0.000	∞	∞
0.2	1.010	0.1005	1.116	3.040
0.4	1.040	0.02040	0.7095	1.391
0.6	1.092	0.314	0.4956	0.829
0.8	1.166	0.433	0.360	0.5486
1.0	1.266	0.565	0.2680	0.383
2.0	2.279	1.591	0.0725	0.0890
3.0	4.881	3.953	0.0221	0.0256
4.0	11.302	9.799	0.0071	0.00795
5.0	27.240	24.336	0.00235	0.00257
6.0	67.2348	61.342	0.00027	0.000688
7.0	168.6	156.04	0.000093	0.000289
8.0	427.6	399.9	0.000032	0.000099
9.0	1093.6	1030.9	0.000032	0.000034
10.0	—	—	0.000011	0.000011

Substituting the above obtained value of C_1 in equation 5.28, we get

$$\theta = \frac{\theta_0}{I_0(2B\sqrt{l})} \times I_0(2B\sqrt{x})$$

$$\text{or} \quad \frac{\theta}{\theta_0} = \frac{I_0(2B\sqrt{x})}{I_0(2B\sqrt{l})} \qquad ...(5.29)$$

The heat flow from the fin is obtained by finding the heat conducted at the base

$$Q = -k\left[A_c\frac{d\theta}{dx}\right]_{x=l} \qquad ...(5.30)$$

The value of $\left(\frac{d\theta}{dx}\right)_{x=l}$ can be worked out by using the following relations pertaining to Bessel functions

$$\frac{d}{dx}[I_n(z)] = I_{n+1}(z)\frac{d}{dx}(z)$$

For zero order Bessel equation, $n = 0$ and therefore

$$\frac{d}{dx}[I_0(z)] = I_1(z)\frac{d}{dx}(z)$$

$$\text{or} \quad \frac{d}{dx}\left[I_0(2B\sqrt{x})\right] = I_1(2B\sqrt{x})\frac{d}{dx}(2B\sqrt{x})$$

$$= I_1(2B\sqrt{x})Bx^{-1/2}$$

An examination of the above identity and equations 5.29 and 5.30 gives

$$Q = kA_c\frac{d}{dx}\left[\theta_0\frac{I_0(2B\sqrt{x})}{T_0(2B\sqrt{l})}\right]$$

$$= \frac{kA_c\theta_0}{I_0(2B\sqrt{l})}\frac{d}{dx}\left[I_0(2B\sqrt{x})\right]_{x=l}$$

$$= \frac{kA_c\theta_0}{I_0(2B\sqrt{l})}\left[I_1(2B\sqrt{x})\,B\,x^{-1/2}\right]_{x=l}$$

$$= \frac{kA_c\theta_0}{I_0(2B\sqrt{l})}\left[I_1(2B\sqrt{l})\frac{B}{\sqrt{l}}\right]$$

$$= \frac{kBA_c\theta_0}{\sqrt{l}}\frac{I_1(2B\sqrt{l})}{I_0(2B\sqrt{l})}$$

Substituting the values : $A_c = b\delta$ and $B = \sqrt{\frac{2hl}{k\delta}}$, we get

$$Q = \frac{k \times \sqrt{2hl/k\delta} \times b\delta \times \theta_0}{\sqrt{l}}\frac{I_1(2B\sqrt{l})}{I_0(2B\sqrt{l})}$$

$$= b\sqrt{2hk\delta}\,\theta_0\frac{I_1(2B\sqrt{l})}{I_0(2B\sqrt{l})} \qquad ...(5.31)$$

EXAMPLE 5.29

An air cooled cylindrical wall is to be fitted with triangular fins of 3 cm thickness at base and 12 cm in height. The fins are made from stainless steel with density 8000 kg/m³ and thermal conductivity 17.5 W/m-deg. The wall temperature is 600°C and the fin is exposed to an environment with t_a = 30°C and h = 20 W/m²-deg. Set up an expression for the temperature distribution along the fin and make calculations for the rate of heat flow per unit mass of fin material used.

Solution : For a triangular fin, the temperature distribution is prescribed by the relation

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{I_0(2B\sqrt{x})}{I_0(2B\sqrt{l})}$$

$$\text{Here} \quad B = \sqrt{2\frac{Hl}{k\delta}} = \sqrt{\frac{2 \times 20 \times 0.12}{17.5 \times 0.03}} = 3.028$$

$$2B\sqrt{l} = 2 \times 3.028 \times \sqrt{0.12} = 2.098$$

From Table 5.1 : $I_0(2.098) = I_0(2) = 2.2796$

$$\therefore \quad \frac{t - 30}{600 - 30} = \frac{I_0(2B\sqrt{x})}{2.2796}$$

$$t = 30 + \frac{570}{2.2796}I_0(2 \times 3.028\sqrt{x})$$

$$= 30 + 250I_0(6.056\sqrt{x})$$

which is the required relation for temperature distribution

(*b*) The heat flow from a triangular fin is

$$Q = b\sqrt{2hk\delta}\,\theta_0\frac{I_1(2B\sqrt{l})}{I_0(2B\sqrt{l})}$$

Considering unit metre width for the fin

$$Q = 1 \times \sqrt{2 \times 20 \times 17.5 \times 0.03} \times (600 - 30) \times \frac{I_1(2.098)}{I_0(2.098)}$$

$$= 1 \times 4.58 \times 570 \times \frac{1.5906}{2.2796}$$

$$= 1822 \text{ W}$$

Mass of fin per metre width

$$= \left(\frac{1}{2} l\delta \times b\right) \times \delta$$

$$= \left(\frac{1}{2} \times 0.12 \times 0.03 \times 1\right) \times 8000$$

$$= 14.4 \text{ kg}$$

$\therefore$ Rate of heat flow per unit mass

$$= \frac{1822}{14.4} = 126.53 \text{ W/kg}$$

5.8.2. Parabolic fins

A parabolic fin is of great practical importance because it dissipates the maximum amount of heat at minimum material cost

Fig. 5.14. shows the notation being used in the analysis of a parabolic fin. The fin has length l, thickness δ at the base and width b perpendicular to the plane of the paper. The thickness of the fin is function of x and

$$y = 0 \quad \text{at} \quad x = 0$$

and $\quad y = \delta/2 \quad \text{at} \quad x = l$

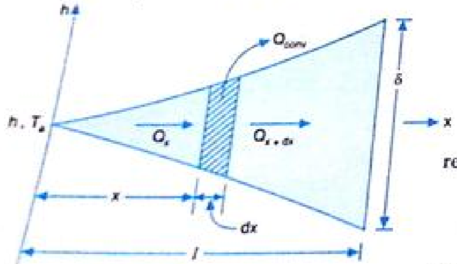


Fig. 5.15. Parabolic fin

As the fin is parabolic, the curve follows the law $y = C x^2$ where C is a constant. From the above stated geometry

$$C = \frac{y}{x^2} = \frac{\delta/2}{l^2} = \frac{\delta}{2l^2}$$

$$\therefore \quad y = \frac{\delta}{2l^2} x^2 = \frac{\delta}{2}\left(\frac{x}{l}\right)^2 \qquad ...(5.32)$$

For an element strip at distance x from the tip, the energy balance gives

$$Q_x = Q_{x+dx} + Q_{conv}$$

$$= \left[Q_x + \frac{d}{dx}(Q_x)dx\right] + Q_{conv}$$

That gives

$$Q_{conv} = -\frac{d}{dx}(Q_x)dx$$

$$h\, dA_s (t - t_a) = -\frac{d}{dx}\left(-kA_c \frac{dt}{dx}\right)dx$$

Now $dA_s = P\,dx = 2b\,dx$ and $A_c = 2y \times b$ where $2y$ is the width of the elemental strip

$$\therefore h\,(2b\,dx) = \frac{-d}{dx}\left[-k(2by)\frac{dt}{dx}\right]dx \qquad ...(5.33)$$

Introducing the parameter temperature excess $\theta = t - t_a$ and upon simplification

$$k\frac{d}{dx}\left(y\frac{d\theta}{dx}\right) = h\,\theta$$

or $$k\left[\frac{dy}{dx}\frac{d\theta}{dx} + y\frac{d^2\theta}{dx^2}\right] = h\,\theta$$

Since, $y = \frac{\delta}{2}\left(\frac{x}{l}\right)^2$; $\frac{dy}{dx} = \frac{x\delta}{l^2}$

$$\therefore\ k\left[\frac{x\delta}{l^2}\frac{d\theta}{dx} + \frac{\delta}{2}\left(\frac{x}{l}\right)^2\frac{d^2\theta}{dx^2}\right] = h\,\theta$$

Upon dividing throughout by $\frac{k\delta}{2l^2}$ and rearanging,

$$x^2\frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - \frac{2hl^2}{k\delta}\theta = 0$$

Making the substitution $m^2 = 2h/k\delta$, we get

$$x^2\frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - m^2 l^2\theta = 0 \qquad ...(5.34)$$

This homogeneous equation can be solved by carrying out the following transformations:

Let $\quad z = \log_e x$

$$\frac{d\theta}{dx} = \frac{d\theta}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{d\theta}{dz}$$

$$\frac{d^2\theta}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{d\theta}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{d\theta}{dz}\right) + \frac{d\theta}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{d\theta}{dx}\right) + \frac{d\theta}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{1}{x}\frac{d\theta}{dz}\right) + \frac{d\theta}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x^2}\frac{d^2\theta}{dz^2} - \frac{1}{x^2}\frac{d\theta}{dz}$$

When these values are substituted in equation 5.34, we obtain

$$x^2\left[\frac{1}{x^2}\frac{d^2\theta}{dz^2} - \frac{1}{x^2}\frac{d\theta}{dz}\right] + 2x\left(\frac{1}{x}\frac{d\theta}{dz}\right) - m^2 l^2\theta = 0$$

or $$\frac{d^2\theta}{dz^2} + \frac{d\theta}{dz} - m^2 l^2\theta = 0$$

or $$D^2 + D - m^2 l^2 = 0 \qquad ...(5.35)$$

Solution of this quadratic equation gives

$$D = \frac{-1 \pm \sqrt{1 + 4m^2 l^2}}{2}$$

If the roots are designated by n_1 and n_2, then temperature distribution can be prescribed as

$$\theta = C_1 e^{n_1 z} + C_2 e^{n_2 z}$$

$$= C_1 e^{n_1 \log_e x} + C_2 e^{n_2 \log_e x}$$

$$= C_1 x^{n_1} + C_2 x^{n_2} \qquad ...(5.36)$$

The constants C_1 and C_2 can be worked out by applying the following boundary conditions at the two ends of the parabolic fin :

(i) At the tip : θ = finite at $x = 0$

Then $\quad \theta = 0 + \frac{C_2}{0} = \infty$

As θ is the finite, C_2 must be zero otherwise θ would take infinite value which is not possible.

$$\therefore \quad \theta = C_1 x^{n_1} \qquad ...(5.37)$$

(ii) At the root (base) :

$$\theta = \theta_0 \quad \text{at} \quad x = l$$

$$\theta_0 = C_1 l^{n_1} \quad \text{or} \quad C_1 = \frac{\theta_0}{l^{n_1}}$$

Substituting the above obtained value of C_1 in equation 5.37, a particular temperature distribution is given by

$$\theta = \theta_0\left(\frac{x}{l}\right)^{n_1} \qquad ...(5.38)$$

where $\quad n_1 = \frac{-1 + \sqrt{1 + 4m^2 l^2}}{2}$

The heat flux at any section is

$$q_x = \left(\frac{Q_x}{A}\right) = k\frac{d\theta}{dx} = \frac{k\theta_0}{l^{n_1}} n_1 x^{n_1 - 1}$$

The negative sign has been omitted as the x-axis is taken in a direction opposite to that of heat flow.

If n_1 is presumed to be unity, then

$$1 = \frac{-1 + \sqrt{1 + 4m^2 l^2}}{2};$$

$m^2 l^2 = 2$ and

$$q_x = \frac{k\theta_0}{l} = \frac{k}{l}(t_0 - t_a) \qquad ...(5.39)$$

Apparently heat flux is independent of x provided $ml = \sqrt{2}$ and this is the most required condition for the fin.

Further, the identity $ml = \sqrt{2}$ implies that

$$m^2 l^2 = 2; \quad \frac{2h}{k\delta}l^2 = 2$$

or $$\frac{(\delta/2)^2}{l^2} = \frac{h(\delta/2)}{2k}$$

or $$\frac{\delta/2}{l} = \sqrt{\frac{h(\delta/2)}{2k}} = \sqrt{\frac{B_i}{2}} \qquad ...(5.40)$$

where $B_i = \sqrt{\frac{h(\delta/2)}{k}}$ is called the Biot number

When a parabolic fin is so designed that

$\frac{\delta/2}{l}$ is equal to $\sqrt{\frac{B_i}{2}}$, then

(*i*) the fin has most economical section, *i.e.*, it has the minimum material requirement

(*ii*) the heat dissipation is independent of the distance from the base of the fin. Heat flux is constant at all sections and the heat convected is equally passed through all sections.

The heat dissipation at any section is then given by

$$Q = q_x \, |A_x|_{x=l}$$

$$= \frac{k}{l}(t_0 - t_a) \times (b \times \delta) \quad ...(5.41)]$$

5.9. THERMOMETRIC WELL

Fig. 5.15 shows an arrangement which is used to measure the temperature of gas flowing through a pipeline. A small tube called thermometric well is welded radially into the pipeline. The well is partially filled with some liquid and the thermometer is immersed into this liquid. When the temperature of the gas flowing through the pipeline is higher than the ambient temperature, the heat flows from the hot gases towards the tube walls along the well. This may cause temperature at the bottom of well to become colder than the gas flowing around. Obviously the temperature indicated by the thermometer will not be the true temperature of the gas. The error in the temperature measurement is estimated with the help of the theory of extended surfaces.

The protective tube (well) can be considered as a hollow fin (internal diameter d, thickness δ and length l), and the temperature distribution obtained by using the relation applicbale to a fin with tip insulated :

$$\frac{\theta_x}{\theta_0} = \frac{(t_x - t_a)}{(t_0 - t_a)} = \frac{\cosh m(l - x)}{\cosh ml}$$

where t_0 is the temperature of the pipe wall, t_a is the temperature of hot gas or air flowing through the pipeline, and t_x is the temperature at any distance x measured from pipe wall along the thermometric well.

If $x = l$ then,

$$\frac{t_l - t_a}{t_0 - t_a} = \frac{\cosh m(l - l)}{\cosh ml} = \frac{1}{\cosh ml}$$

where t_l is the temperature recorded by the thermometer at the bottom of the well.

The perimeter of the protective well $P = \pi(d + 2\delta) = \pi d$, and its cross-sectional area $A_c = \pi d \delta$. Therefore,

$$\frac{P}{A_c} = \frac{\pi d}{\pi d \delta} = \frac{1}{\delta}$$

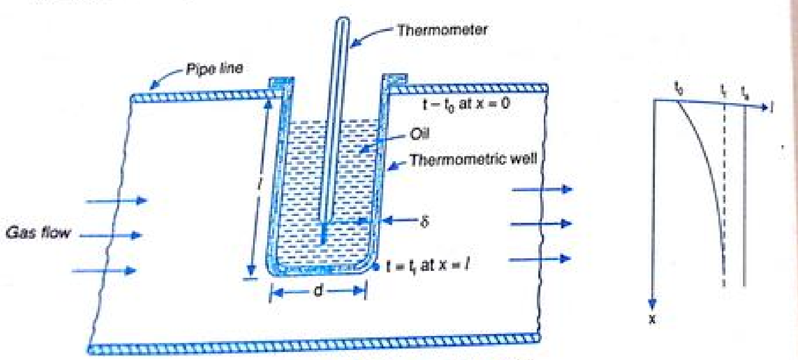


Fig. 5.16. Thermometric well

then, $m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h}{k\delta}}$

Apparently, diameter of the well does not have any effect on temperature measurement by the thermometer.

The error can be minimized by :

(*i*) lagging the tube so that conduction of heat along its length is arrested.

(*ii*) increasing the value of parameter ml

For a rectangular fin $m = \sqrt{2h/k\delta}$. An increase in m can be affected by using a thinner tube of low thermal conductivity or by increasing the convection coefficient through finning the manometric well. The operative length l is increased by inclining the pocket and setting its projection beyond the pipe axis. The need to locate the protective tube oblique (inclined) would arise if the desired length is more than the diameter of pipeline. The diameter of tube does not have any effect on increasing the accuracy of temperature measurement.

EXAMPLE 5.30

Measurements of temperature of gas flowing through a pipe has been made by mercury-in-glass thermometer dipped into an oil-filled steel tube (protective well) welded radially to the pipe line. The thermometer indicates a temperature at the end of the steel tube which is lower than the gas temperature due to transfer of heat by conduction along the protective well. How large is the measurement error if the thermometer reads t_1 = 85°C and the temperature at the base of the protective well (pipe wall) is t_0 = 40°C. The protective tube is 125 mm long and has 1.5 mm thick wall. It may be presumed that the thermal conductivity of the tube material is 56 W/mK and the local coefficient of heat transfer from gas to the protective tube is 23.5 W/m²K.

Solution : The temperature distribution along the length of the pocket is given by

$$\frac{t_1 - t_a}{t_0 - t_a} = \frac{1}{\cosh ml}$$

where t_1 is the temperature at the bottom of the pocket, t_a is the temperature of gas flowing and t_0 is the temperature of the pipe wall.

$$\frac{P}{A_c} = \frac{\pi d}{\pi d \delta} = \frac{1}{\delta}$$

then, $m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h}{k\delta}}$

$$= \sqrt{\frac{23.5}{56 \times 0.0015}} = 16.73 \text{ m}^{-1}$$

$$ml = 16.73 \times 0.125 = 2.0912$$

$$\cosh ml = \cosh (2.0912) = 4.109$$

$$\therefore \frac{85 - t_a}{40 - t_a} = \frac{1}{4.109}$$

$$4.109\,(85 - t_a) = 40 - t_a$$

$$3.109\, t_a = 85 \times 4.109 - 40 = 309.265$$

Therefore, the temperature of gas,

$$t_a = \frac{309.265}{3.109} = 99.47°C$$

Error in measurement of temperature is equal to

$$99.47 - 85 = 14.47°C$$

Percentage of error

$$= \frac{14.47}{99.47} \times 100 = 14.55\%$$

EXAMPLE 5.31

A thermometric pocket is a hollow brass tube (k = 75 W/m-deg) having outer and inner diameter of 15 mm of 10 mm respectively. The pocket extends to 5 cm depth from the wall of a 15 cm diameter pipe which carries hot air. The heat transfer coefficient between the pocket and air is prescribed by the relation.

Nusselt number $N_u = 0.175\,(Re)^{0.62}$

Make calculations for the error in temperature measurement. Presume the following data :

Air temperature 160°C and pipe wall temperature 40°C

Reynolds number 25000 and thermal conductivity of air 0.036 W/m-deg

Solution : Refer Fig. 5.16. for the arrangement

Nusselt number,

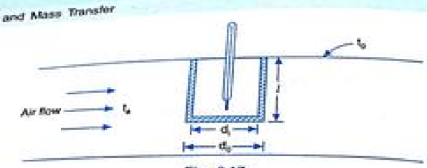


Fig. 5.17.

$$N_u = 0.175\,(R_e)^{0.62}$$
$$= 0.175\,(25000)^{0.62} = 93.27$$

In terms of convective heat transfer coefficient h, the Nusselt number is

$$N_u = \frac{h d_0}{k_{air}}$$

$$\therefore \quad h = \frac{93.27 \times 0.036}{0.015}$$
$$= 223.85 \text{ W/m}^2\text{-deg}$$

The temperature distribution along the length of pocket is

$$\frac{t - t_a}{t_0 - t_a} = \frac{1}{\cosh ml}$$

where t is the temperature at the bottom of pocket, t_a is the temperautre of air flowing and t_0 is the temperature of the pipe wall.

Further,

$$m = \sqrt{\frac{hP}{kA_c}}$$

$$\text{where} \quad \frac{P}{A_c} = \frac{\pi d_0}{\frac{\pi}{4}\left(d_0^2 - d_i^2\right)} = \frac{4 d_0}{d_0^2 - d_i^2}$$

$$= \frac{4 \times 0.015}{0.015^2 - 0.01^2} = 480$$

$$\therefore \quad m = \sqrt{\frac{223.85}{75} \times 480} = 37.85$$

$$ml = 37.85 \times 0.05 = 1.892$$

$$\cosh ml = \cosh 1.892 = 3.391$$

$$\therefore \frac{t - 160}{40 - 160} = \frac{1}{3.391}$$

Therefore, the temperature of air as measured by the thermometric pocket is

$$t = 160 + \frac{40 - 160}{3.391} = 124.61°\text{C}$$

Error in measurement of temperature,

$$= 160 - 124.61 = 35.39°\text{C}$$

$$\% \text{ age of error} = \frac{35.39}{160} \times 100 = 22.12\%$$

EXAMPLE 5.32

The temperature of dry saturated steam flowing in a 10 cm diameter steel pipe has been measured in the laboratory by means of a mercury-in-glass thermometer immersed in an oil-filled steel well. The well is of 1.25 cm diameter and 3 mm wall thickness. A reliable pressure gauge fitted to the pipe line reads 10.5 bar and a reference to steam table indicates that the saturation temperature of steam at 10.5 bar is 181.1°C. What should be the depth (length) of the well if the limit on maximum error in temperature measurement is 2.5%. The temperature at the pipe wall is 120°C. The heat transfer co-efficient between steam and steel well is 420 kJ/m²-hr-deg and thermal conductivity of the wall material is 160 kJ/m-hr-deg.

Solution : Treating the steel as fin with insulated tip

$$\frac{\theta_x}{\theta_0} = \frac{(t_x - t_a)}{(t_0 - t_a)} = \frac{\cosh m(l - x)}{\cosh ml}$$

For the protective well,

Perimeter $P = \pi\,(d + 2\delta)$,

Cross-sectional area $A_c = \pi d \delta$

Therefore

$$\frac{P}{A_c} = \frac{\pi d}{\pi d \delta} = \frac{1}{\delta}$$

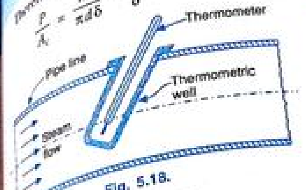


Fig. 5.18.

$$\text{Then, } m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h}{k\delta}}$$

$$= \sqrt{\frac{420}{160 \times 0.003}} = 29.58 \text{ m}^{-1}$$

$$t_0 = 120°\text{C}\ ; \quad t_a = 181.1°\text{C}\ ;$$
$$t_x = 0.975 \times 181.1°\text{C}$$
$$= 176.57°\text{C} \quad \text{at} \quad x = l$$

$$\therefore \frac{176.57 - 181.1}{120 - 181.1} = \frac{\cosh m(l - l)}{\cosh ml}$$
$$= \frac{1}{\cosh ml}$$

$\cosh ml - 13.488\ ;\ ml = 3.293$

$\therefore$ length (depth) of the well, l

$$= \frac{3.293}{29.58} = 0.1113 \text{ m} = \mathbf{11.13 \text{ cm}}$$

Since the length of protective well required is more than the diameter of the steel pipe, it is necessary to locate the well obliquely (inclined) to the pipe as shown in Fig. 5.18.

5.10. HEAT TRANSFER FROM A BAR CONNECTED TO TWO HEAT SOURCES AT DIFFERENT TEMPERATURES

Consider heat flow along a bar between two thermal reservoirs; the reservoir A at temperature t_1 and the reservoir B at temperature t_2. The system will also be subjected to convective heat flow from the bar to the surroundings.

Let attention be focussed on an infinitesimal element of the bar; the element has thickness δx and is located at a distance x from the reservoir A. The heat flow terms are :

$$Q_{in} = -kA_c \frac{dt}{dx}$$

$$Q_{out} = -kA_c \frac{d}{dx}\left(t + \frac{dt}{dx}\delta x\right)$$

$$Q_{conv} = hP\delta x\,(t - t_a)$$

where A_c is the cross-sectional area of the bar, P is the perimeter of the bar, h is the convective heat transfer coefficient, k is the thermal conductivity of the bar material, and t_a is the temperature of surroundings.

A heat balance on the element gives :

$$Q_{in} = Q_{out} + Q_{conv}$$

$$-kA_c \frac{dt}{dx} = -kA_c \frac{d}{dx}\left(t + \frac{dt}{dx}\delta x\right) + hP\delta x\,(t - t_a)$$

Upon simplification and re-arrangement

$$\frac{d^2 t}{dx^2} - \frac{hP}{kA_c}(t - t_a) = 0 \qquad ...(5.42)$$

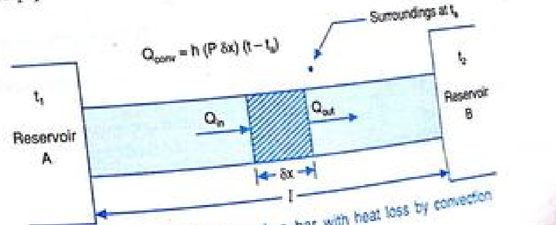


Fig. 5.19. Conduction through a bar with heat loss by convection

Since the ambient temperature is assumed constant, the temperature excess $(t - t_a)$ can be replaced by θ and

$$\frac{d^2t}{dx^2} \text{ becomes } \frac{d^2\theta}{dx^2}$$

Equation 5.42 gets transformed to

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA_c}\theta = 0 \qquad \ldots(5.43)$$

and solution to this differential equation is

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \qquad \ldots(5.44)$$

$$\text{where } m = \sqrt{\frac{hP}{kA_c}}$$

and the constants C_1, C_2 are to be determined from the boundary conditions.

The relevant boundary conditions for the given arrangement are :

(i) $\theta = \theta_1$ at $x = 0$

(ii) $\theta = \theta_2$ at $x = l$

Applying these boundary conditions to equation 5.44

$$\theta_1 = C_1 + C_2 \qquad \ldots(a)$$

$$\theta_2 = C_1 e^{ml} + C_2 e^{-ml} \qquad \ldots(b)$$

Solving expressions (a) and (b), the constants are determined as follows

$$C_1 = \frac{\theta_2 - \theta_1 e^{-ml}}{e^{ml} - e^{-ml}}$$

$$C_2 = \frac{\theta_1 e^{ml} - \theta_2}{e^{ml} - e^{-ml}}$$

Substituting these values of constants C_1 and C_2 in equation 5.44, one obtains the following expression for temperature distribution along the length of bar.

$$\theta = \frac{\theta_2 - \theta_1 e^{-ml}}{e^{ml} - e^{-ml}} e^{mx} + \frac{\theta_1 e^{ml} - \theta_2}{e^{ml} - e^{-ml}} e^{-mx}$$

$$= \frac{\theta_1\left[e^{ml}e^{-mx} - e^{-ml}e^{mx}\right]}{e^{ml} - e^{-ml}} + \frac{\theta_2\left[e^{mx} - e^{-mx}\right]}{e^{ml} - e^{-ml}}$$

$$= \frac{\theta_1 \sinh m(l-x)}{\sinh ml} + \frac{\theta_2 \sinh mx}{\sinh ml}$$

$$= \frac{\theta_1 \sinh m(l-x) + \theta_2 \sinh mx}{\sinh ml} \qquad \ldots(5.45)$$

The rate of heat loss can be worked out from the relation

$$Q = \int_0^l hP\,dx(t - t_a) = \int_0^l hP\,dx\,\theta$$

$$= hP\int_0^l \frac{\theta_1 \sinh m(l-x) + \theta_2 \sinh mx}{\sinh ml}\,dx$$

$$= \frac{hP}{\sinh ml}\left[-\theta_1 \frac{\cosh m(l-x)}{m} + \theta_2 \frac{\cosh mx}{m}\right]_0^l$$

$$= \frac{hP}{\sinh ml}\left[-\frac{\theta_1}{m}(1 - \cosh ml) + \frac{\theta_2}{m}\{\cosh ml - 1\}\right]$$

$$= \frac{hP}{m \sinh ml}(\theta_1 + \theta_2)\{\cosh ml - 1\}$$

Substituting back the value of m

$$= \sqrt{Ph / kA_c}$$

$$Q = \sqrt{phkA_c}\,(\theta_1 + \theta_2)\left[\frac{\cosh ml - 1}{\sinh ml}\right] \qquad \ldots(5.46)$$

The position of minimum temperature can be worked out by differentiating the expression for temperature distribution with respect to x and setting the derivative equal to zero. Thus

$$\frac{d\theta}{dx} = \frac{1}{\sinh ml}\left[-\theta_1 m \cosh m(l-x) + \theta_2 m \cosh mx\right] = 0$$

$$\therefore \theta_1 \cosh m(l-x) = \theta_2 \cosh mx$$

$$\text{or} \quad \frac{\theta_1}{\theta_2} = \frac{\cosh mx}{\cosh m(l-x)} \qquad \ldots(5.47)$$

When both the reservoirs are at the same temperature, $\theta_1 = \theta_2$, the value of x becomes $l/2$, *i.e.*, the maximum temperature occurs at the middle of the rod.

Temperature excess at the middle of the bar is then obtained by substituting $\theta_1 = \theta_2$ and $x = l/2$ in equation 5.45.

$$(\theta)_{l/2} = \frac{\theta_1 \sinh m(l - l/2) + \theta_2 \sinh ml}{\sinh ml}$$

$$= 2\theta_1 \frac{\sinh ml/2}{\sinh ml} \qquad \ldots(5.48)$$

EXAMPLE 5.33

A thin copper rod (k = 95 W/m-deg) is 12 mm in diameter and spans between two plates 150 mm apart. Air flows over the plates providing convective heat transfer coefficient equal to 50 W/m²-deg. If surface temperature of the plates exceeds the air temperature by 45°C, calculate excess temperature at mid length of the rod over that of air and the heat loss from the rod.

Solution : For a circular rod of diameter d,

$$P = \pi d \; ; \quad A_c = \frac{\pi}{4}d^2$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h \times \pi d}{k \times \frac{\pi}{4}d^2}}$$

$$= \sqrt{\frac{4h}{kd}} = \sqrt{\frac{4 \times 50}{95 \times 0.012}}$$

$$= 13.245 \text{ m}^{-1}$$

$$ml = 13.245 \times 0.15 = 1.987$$

Temperature distribution along the length of the bar is given by,

$$\theta = \frac{\theta_1 \sinh m(l-x)}{\sinh ml} + \frac{\theta_2 \sinh mx}{\sinh ml}$$

$\theta_1 = \theta_2$ as both the plates are at the same temperature. Temperature excess at the middle of the bar is then obtained by substituting $\theta_1 = \theta_2$ and $x = l/2$ in the above identity

$$(\theta)_{l/2} = \frac{\theta_1 \sinh m(l - l/2)}{\sinh ml} + \frac{\theta_1 \sinh ml/2}{\sinh ml}$$

$$= 2\theta_1 \frac{\sinh ml/2}{\sinh ml}$$

$$= 2(t_0 - t_a)\frac{\sinh ml/2}{\sinh ml}$$

$$= 2 \times 45 \times \frac{\sinh(1.987/2)}{\sinh 1.987}$$

$$= 90 \times \frac{1.1652}{3.580} = 29.29°\text{C}$$

Heat flow through the rod is given by

$$Q = \sqrt{phkA_c}\,(\theta_1 + \theta_2)\frac{\cosh(ml) - 1}{\sinh ml}$$

$$= \sqrt{\pi d hk \frac{\pi}{4}d^2} \times 2\theta_1 \times \frac{\cosh(ml) - 1}{\sinh ml} \qquad (\because \theta_1 = \theta_2)$$

$$= \sqrt{kh\pi^2 d^3} \times \theta_1 \times \frac{\cosh(ml) - 1}{\sinh ml}$$

$$= \sqrt{95 \times 50 \times \pi^2 \times 0.012^3} \times 45 \times \frac{\cosh(1.987) - 1}{\sinh 1.987}$$

$$= 0.2845 \times 45 \times \frac{2.715}{3.578}$$

$$= 9.714 \text{ W}$$

EXAMPLE 5.34

An iron bar 15 mm in diameter spans the distance between two plates 50 cm apart. Air at 25°C flows in the space between the plates providing a heat transfer coefficient of 17.5 W/m² K. Determine the heat flowing and temperature at the middle of the bar if

(a) the plates are maintained at 125°C each

(b) the plates are maintained at 125°C and 40°C respectively

Thermal conductivity of the iron bar k = 46.5 W/mK.

Solution : For the circular bar,

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4}(d)^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kd}}$$

$$= \sqrt{\frac{4 \times 17.5}{46.5 \times 0.015}} = 10.02 \text{ m}^{-1}$$

$$ml = 10.02 \times 0.5 = 5.01$$

(*a*) Temperature excess for each plate

$$\theta_1 = \theta_2 = 125 - 25 = 100°C$$

Heat flowing through the bar,

$$Q = \sqrt{phkA_c}\,(\theta_1 + \theta_2)\frac{\cosh ml - 1}{\sinh ml}$$

$$= kA_c m(\theta_1 + \theta_2)\frac{\cosh ml - 1}{\sinh ml}$$

$$= 46.5 \times \left(\frac{\pi}{4} \times 0.015^2\right) \times 10.02$$

$$\times (100 + 100)\frac{\cosh (5.01) - 1}{\sinh 5.01}$$

$$= 16.24 \text{ W}$$

Temperature excess at the middle of the bar,

$$\theta = 2\theta_1 \frac{\sinh ml/2}{\sinh ml}$$

$$= 2 \times 100 \times \frac{\sinh 2.505}{\sinh 5.01} = 16.22°C$$

$\therefore$ Temperature at the middle of the rod

$$= 16.22 + 25 = 41.22°C$$

(*b*) The temperature excesses at the two plates are :

$$\theta_1 = 125 - 25 = 100°C\ ;$$

$$\theta_2 = 40 - 25 = 15°C$$

Heat flowing through the bar,

$$Q = kA_c m(\theta_1 + \theta_2)\frac{\cosh ml - 1}{\sinh ml}$$

$$= 46.5 \times \left(\frac{\pi}{4} \times 0.015^2\right) \times 10.02$$

$$\times (100 + 15)\frac{\cosh 5.01 - 1}{\sinh 5.01}$$

$$= 9.34 \text{ W}$$

Temperature excess at the middle of the bar,

$$\theta = \frac{\theta_1 \sinh m(l - x) + \theta_2 \sinh mx}{\sinh ml}$$

$$= \frac{100 \sinh 10.02\,(0.5 - 0.25) + 15 \sinh (10.02 \times 0.25)}{\sinh (10.02 \times 0.5)}$$

$$= \frac{100 \times 6.08 + 15 \times 6.08}{74.95}$$

$$= 9.33°C$$

$\therefore$ Temperature at the middle of the bar,

$$= 25 + 9.33 = 34.33°C$$

EXAMPLE 5.35

Both ends of a 5 mm dimeter U-shaped copper (k = 300 W/m-deg) rod are rigidly fixed to a vertical wall which is at 120°C temperature. The length of U-shaped rod is 50 cm and it is exposed to air at 30°C. The combined radiative and convective heat trousfer coefficient for the rod is 25 W/m²-deg. Make calculations for the temperature at the centre of U-shaped rod and the heat transfer.

Solution : Refer Fig. 5.20 for the configuration of the system

The temperature t at the centre of the rod can be worked out for the relation

$$\theta = 2\,\theta_0\,\frac{\sinh ml/2}{\sinh ml}$$

where $\theta = t - t_a\ ;\quad \theta_0 = (t_w - t_a)$

$$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{h}{k}\frac{\pi d}{\frac{\pi}{4}d^2}} = \sqrt{\frac{4h}{kd}}$$

$$= \sqrt{\frac{4 \times 25}{300 \times 0.005}} = 8.165 \text{ m}^{-1}$$

$$ml = 8.165 \times 0.5 = 4.082$$

$$\frac{ml}{2} = \frac{4.082}{2} = 2.041$$

$$\therefore\ t - 30 = 2\,(120 - 30) \times \frac{\sinh 2.041}{\sinh 4.082}$$

$$= 22.36$$

Thus the temperature at the centre of U-shaped rod is,

$$t = 30 + 22.36 = 52.36°C$$

(*b*) The heat transfer rate from the fin can be worked out by considersing two fins of 25 cm length each with insulated tip.

$$Q = 2\,\sqrt{hPkA}\,(t_0 - t_a)\tanh ml$$

Fig. 5.20.

$$= 2\sqrt{25 \times \pi \times 0.006 \times 300 \times \pi/4\,(0.006)^2}\ (120 - 30)\tanh(4.082)$$

$$= 11.37 \text{ W}$$

EXAMPLE 5.36

The handle of a laddle used for pouring molten lead at 325°C is 35 cm long and is made of 25 mm × 15 mm mild steel bar stock (k = 38 W/m-deg). In order to reduce the grip temperature (temperature at the centre of the handle), it is proposed to make a hollow handle of mild steel plate 1.5 mm thick to same rectangular shape. The surface heat transfer coefficient is 12 W/m²-deg and the ambient air temperature is 25°C. Neglecting heat transfer from inner surface of hollow shape, make calculations for reduction in grip temperature.

Solution : Refer Fig. 5.21 for the configuration of solid and hollow cylinders. All dimensions are in mm

The handle can be consideral a rod welded to the two plates which are at the same temperature. Then the grip temperature (temperature at the centre of the handle) can be worked out from the relation.

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = 2\,\frac{\sinh ml/2}{\sinh ml}$$

where $m = \sqrt{hP/kA}$

(*a*) $P = 2\,(0.025 + 0.015) = 0.08$ m

$A = 0.025 \times 0.015 = 0.000375 \text{ m}^2$

$$m = \sqrt{\frac{12 \times 0.08}{38 \times 0.000375}} = 8.208$$

$$ml = 8.208 \times 0.35 = 2.873$$

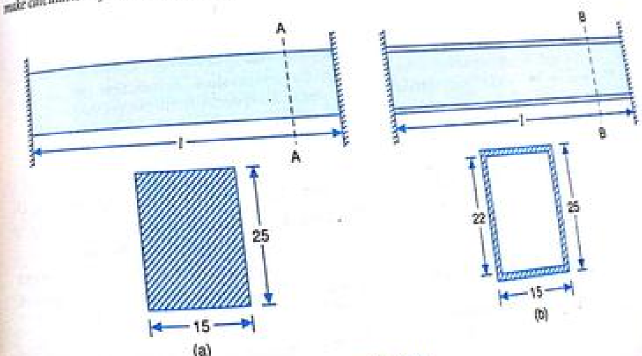


Fig. 5.21.

$$\therefore \frac{t-25}{325-25} = 2\frac{\sinh(2.873/2)}{\sinh 2.873}$$

$$= \frac{2\times 1.984}{8.816} = 0.45$$

$$t = 25 + (325-25)\times 0.45 = 160°C$$

(ii) $P = 2\,(0.025 + 0.015) = 0.08$ m

$$A = (0.025\times 0.015) - (0.022\times 0.012)$$

$$= 1.11\times 10^{-4}\ m^2$$

$$m = \sqrt{\frac{12\times 0.08}{38\times 1.11\times 10^{-4}}} = 15.086$$

$$ml = 15.086\times 0.35 = 5.28$$

$$\therefore \frac{t-25}{325-25} = 2\frac{\sinh(5.28/2)}{\sinh 5.28}$$

$$= \frac{2\times 6.971}{98.18} = 0.142$$

$$t = 25 + (325-25)\times 0.142$$

$$= 67.60°C$$

Thus drop in grip temperature are due to hollow section is

$$= 160 - 67.60 = 92.4°C$$

SALIENT POINTS

1. *A fin is an attachment that protrudes from the surface of a large body dissipating heat to the surroundings. These protrusions increase the surface area and that causes substantial improvement in the heat transfer rate.*
2. *The fins can take a variety of forms:*
   (i) longitudinal fins of rectangular or triangular cross-section attached to a wall (uniform and tapered straight fin)
   (ii) cylindrical tubes with radial fins (annular fin)
   (iii) cylindrical or truncated conical rod protruding from a wall (pin fins or spines)
3. *Common applications of finned surfaces are with*
   - *economisers for steam power plants*
   - *electrical transforms and motors*
   - *air cooled cylinders of aircraft engines, IC engines and air compressors*
   - *convectors for steam and hot-water heating systems*
4. *When the base of a fin of uniform cross-section A and perimeter P is held at temperature t_o and the fin is exposed to a fluid at temperature t_a, then the temperature distribution in the fin is*

$$t - t_a = C_1\,e^{mx} + C_2\,e^{-mx}$$

where $m = \sqrt{\dfrac{hP}{kA}}$. *The constant C_1 and C_2 are determined with the aid of relevant boundary conditions.*

5. Heat dissipation from an infinitely long fin. The boundary conditions are

$$t = t_0 \quad \text{at} \quad x = 0$$

and $\quad t = t_a \quad$ at $\quad x = \infty$

This condition may be approached when $ml > 5$. The temperature distribution and the rate of heat transfer from the fin then workout as

$$\frac{t-t_a}{t_0-t_a} = e^{-mx}$$

and $\quad Q_{fin} = \sqrt{hPkA}\,(t_0 - t_a)$

6. Heat dissipation from a fin insulated at the tip. The relevant boundary conditions are:

$$t = t_0 \quad \text{at} \quad x = 0$$

and $\quad \dfrac{dt}{dx} = 0 \quad$ at $\quad x = l$

The temperature distribution and the rate of heat dissipation from the fin then workout as

$$\frac{t-t_a}{t_0-t_a} = \frac{\cosh m(l-x)}{\cosh ml}$$

and $\quad Q_{fin} = \sqrt{hPkA}\,(t_0 - t_a)\tanh ml$

7. Heat dissipation from a fin losing heat at tip the relevant boundary conditions are:

$$t = t_0 \quad \text{at} \quad x = 0$$

and $\quad \left[-kA\dfrac{dt}{dx}\right]_{x=l} = hA\,(t_{x=l} - t_a)$

The temperature distribution and rate of heat transfer from the fin then workout as

$$\frac{t-t_a}{t_0-t_a} = \frac{\cosh m(l-x) + \frac{h}{km}\sinh m(l-x)}{\cosh ml + \frac{h}{km}\sinh ml}$$

and $\quad Q_{fin} = \sqrt{hPkA}\,(t_0 - t_a) \times \dfrac{\sinh ml + \frac{h}{km}\cosh ml}{\cosh ml + \frac{h}{km}\sinh ml}$

8. Fin efficiency is defined as the ratio of actual heat transferred by the fin to the heat which would be transferred if the entire fin area were at the base temperature.

(i) For an infinitely long fin

$$\eta_f = \frac{1}{ml}$$

(ii) For a fin of finite length with an insulated end

$$\eta_f = \frac{\tanh ml}{ml}$$

(iii) For a fin of finite length with convection heat transfer from the end

$$\eta_f = \frac{1}{ml}\left[\frac{\sinh ml + \frac{h}{km}\cosh ml}{\cosh ml + \frac{h}{km}\sinh ml}\right]$$

9. Effectiveness of fin is the ratio of the fin heat transfer rate (heat dissipation with fin) to the heat transfer rate that would exist without a fin.

For an infinitely long fin,

$$\epsilon_f = \sqrt{\frac{Pk}{hA_c}}$$

For a straight rectangular fin of thickness δ and width b,

$$\frac{P}{A_c} = \frac{2(b+\delta)}{b\delta} = \frac{2}{\delta}$$

That gives

$$\epsilon_f = \sqrt{\frac{2k}{h\delta}}$$

10. The fins would be effective for heat dissipation if

(i) ratio $\dfrac{Pk}{hA_c}$ is greater than 5

(ii) fins are made from a material of high thermal conductivity

(iii) fins are thin and are closely spaced

(iv) convective heat transfer coefficient is small

An increase in fin effectiveness can be obtained by extending the length of fin but that rapidly becomes a losing proposition in terms of fin efficiency.

11. A thermometer well is a small tube welded radially into a pipeline through which flows the fluid whose temperature is to be measured. This tube is considered as a hollow fin and the temperature distribution is obtained by using the relation applicable to a fin with insulated tip.

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. What is the utility of extended surfaces ?
2. Give a few practical and specific examples of use of fin in heat transfer.
3. How does a fin enhance heat transfer at a surface ?
4. State the mode of heat transfer through a fin.
5. Mention the most common types of fins and sketch them.
6. What type of boundary condition is used at the fin edge ?
7. List the assumptions made while analysing the heat flow from a finned surface.
8. Enumerate the various assumptions made in the formation of energy equation for one-dimensional heat dissipation from an extended surface.
9. Derive the governing differential equation for temperature distribution of constant area extended surface in the following form :

$$\frac{d^2t}{dx^2} = \frac{hP}{kA_c} = m^2\theta$$

where θ is the temperature excess above ambient air of the fin temperature at distance x from the root; P is the perimeter ; A_c the cross-sectional area of the fin; h is the heat

transfer coefficient and k is the thermal conductivity of the material.
Proceed to develop expressions for temperature distribution and total heat flow rate under steady state conditions for an infinitely long fin.

10. For a constant cross-sectional area fin, obtain the temperature distribution and total heat flow rate under steady state conditions when one end of the fin is attached to a body at high temperature and other end of the fin is insulated.
11. Fins are generally made of aluminium. Why?
12. Fins are preferred to be provided on a surface exposed to condensing steam. Give the reason.
13. With which medium, gas or liquid, the use of fin will be more effective?
14. When is the use of fins not justified?
15. State the difference between fin effectiveness and fin efficiency, and set up the relation between these performance parameters.
16. Determine the optimum shape of a fin having the minimum weight for a give that duty. Proceed to explain how the triangular shape is best for a fin.
17. Justify the preferences for the use of thin and closely spaced fins.
18. A fin 30 cm long and 10 mm diameter throughout is made of steel alloy of thermal conductivity 43 W/m-deg. The fin attached to a plane heated wall at 200°C temperature, extends into surroundings at 25°C and unit surface conductance of 120 W/m^2-deg. Work out the heat flow rate from the fin to the surroundings. Presume that the tip of the fin is insulated and thermal radiation effects are negligible.

    (Ans. 15.1 W)

19. Two long pieces of copper (k = 400 W/m-deg) wire 1.5 mm diameter are to be soldered together end to end. The surrounding air temperature is 30°C and the melting point of the solder is 230°C. If the convective heat transfer coefficient between the copper wire and air is 18W/m^2-deg, find the minimum energy input in watts to keep the soldered surface at 230°C.

    (Ans. 3.096 watts)

20. A rectangular fin measuring 4 cm length, 0.03 cm thickness and 30 cm width is made of cast iron having thermal conductivity k = 180 kJ/m-hr-deg. The base temperature of fin is 130°C, the fin is exposed to surrounding air at 30°C and unit surface conductance is 120 kJ/m^2-hr-deg. Make calculations for the rate of heat flow through the cross-sectional area at the root of the fin. Assume uniform temperature distribution at any cross-section perpendicular to the length of fin and neglect heat flow in the direction perpendicular to the fin profile area.

    (Ans. 392 kJ/hr)

21. Show that the heat flow rate per unit width from a straight fin of rectangular cross-section is governed by the relation

    $$Q = k\, m\, \delta\, \theta_0 \tanh ml$$

    where k is thermal conductivity of fin material, $m = \sqrt{2h/k\delta}$, h is the convective coefficient, δ is the fin thickness, θ_0 is the temperature excess at the base and l denotes the length of the fin. Neglect the heat flow through the tip of the fin.
    Fin of this type project from a plane wall at 10.2 cm intervals. Each fin is 1.27 cm thick and 18.2 cm long. Assume that the surface coefficient, both with and without fins, is 45.5 W/m^2-deg. Find the ratio of heat loss from the wall under same temperature conditions.

    (Ans. 2.37 : 1)

22. One end of the long rod is inserted into a furnace and the other end projects into the surrounding air at 20°C. Under steady state conditions, the temperature of the rod measured at two points, 100 mm apart, was found to be 120°C and 100°C respectively. If the diameter of the rod is 25 mm and thermal conductivity of the fin material is 120 W/m-deg, make calculations for the surface heat transfer coefficient.

    (Ans. 3.725 W/m^2-deg)

23. A heating unit in the form of a vertical tube, 120 cm high and 6 cm outside diameter, has its surface maintained at 80°C whilst the surrounding air is at 18°C. The tube is provided with 20 equally spaced longitudinal fins of rectangular section; the fins are 5 cm long and 0.03 cm thick. Calculate the amount of heat transferred from the finned wall to the surroundings. Take thermal conductivity of fin material k = 55.7 W/m-deg and the surface heat transfer coefficient h = 9.3 W/m^2-deg.

    (Ans. 1356 W)

24. A 12.5 mm diameter rod of iron (k = 45 W/m-deg) is heated to 260°C at its base and protrudes into air 35°C where surface conductance h = 8.5 W/m^2-deg. How long must be the rod in order that its end temperature may be computed, with less than 2.75°C error, by the use of infinite rod equations ? What error in heat dissipation is made, at this length, by the use of the infinite rod equation ?
25. A glass rod (k = 3.45 kJ/m-hr-deg) of 15 mm diameter and 65 cm length is heated to 150°C at its base and extends into air at 25°C. If the surface film coefficient is known to be

    26 kJ/m^2-hr-deg

    determine the temperature distribution and heat loss from the rod. Neglect heat flux through the tip of rod.

    (Ans. 3.4 kJ/hr)

26. Point out the various factors which need consideration for the optimum design of fins. Consider a straight rectangular fin of length l, thickness δ and width b. Let the profile area be the area taken in a plane parallel to the fin length and normal to the width

    $$A_p = bl$$

    For a fixed amount of material, i.e., A_p = constant, show that the fin would dissipate maximum amount of heat if its length and thickness are related by the expression

    $$\tan u = 2u \operatorname{sech}^2 u$$

    where $u = \left(\sqrt{\dfrac{2h}{k\delta}}\,\dfrac{A}{\delta}\right)$

27. Explain the different parameters which form a basis for assessing the utility of a fin in dissipating a given quantity of heat.
    A longitudinal copper fin (k = 14 kJ/m-hr-deg) 60 mm long and 5 mm in diameter is exposed to air stream at 20°C. The air moving past the fin has a uniform convection coefficient h = 80 kJ/m^2-hr-deg. If the fin has a base temperature of 150°C, determine the heat given up by the rod and its effectiveness in dissipating heat.

    (Ans. 9.2 kJ/m^2-hr, 0.94)

28. A thin rod of lenth l has its two ends connected to the two walls which are maintained at temperatures t_1 and t_2 respectively. The rod loses heat to the environment at t_a by convection. Setup an expression for temperature distribution in the rod and the total heat lost by it.
29. A 2 cm diameter bar is used as a support in a stack through which pass the fluid gases at 200°C. The bar is 1 meter long and thermal conductivity of its material is 150 W/m-deg. Determine the maximum temperature along the bar if the chimney walls where the bar is attached are maintained at 60°C. Assume the heat transfer coefficient as 8 W/m^2-deg.

    (Ans. 115°C)

30. Two thermal reservoirs maintained at constant and equal temperatures are connected by a metallic rod. Heat is generated by the passage of current in the rod which is not insulated. Develop an expression for the temperature difference between the mid-point of the rod and the reservoirs. State any assumptions made.
31. A bar of square cross-section 20 mm × 20 mm connects two metallic structures which are maintained at constant temperature of 200°C and 50°C respectively. The bar is 100 m long and is made of steel with thermal conductivity 0.06 kW/m-deg. The surroundings are at 20°C and the heat transfer coefficient between the bar and the surroundings is 0.01 kW/m^2-deg. Deduce an expression for the temperature distribution along the length of bar, and hence, calculate the heat flow from the bar to the surroundings.

    (Ans. 7.6 W)

32. A bar of length l, cross-sectional area A_c and perimeter P is made from material of thermal conductivity k. The ends of the bar are maintained at temperatures θ_1 and θ_2 above of the surroundings. The air moving past the bar has a uniform convective coefficient h. Show that :

    (i) heat emitted from the surface of the bar is

    $$\sqrt{PhkA_c}\,(\theta_1 + \theta_2)\frac{\cosh(ml - 1)}{\cosh ml}$$

    where $m = \sqrt{\dfrac{hP}{kA_c}}$

    (ii) If there is no heat flow through the end of the bar at θ_2 excess temperature, then

    $$\theta_1 = \theta_2 \cosh ml$$

    (iii) If both ends of bar are at the same temperature, then the temperature at the mid point of the bar is

$$\frac{\theta_2}{\cosh(ml/2)}$$

(iv) For unequal temperatures θ_1 and θ_2 the minimum temperatures occurs when

$$\frac{\theta_1}{\theta_2} = \frac{\cosh mx}{\cosh m(l-x)}$$

B. Fill in the blanks with appropriate word/words :

1. A .... is an extended surface used for the enhancement of heat dissipation.
2. Fins are provided to a heat exchanger surface in order to .... heat transfer by increasing the ....
3. A metallic rod protrudes from a wall maintained at 100°C temperture. The temperature at the tip will be minimum when the rod is made of a material with .... thermal conductivity.
4. Fins are usually made of a material which has .... thermal conductivity.
5. For a finned surface, it is considered appropriate that area of cross-section of the fin be .... along the length.
6. A .... fin is considered better as its lateral area is more near the base/root where temperature difference is high.
7. An increase in convective coefficient over a fin .... effectiveness.
8. The ratio of actual heat transferred by the fin to the heat which would be transferred if the entire fin area were at the base temperature is called ....
9. The thermometer well is treated as a fin of .... length with heat dissipation from the ....
10. In boiling and condensation, the convective coefficient is .... and that may produce .... in heat transfer rate.

Answers : 1. fin; 2. increase, surface area; 3. low; 4. high; 5. reduced; 6. tapered; 7. decreases; 8. fin efficiency; 9. finite, end; 10. high, decrease.

C. Multiple choice questions :

1. On a heat transfer surface, fins are provided to
   (a) increase temperature gradient so as to enhance heat transfer
   (b) increase turbulence in flow for enhancing heat transfer.
   (c) increase surface area to promote the rate of heat transfer
   (d) decrease the pressure drop of the fluid
2. In order to achieve maximum heat dissipation the fin should be designed in such a way that it has
   (a) maximum lateral surface at the root side of fin
   (b) maximum lateral surface towards the tip side of fin
   (c) maximum lateral surface near the centre of fin
   (d) minimum lateral surface near the centre of fin
3. Three fins of equal length and diameter but made of aluminium, brass and cast iron are heated to 200°C at one end. If the fins dissipate heat to the surrounding air at 25°C, the temperature at the free end will be least in case of
   (a) aluminium fin
   (b) brass fin
   (c) cast iron fin
   (d) each fin will have the same temperature at the free end
4. Two long rods A and B of the same diameter have thermal conductivities k and $4k$ and have one of their ends inserted into a furnace at 400 K. At a section 9.5 m away from the furnace, the temperature of rod B is 120°C. At what distance from the furnace end, the same temperature would be reached in the rod A?
   (a) 0.15 m (b) 0.25 m
   (c) 0.50 m (d) 0.75 m
5. The temperature distribution $(t - t_a)/(t_0 - t_a)$ for a fin with insulated tip is given by
   (a) $\exp(-mx)$
   (b) $\dfrac{\exp(mx)+\exp(-mx)}{2}$
   (c) $\dfrac{\cosh m(l-x)}{\cosh ml}$
   (d) $\cosh m(l-x) + \cosh ml$

   The symbols have their usual meanings.

6. A fin of length l protrudes from a surface held at temperature t_0; it being higher than the ambient temperature t_a. The heat dissipation from the free end of the fin is stated to be negligibly small. The temperature gradient at the fin tip $(dt/dx)_{x=l}$ can then be expressed as
   (a) 0 (b) $\dfrac{t_0 - t_1}{l}$
   (c) $h(t_0 - t_a)$ (d) $\dfrac{t_l - t_a}{t_0 - t_a}$
7. A fin protrudes from a surface which is held at a temperature higher than that of its environments. The heat transferred away from the fin is
   (a) heat escaping from the tip of the fin
   (b) heat conducted along the fin length
   (c) convective heat transfer from the fin surface
   (d) sum of heat conducted along the fin length and that convected from the surface
8. A straight fin of cross-sectional area A for all along its length and made of a material of thermal conductivity k serves to dissipate heat to the surroundings from a surface held at a constant temperature. What additional data is required to workout the rate of heat dissipation?
   (a) the root and tip temperatures
   (b) the temperature gradient at the root
   (c) the temperature gradient at the tip
   (d) the convective heat transfer coefficient and the fin perimeter
9. The heat dissipation from an infinitely long fin is given by
   (a) $\sqrt{PhkA_c}\,(t_0 - t_a)$
   (b) $hPl\,(t_0 - t_a)$
   (c) $\sqrt{PhkA_c}\,(t_0 - t_a)\tanh ml$
   (d) $\sqrt{PhkA_c}\,(t_0 - t_a)\dfrac{\tanh ml + (h/km)}{1+(h/km)\tanh ml}$
10. The parameter $m = \sqrt{hP/kA_c}$ has been stated to increase in a long fin. If all other parameters are maintained constant, then
    (a) temperature profile will remain the same
    (b) temperature drop along the length will be at a lower rate.
    (c) the temperature drop along the length will be steeper
    (d) the parameter m influences the heat flow only.
11. In a particular heat transferring situation, a cast iron fin has been replaced by a copper fin of identical configuration. If all other parameters are maintained constant, such a replacement will
    (a) increase the total heat flow
    (b) decrease the total heat flow
    (c) heat flow is influenced only by the base temperature and sectional area
    (d) will effect only the temperature distribution
12. For a finned surface, it is considered appropriate that area of cross-section be
    (a) maintained constant along the length
    (b) increased along the length
    (c) reduced along the length
    (d) it is considered better to vary the convection coefficient rather than area
13. Provision of fins on a given heat transfer surface will be more effective if there are .... number of .... fins.
    (a) fewer, thick (b) large, thick
    (c) large, thin (d) fewer, thin
14. An increase in fin effectiveness is caused by high value of
    1. convective coefficient
    2. thermal conductivity
    3. sectional area
    4. circumference

    Identify the correct statement
    (a) 1 and 3 (b) 2 and 3
    (c) 3 and 4 (d) 2 and 4
15. Fin efficiency is defined as the ratio of the heat transferred across the fin surface to the theoretical heat transfer across an equal area held at
    (a) temperature of fin end
    (b) constant temperature equal to that of base
    (c) average temperature of fin
    (d) none of the above

16. Consider a square section fin split longitudinally and used as two fins. This will result in
(a) increase in heat transfer
(b) decrease in heat transfer
(c) increase or decrease in heat transfer depending on material of fin
(d) heat flow remains constant

17. Mark the false statement regarding effectiveness of fin :
(a) fin effectiveness represents the ratio of heat dissipation with a fin to the heat transfer that would exist without a fin.
(b) fin effectiveness represents the ratio of heat transfer rate from the fin to the heat that would be dissipated if the entire fin surface area were maintained at the base temperature
(c) fin effectiveness is improved if the fin is made from a material of low thermal conductivity.
(d) a high value of film coefficient adversely affects the fin effectiveness
(e) fin effectiveness is improved by having thin but closely spaced fins.

18. Consider the following statements pertaining to heat transfer through fins :
1. Fins must be arranged at right angles to the direction of flow of the working fluid.
2. The temperature along the fin is variable and accordingly the heat transfer rate varies along the fin elements
3. Fins are equally effective irrespective whether they are on the hot side or cold side of the fluid
4. Fins are made of materials that have thermal conductivity higher than that of the wall.

Identity the correct statements
(a) 1 and 2 (b) 1 and 3
(c) 2 and 3 (d) 3 and 4

19. Usually fins are provided to increase the rate of heat transfer, but fin also acts as an insulation. Which one of the following non-dimensional number decides this factor?
(a) Eckert number (b) Biot number
(c) Fourier number (d) Peclet number

Answers :

1. (c)	2. (a)	3. (a)	4. (b)	5. (c)
6. (a)	7. (c)	8. (b)	9. (a)	10. (c)
11. (a)	12. (c)	13. (c)	14. (d)	15. (b)
16. (a)	17. (b, c)	18. (a)	19. (b)	

HINTS AND COMMENTS

1(c):

Fins are provided to a heat exchanger surface to augment the heat transfer by increasing the surface area exposed to the surroundings.

2(a):

Fins are so designed that lateral surface at the root side of the fin is maximum. This aspect results into higher heat dissipation.

3(a):

Thermal conductivity of aluminium is higher than that of brass and cast iron.

4(b):

Thermal conductivity of the material of a rod is directly proportional to the square of the length at which the same temperature is reached on the rod

That is :

$$\frac{k_A}{l_A^2} = \frac{k_B}{l_B^2}$$

$$\therefore \quad l_A = l_B \sqrt{\frac{k_A}{k_B}} = 0.5 \sqrt{\frac{k}{4k}} = 0.25 \text{ m}$$

8(b):

Refer to the following relation for the rate of heat dissipation from a fin

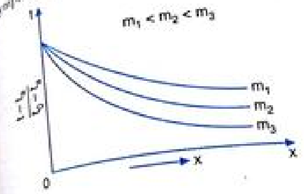
$$Q_{fin} = -kA \left(\frac{dt}{dx}\right)_{x=0}$$

10(c):

For an infinitely long fin,

$$\frac{t - t_a}{t_o - t_a} = e^{-mx}$$

The figure given below shows the dependence of dimensionless temperature $\frac{t - t_a}{t_o - t_a}$ along the length of fin for different values of parameter m ($m_1 < m_2 < m_3$). The plot indicates that the dimensionless temperature falls more with increase in factor m. With fin length extending to infinity, $x \to \infty$, all the curves approach $\frac{t - t_a}{t_o - t_a} = 0$ asymptotically.



11(a):

$$Q_{fin} = \sqrt{P\,h\,k\,A_c}\;(t_o - t_a)$$

With replacement of cast iron fin with a copper fin (k increases) and that will increase the total heat dissipation.

12(c):

A tapped fin is considered to be a better design as it has more lateral area near the base where the difference in temperature is high.

13(c):

$$\text{Fin effectiveness} = \sqrt{\frac{Pk}{hA_c}}$$

Increase in the ratio of perimeter P to be cross-sectional area A_c brings about improvement in the effectiveness of fin. That suggests for thin but closely spaced fin.

14(d):

Refer to the expression for fin effectiveness

$$\text{Fin effectiveness} = \sqrt{\frac{Pk}{hA_c}}$$

An increase in fin effectiveness is caused by high value of thermal conductivity and circumference.

18(a):

The statements made at serial number 3 and 4 are wrong. Fins are located on the side where the convective coefficient has a low value. Further, the fin and wall are made of the same material.

Chapter

6 Transient (Unsteady State Heat Conduction)

Learning objectives : A study of the subject matter included in this chapter will enable the readers to deal with the following aspects:

- *transient heat conduction; periodic and non-periodic temperature variation*
- *transient heat conduction in solids with infinite thermal conductivity; time constant and response of a thermocouple*
- *transient heat conduction in solids with finite conduction and convective resistance; Heisler charts*
- *transient heat conduction in infinite thick solids and heat conduction where the surface temperature variations are periodic in nature*

The term transient or unsteady state designates a phenomenon which is time dependent. Conduction of heat in unsteady state refers to the transient conditions wherein the heat flow and the temperature distribution at any point of the system vary continuously with time. The temperature and rate of heat conduction are then undoubtedly dependent both on the time and space coordinates, *i.e.*, $t = f(x, y, z, \tau)$. Transient conduction occurs in :

(*i*) heating or cooling of metal billets
(*ii*) cooling of IC engine cylinder
(*iii*) cooling and freezing of food
(*iv*) brick burning and vulcanisation of rubber
(*v*) starting and stopping of various heat exchange units in power installations

Change in temperature during unsteady state may follow a periodic or a non-periodic variation.

(*i*) **Periodic variation** : The temperature changes in repeated cycles and the conditions get repeated after some fixed time interval. Temperature variations in the cylinder of an IC engine are considered periodic. During each cycle, a definite variation of temperature occurs with respect to the crank angle and this change continues as long as the engine continues to operate. The profile of temperature variation with crank angle for one cycle is called *temperature wave* and the duration of each temperature wave is called *period*. Other notable examples of periodic variation are :

(*a*) variation of temperature of a building during a full day period of 24-hours
(*b*) temperature variation in surface of earth during a period of 24 hours
(*c*) heat processing of regenerators whose packings are alternately heated by fuel gases and cooled by air

(*ii*) **Non-periodic variation** : The temperature changes as some non-linear function of time. This variation is neither according to any definite pattern nor is in repeated cycles. Such a variation includes the processes where the medium is heated or cooled by exposing it to another medium of given thermal state. Examples are :

(*a*) heating of an ingot in a furnace
(*b*) cooling of bars, blanks and metal billets in steel works

Undoubtedly the time-dependent effects occur in many industrial heating, cooling and drying processes. An increase or decrease in temperature at any instant continues until steady temperature distribution is attained. For example during quenching of steel, there occurs a gradual decrease in the temperature of hot steel rod until the rod and the quenching medium attain the same temperature.

6.1. TRANSIENT CONDUCTION IN SOLIDS WITH INFINITE THERMAL CONDUCTIVITY K → ∞ (LUMPED PARAMETER ANALYSIS)

Solutions to the many of the transient heat flow problems are obtained by the lumped parameter analysis which presumes that the solid possesses infinitely large thermal conductivity. Internal conduction resistance is then so small that heat flow to or from the solid is controlled primarily by the convective resistance. Temperature gradients are negligible within the solid. Consequently the solid is spacewise isothermal $t \neq f(x, y, z)$ with temperature varying only with the time $t = f(\tau)$. Temperature, though changing with time, is never-the-less uniform throughout the solid at any time. Typical examples of this type of heat flow are :

(*i*) cooling of a small metal casting or a billet in quenching bath after its removal from the furnace

(*ii*) heating or cooling of a fine thermocouple wire due to change in ambient temperature

Fig. 6.1. shows a general lump of material comprising the system of interest. A body of surface area A, volume V, density ρ, thermal conductivity k, specific heat c and initial temperature t_i has been exposed to the surroundings maintained at temperature t_a. The transient response of the solid can be determined by relating its rate of change of internal energy with convective heat exchange at the surface. That is :

$$-\rho V c \frac{dt}{d\tau} = h A (t - t_a) \quad ...(6.1)$$

This expression can be rearranged and integrated; temperature t and the time τ are the two variables.

$$\int \frac{dt}{(t - t_a)} = -\frac{hA}{\rho V c} \int d\tau$$

$$\text{or } \log_e (t - t_a) = -\frac{hA}{\rho V c} \tau + C_1$$

The integration constant C_1 is evaluated from the initial conditions: $t = t_i$ at $\tau = 0$; t_i symbolizes the body temperature at the commencement of the cooling or heating process. Therefore $C_1 = \log_e (t_i - t_a)$ and hence

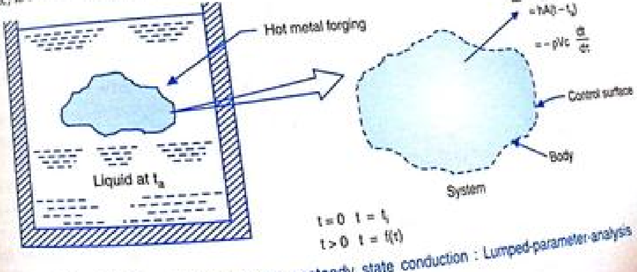


Fig. 6.1. General system for unsteady state conduction : Lumped-parameter-analysis

$$\log_e (t - t_a) = -\frac{hA}{\rho Vc}\tau + \log_e (t_i - t_a)$$

$$\text{or } \log_e\left(\frac{t - t_a}{t_i - t_a}\right) = -\frac{hA}{\rho Vc}\tau$$

$$\text{or } \quad \frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right) \qquad \text{...(6.2)}$$

Following points are worth noting:

1. The body temperature falls or rises exponentially with time and the rate depends on the parameter ($hA/\rho Vc$). Theoretically the body takes infinite time to approach the temperature of surroundings and thus attain the steady state conditions. However the difference between t and t_a becomes extremely small after a short time and beyond that period the body temperature becomes practically equal to the ambient temperature. The change in temperature of a body with respect to time is shown in Fig. 6.2 for both cases of heating (body exposed to gas temperature t_g higher than the body temperature t_i) and cooling (body placed in a liquid with temperature t_l lower than the body temperature).

2. The quantity ($\rho Vc/hA$) has the dimensions of time and is called the thermal time constant. Its value is indicative of the rate of response of a system to a sudden change in the environmental temperature; how fast a body will respond to a change in the environmental temperature.

3. The lumped-system analysis can be visualised as an electrical analogy in terms of a resistance-capacitance RC circuit and the process of heating/cooling as charging/discharging the capacitor (Fig. 6.3).

Here $C_{th} = \rho Vc$ is thermal capacitance and $R_{th} = \frac{1}{hA}$ is resistance to convective heat transfer. When the switch S is closed, the solid is charged to high temperature potential. On opening the switch, the thermal energy C_{th} gets dissipated through thermal resistance R_{th} and the temperature of the body decays with time.

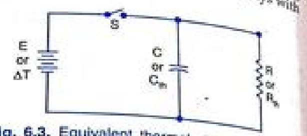


Fig. 6.3. Equivalent thermal circuit for lumped capacity method

4. The dimensionless argument of the exponential can be arranged in different forms such as :

$$\frac{hA}{\rho Vc}\tau = \left(\frac{hV}{kA}\right)\left(\frac{A^2k}{\rho V^2c}\tau\right)$$

$$= \left(\frac{hl}{k}\right)\left(\frac{\alpha\tau}{l^2}\right) \qquad \text{...(6.3)}$$

where $\alpha = (k/\rho c)$ is the thermal diffusivity of the solid, and l is a characteristic length equal to the ratio of the volume of the solid to its surface area.

5. For simple geometrical shapes, the values of characteristic length l are :

Sphere : $l = \dfrac{\frac{4}{3}\pi R^3}{4\pi R^2} = \dfrac{R}{3}$

Cylinder : $l = \dfrac{\pi R^2 L}{2\pi RL} = \dfrac{R}{2}$

Cube : $l = \dfrac{L^3}{6L^2} = \dfrac{L}{6}$

For a flat plate (thickness δ, breadth b, height h) the heat exchange occurs from both the sides; area exposed for heat transfer is $2bh$. The characteristics length then equals

$l = (\delta bh/2bh) = \delta/2$, *i.e.*, half the plate thickness.

• The non-dimensional factor ($\alpha\tau/l^2$) is called the **Fourier number**, F_0. It signifies the degree of penetration of heating or cooling effect through a solid. For instance, a large time τ would be required to obtain a significant temperature change for small values of (α/l^2).

• The non-dimensional factor (hl/k) is called the **Biot number**, B_i. It gives an indication of the ratio of internal (conduction) resistance to the surface (convection) resistance. A small value of B_i implies that the system has a small conduction (internal) resistance, *i.e.*, relatively small temperature gradient or the existence of a practically uniform temperature within the system. The convective resistance then predominates and the convective heat exchange controls the transient phenomenon. Essentially this has been the basic assumption in the lumped parameter analysis made above.

The application of the lumped parameter approach to bodies wih shapes similar to plates, cylinders or spheres does indicate that the temperatures in the body differ by less than 5% at any time for a value of B_i < 0.1. Small Biot numbers can hold with thin plates, and with large thermal conductivity k and small heat transfer coefficient h.

The lumped parameter solution for transient conduction can be conveniently stated as

$$\frac{t - t_a}{t_i - t_a} = \exp(-B_i F_0) \qquad \text{...(6.4)}$$

Figure 6.4 is a graphical portrayal of equation 6.4 for solids of different geometrical shapes: infinite cylinders and infinite square rods, and cubes and spheres.

Instantaneous and total heat flow rate : The instantaneous heat flow rate Q_i may be computed as follows :

$$Q_i = \rho Vc \frac{dt}{d\tau}$$

$$= \rho Vc \frac{d}{d\tau}\left[t_a + (t_i - t_a)\exp\left(-\frac{hA}{\rho Vc}\tau\right)\right]$$

$$= \rho Vc\left[(t_i - t_a)\left(-\frac{hA}{\rho Vc}\right)\exp\left(-\frac{hA}{\rho Vc}\tau\right)\right]$$

$$= -hA(t_i - t_a)\exp\left(-\frac{hA}{\rho Vc}\tau\right) \qquad \text{...(6.5)}$$

and the total heat flow (loss or gain) is obtained by integrating equation 6.5 over the time interval $\tau = 0$ to $\tau = \tau$.

$$Q_t = \int_0^\tau Q_i \, d\tau$$

$$= \int_0^\tau -hA(t_i - t_a)\exp\left(-\frac{hA}{\rho Vc}\tau\right)d\tau$$

$$= \left[-hA(t_i - t_a)\frac{\exp[-(hA/\rho Vc)\tau]}{-hA/\rho Vc}\right]_0^\tau$$

$$= \rho Vc(t_i - t_a)\left[\exp\left(-\frac{hA}{\rho Vc}\tau\right)\right]_0^\tau$$

$$= \rho Vc(t_i - t_a)\left[\exp\left(-\frac{hA}{\rho Vc}\tau\right) - 1\right] \qquad \text{...(6.6)}$$

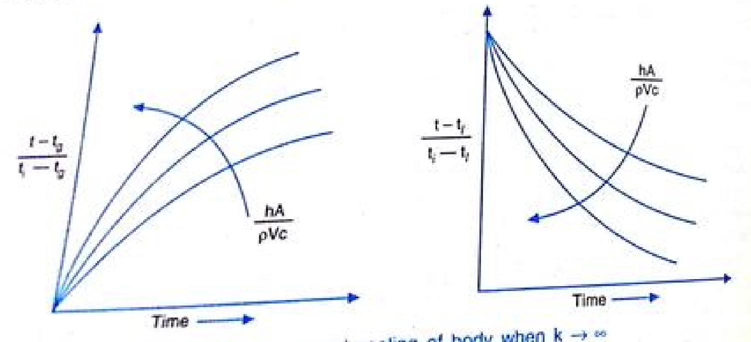


Fig. 6.2. Heating and cooling of body when k → ∞

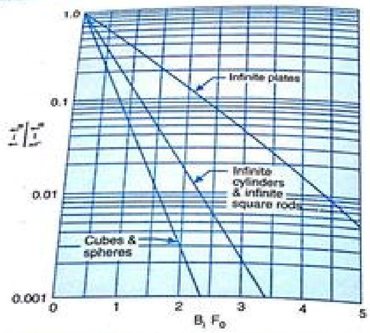


Fig. 6.4. Temperature variation in a lumped-parameter system

In terms of non-dimensional Biot and Fourier numbers, we may write,:

$$Q_i = -hA\,(t_i - t_a)\exp\left[-B_i F_0\right]$$

$$\text{and}\quad Q_t = \rho Vc\,(t_i - t_a)\left[\exp(-B_i F_0) - 1\right] \qquad ...(6.7)$$

EXAMPLE 6.1

An iron (k = 65 W/mK) billet measuring 20 × 15 × 80 cm is exposed to a convective flow resulting in convection coefficient h = 11.5 W/m²K. Determine the Biot number and the suitability of a lumped analysis to represent the cooling rate if the billet is initially hotter than the environment.

Solution: The characteristic linear dimension defined as the ratio of the volume of the billet to its surface area works out to be

$$l = \frac{20\times15\times80}{2(20\times15)+2(15\times80)+2(80\times20)}$$

$$= 3.87 \text{ cm}$$

$$\text{Biot number } B_i = \frac{hl}{k}$$

$$= \frac{11.5\times(3.87\times10^{-2})}{65} = 0.006846$$

Since the Biot number is less than 0.1, the internal temperature gradients are small. Consideration of the billet as a lumped system would be quite accurate; it will introduce an error of no more than 5%.

EXAMPLE 6.2

A 2 cm thick steel slab heated to 525°C is held in air stream having a mean temperature of 25°C. Estimate the time interval when the slab temperature would not depart from the mean value of 25°C by more than 0.5°C at any point in the slab. The steel plate has the following thermo-physical properties:

$$\rho = 7950 \text{ kg/m}^3$$

$$c_p = 455 \text{ J/kg-deg}$$

$$k = 46 \text{ W/m-deg}$$

h (heat transfer coefficient on plate surface)

$$= 36 \text{ W/m}^2\text{-deg}$$

Solution : For a flat plate (thickness δ, breadth *b*, height *h*), the heat exchange occurs from both the sides; the area exposed for heat transfer is $2bh$. The characteristic length then equals

$$l = \frac{\text{volume of plate}}{\text{surface area}}$$

$$= \frac{\delta b h}{2bh} = \frac{\delta}{2} = \frac{0.02}{2} = 0.01 \text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{36\times0.01}{46} = 0.0078 < 0.1$$

Since the Biot number is less than 0.1, the internal temperature gradients are small. Consideration of the steel slab as a lumped system would be quite accurate; it will produce an error of no more than 5 percent. The lumped-parameter solution for transient conduction states as,

$$\frac{t-t_a}{t_i-t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

$$\text{Now: } \frac{A}{V} = \frac{2bh}{\delta bh} = \frac{2}{\delta} = \frac{2}{0.02} = 100 \text{ m}^{-1}$$

$$\frac{h}{\rho c}\left(\frac{A}{V}\right) = \frac{36}{7950\times455}\times100$$

$$= 9.952\times10^{-4}$$

$$\therefore \frac{0.5}{525-25} = \exp\left[-9.952\times10^{-4}\,\tau\right]$$

$$\text{or } \exp\left[9.952\times10^{-4}\,\tau\right] = \frac{525-25}{0.5} = 1000$$

$$\text{or } 9.952\times10^{-4}\,\tau = \log_e(1000) = 6.9077$$

$$\therefore \quad \tau = \frac{6.9077}{9.952\times10^{-4}} = \mathbf{6941\ s}$$

EXAMPLE 6.3

An average convective heat transfer coefficient for flow of air over a sphere has been measured by observing the temperature-time history of a 12 mm diameter copper sphere (ρ = 9000 kg/m³ and c = 0.4 kJ/kg k) exposed to air at 30°C. The temperature of the sphere was measured by two thermocouples one located at the centre and other near the surface. Both the thermocouples registered, within accuracy of recording instruments the same temperature at a given constant. In one such tests, the initial temperature of the ball was 75°C and it decreased by 10°C in 1.2 minutes. Make calculations for the heat transfer coefficient.

Solution : The calculations will be made by assuming that the internal resistance is negligible ($B_i < 0.1$) and the lumped parameter analysis applies. Then

$$\frac{t-t_a}{t_i-t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

$$\text{or } \log_e\frac{t-t_a}{t_i-t_a} = -\frac{hA}{\rho Vc}\tau$$

$$\text{or} \quad h = \frac{\rho Vc}{A\tau}\log_e\frac{t_i-t_a}{t-t_a}$$

$$= \frac{\frac{4}{3}\pi r^3}{4\pi r^2}\times\frac{\rho c}{\tau}\log_e\frac{t_i-t_a}{t-t_a}$$

$$= \frac{r}{3}\frac{\rho c}{\tau}\log_e\frac{t_i-t_a}{t-t_a}$$

Substituting the given data, we obtain

$$h = \frac{0.006}{3}\times\frac{9000\times0.4\times1000}{1.2\times60}\times\log_e\frac{75-30}{65-30}$$

$$= 25.13 \text{ W/m}^2\text{-deg}$$

EXAMPLE 6.4

A cylindrical stainless steel (k = 25 W/mK) ingot, 10 cm in diameter and 25 cm long, passes through a heat treatment furnace which is 5 metre in length. The initial ingot temperature is 90°C, the furnace gas is at 1260°C and the combined radiant and convective surface coefficient is 100 W/m²K. Determine the maximum speed with which the ingot moves through the furnace if it must attain 830°C temperature.

Take thermal diffusivity

$$\alpha = 0.45\times10^{-5} \text{ m}^2\text{/s}$$

Solution: The characteristic linear dimension defined as the ratio of the volume of ingot to its surface is worked out to be

$$l = \frac{\pi r^2 L}{2\pi r(r+L)} = \frac{rL}{2(r+L)}$$

$$= \frac{5 \times 25}{2(5+25)} = 2.08 \text{ cm}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{100 \times 0.0208}{25} = 0.0832 < 0.1$$

Since the Biot number is less than 0.1, the internal temperature gradients are small. Consideration of the ignot as a lumped system would be quite accurate; it will introduce an error of no more than 5%.

The lumped-parameter solution for transient conduction is stated as

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

$$\text{Now: } \frac{A}{V} = \frac{2\pi r(r+L)}{\pi r^2 L} = \frac{2(r+L)}{rL}$$

$$= \frac{2(5+25)}{5 \times 25} = 0.48 \text{ cm}^{-1}$$

$$\frac{hA}{\rho V c} = \frac{k}{\rho c} \times \frac{h}{k} \times \frac{A}{V} = \alpha \times \frac{h}{k} \times \frac{A}{V}$$

$$= (0.45 \times 10^{-5}) \times \left(\frac{100}{25}\right) \times (0.48 \times 100)$$

$$= 8.64 \times 10^{-4}$$

$$\therefore \frac{830 - 1260}{90 - 1260} = \exp[-8.64 \times 10^{-4}\,\tau]$$

or $\exp[8.64 \times 10^{-4}\,\tau]$

$$= \frac{90 - 1260}{830 - 1260} = 2.7209$$

or $8.64 \times 10^{-4}\,\tau$

$$= \log_e (2.7209) = 1.00097$$

$$\therefore \quad \tau = \frac{1.00097}{8.64 \times 10^{-4}} = \mathbf{1158.53 \ s}$$

The reqired ingot velocity then becomes

$$V = \frac{\text{furnace length}}{\text{time}}$$

$$= \frac{0.25}{1158.53} = \mathbf{0.216 \times 10^{-3} \ m/s}$$

EXAMPLE 6.5

Glass spheres of 2 mm radius and at 500°C are to be cooled by exposing them to an air stream at 25°C. Make calculations for the maximum value of convection coefficient that is permissible, and the minimum time required for cooling to a temperature of 60°C. Assume the following property values :

density 2250 kg/m³; sp heat 850 J/kgK and conductivity 1.5 W/m-deg

Solution : Surface cracks are caused by temperature difference within a solid. No cracks will appear during cooling if the temperature gradients are small. For that, Biot number should be less than 0.1. In the limit $B_i = hl/k = 0.1$.

The characteristic linear dimension defined as the ratio of volume of sphere to its surface area is worked out to be

$$l = \frac{\text{volume}}{\text{surface area}} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$\therefore \frac{h \times \frac{r}{3}}{k} = 0.1 \ ; \ h = \frac{0.1 \times k \times 3}{r}$$

Maximum permissible value of convective coefficient

$$h = \frac{0.1 \times 1.5 \times 3}{0.002} = 225 \text{ W/m}^2\text{-deg}$$

(*b*) The temperature variation with time is given by

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

$$\frac{hA}{\rho V c}\tau = \frac{h}{\rho c}\left(\frac{A}{V}\right)\tau = \frac{h}{\rho c}\left(\frac{3}{r}\right)\tau$$

$$= \frac{225}{2250 \times 850}\left(\frac{3}{0.002}\right)\tau$$

$$= 0.17647\,\tau$$

$$\therefore \frac{60 - 25}{500 - 25} = \exp[-0.17647\,\tau]$$

$$\text{or } \exp[0.17647\,\tau] = \frac{500 - 25}{60 - 25} = 13.57$$

or $0.17647\,\tau = \log_e 13.57 = 2.608$

$\therefore$ Time required for cooling, τ

$$= \frac{2.608}{0.17647} = \mathbf{14.78 \ sec}$$

EXAMPLE 6.6

During heat treatment, cylindrical pieces of 25 mm diameter, 30 mm height and at 30°C are placed in a furnace at 750°C with convection coefficient 80 W/m²-deg. Calculate the time required to heat the pieces to 600°C. What will be the shortfall in temperature if the pieces are taken out from the furnace after 280 seconds ? Assume the following property values :

density 7850 kg/m³ ; specific heat 480 J/kgK; conductivity 40 W/m-deg.

Solution : For a cylindrical piece, the characteristic linear dimension is

$$l = \frac{\text{volume}}{\text{surface area}}$$

$$= \frac{\pi r^2 h}{2\pi r(r+h)} = \frac{rh}{2(r+h)}$$

$$= \frac{0.0125 \times 0.03}{2(0.0125 + 0.03)}$$

$$= 4.41 \times 10^{-3} \text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{80 \times 4.41 \times 10^{-3}}{40} = 0.00882 < 0.1$$

Since $B_i < 0.1$, the lumped parameter model can be adopted. Therefore

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

$$\frac{hA}{\rho V c} = \frac{h}{\rho c}\left(\frac{A}{V}\right) = \frac{h}{\rho c l}$$

$$= \frac{80}{7850 \times 480 \times 4.41 \times 10^{-3}}$$

$$= 0.004814$$

$$\therefore \frac{600 - 750}{30 - 750} = \exp[-0.004814\,\tau] \ ;$$

$$\frac{30 - 750}{600 - 750} = \exp[0.004814\,\tau]$$

$$\text{or } 0.004814\,\tau = \log_e\left(\frac{30 - 750}{600 - 750}\right)$$

$$= \log_e 4.8 = 1.5686$$

$\therefore$ Time required for heating, τ

$$= \frac{1.5686}{0.004814} = 326 \text{ sec}$$

Let t be the temperature attained when the pieces are taken out from the furnace after 280 seconds. Then

$$\frac{t - 750}{30 - 750} = \exp[-0.004814 \times 280]$$

$$= \exp[-1.348] = \frac{1}{3.849}$$

$$\text{or} \quad t = 750 + \frac{30 - 750}{3.849} = 563°\text{C}$$

$\therefore$ Shortfall in temperature

$$= 600 - 563 = 37°\text{C}$$

EXAMPLE 6.7

An egg with mean diameter of 4 cm and initially at 25°C is placed in a boiling water pan for 4 minutes and found to be boiled to the consumer's taste. For how long should a similar egg for same consumer be boiled when taken from a refrigerator at 5°C ? Use lumped parameter theory and presume the following properties for egg :

$k = 12$ W/m-deg

$h = 125$ W/m²-deg

$c = 2$ kJ/kg K

and $\rho = 1250$ kg/m³

Solution : Characteristic length l

$$= \frac{\text{volume}}{\text{surface area}} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$= \frac{0.02}{3} = 0.00666 \text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{125 \times 0.00666}{12} = 0.0694$$

Since Biot number is less than 0.1, the solution can be worked out by applying lumped-parameter theory which states that

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

$$\text{Now } \frac{hA}{\rho V c} = \frac{h}{\rho c}\left(\frac{A}{V}\right) = \frac{h}{\rho c} \times \frac{3}{r}$$

$$= \frac{125}{1250 \times (2 \times 1000)} \times \frac{3}{0.02}$$
$$= 0.0075$$

$$\therefore \quad \frac{hA}{\rho Vc}\tau = -0.0075 \times (4 \times 60) = 1.8$$

Let t_a = temperature of boiling water in the pan = 100°C. Then

$$\frac{t-100}{25-100} = \exp[-1.8] = \frac{1}{\exp[1.8]} = \frac{1}{6.05}$$

$$\therefore \quad t = \frac{25-100}{6.05} + 100 = \mathbf{87.6°C}$$

(*b*) Now, we have to find time τ for the temperature values:

$$t_i = 5°C;\ t_a = 100°C \text{ and } t = 87.6°C$$

$$\therefore \quad \frac{87.6-100}{5-100} = \exp[-0.0075\ \tau]$$

$$\text{or } \exp[0.0075\ \tau] = \frac{5-100}{87.6-100} = 7.66$$

$$\text{or } 0.0075\ \tau = \log_e 7.66 = 2.036$$

$$\therefore \quad \tau = \frac{2.036}{0.0075}$$
$$= 271.47 \text{ s} = \mathbf{4.524\ minutes}$$

EXAMPLE 6.8

During a heat treatment process, alloy steel spherical balls of 12 mm diameter are initially heated to 800°C in a furnace. Subsequently these are cooled to 100°C by keeping them immersed in an oil bath at 35°C with convection coefficient 20 W/m²-deg. Determine the time required for the cooling process. Proceed to calculate the value of convection coefficient if it is desired to complete the cooling process in a period of 10 minutes. The thermo-physical properties of steel balls are :

Density 7750 kg/m³ ; specific heat 520 J/kg K and conductivity 50 W/m-deg.

Solution : For a sphere, the characteristic linear dimension is

$$l = \frac{\text{volume}}{\text{surface area}} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$= \frac{0.006}{3} = 0.002 \text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{20 \times 0.002}{50} = 0.0008 < 0.1$$

Since $B_i < 0.1$, the lumped parameter model can be adopted. Therefore

$$\frac{t-t_a}{t_i-t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

$$\frac{hA}{\rho Vc}\tau = \frac{h}{\rho c}\left(\frac{A}{V}\right)\tau$$

$$= \frac{h}{\rho cl}\tau$$

$$= \frac{20}{7750 \times 520 \times 0.002} \times \tau$$
$$= 0.00248\ \tau$$

$$\therefore \quad \frac{100-35}{800-35} = \exp[-0.00248\ \tau]$$

$$\text{or } \exp[0.00248\ \tau] = \frac{800-35}{100-35} = 11.769$$

$$\text{or } 0.00248\ \tau = \log_e 11.769 = 2.465$$

$\therefore$ Time required for cooling

$$= \frac{2.465}{0.00248} = \mathbf{993.95\ sec}$$

(*b*) If cooling is to be achieved in 10 minutes, then

$$\frac{hA}{\rho Vc}\tau = \frac{h}{\rho c} \times \frac{1}{l} \times \tau$$

$$= \frac{h}{7750 \times 520} \times \frac{1}{0.002} \times (10 \times 60)$$
$$= 0.0744\ h$$

$$\therefore \quad \frac{100-35}{800-35} = \exp[-0.0744\ h];$$

$$\frac{800-35}{100-35} = \exp[0.0744\ h]$$

$$\text{or } 0.0744\ h = \log_e\left(\frac{800-35}{100-35}\right) = 2.465$$

$\therefore$ Convective coefficient h

$$= \frac{2.465}{0.0744} = \mathbf{33.13\ W/m^2\text{-}deg}$$

Biot number may be calculated to check the validity of lump-parameter model

Biot number B_i

$$= \frac{hl}{k} = \frac{33.13 \times 0.002}{50}$$
$$= 0.00132 < 0.1$$

EXAMPLE 6.9

A cylindrical ingot, 25 mm radius and 250 mm long, initially at 800°C is dipped in water at 25°C with convective heat transfer coefficient of 215 W/m²-deg and dipping continues till the temperature drops to 400°C. Subsequently the ingot is exposed to air at 25°C with convective coefficient of 27.5 W/m²-deg till it attains a temperature of 80°C. If the ingot material has thermal conductivity 65 W/m-deg, specific heat 250 J/kg K and density 820 kg/m³, make calculations for the total time required for the ingot to reach the temperature from 800°C to 80°C.

Solution : For a cylindrical ingot the characteristic length is,

$$l = \frac{\text{volume}}{\text{surface area}} = \frac{\pi r^2 L}{2\pi r L}$$

$$= \frac{r}{2} = \frac{0.025}{2} = 0.0125 \text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{215 \times 0.0125}{65}$$
$$= 0.04135 < 0.1$$

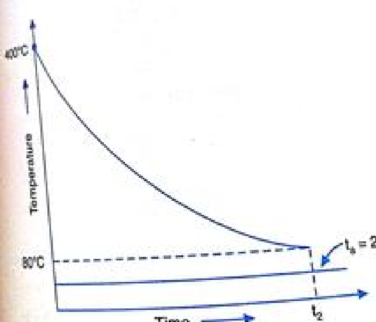


Fig. 6.5.

As Biot number is less than 0.1, the internal thermal resistance can be neglected and the lump theory can be adopted.

(i) The temperature variation with respect to time when cooled in water (Fig. 6.5) is given by

$$\frac{t-t_w}{t_i-t_w} = \exp\left(-\frac{h_w A}{\rho V c}\tau_1\right)$$

$$\text{where } \frac{h_w A}{\rho V c} = \frac{h_w}{\rho c}\left(\frac{A}{V}\right) = \frac{h_w}{\rho c l}$$

$$= \frac{2.5}{820 \times 250 \times 0.0125}$$
$$= 0.0839$$

$$\therefore \quad \frac{400-25}{800-25} = \exp[-0.0839\ \tau_1]$$

$$= \frac{1}{\exp[0.0839\ \tau_1]}$$

$$\text{or } \exp[0.0839\ \tau_1] = \frac{800-25}{400-25}$$

$$= \frac{775}{375} = 2.067$$

$$\text{or } 0.0839\ \tau_1 = \log_e 2.067 = 0.726$$

$$\therefore \quad \tau_1 = \frac{0.726}{0.0839} = \mathbf{8.65\ sec}$$

(ii) The temperature variation with respect to time when cooled in air (Fig. 6.6) is given by

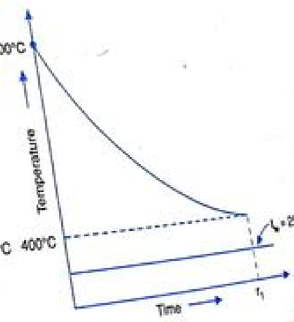


Fig. 6.6.

$$\frac{t-t_a}{t_i-t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

where $\frac{h_a A}{\rho Vc} = \frac{h_a}{\rho c}\left(\frac{A}{V}\right) = \frac{h_a}{\rho c l}$

$$= \frac{27.5}{820 \times 250 \times 0.0125}$$
$$= 0.01073$$

$$\therefore \frac{80-25}{400-25} = \exp[-0.01073\,\tau_2]$$
$$= \frac{1}{\exp[0.01073\,\tau_2]}$$

or $\exp[0.01073\,\tau_2]$
$$= \frac{400-25}{80-25} = \frac{375}{55} = 6.818$$

or $0.01073\,\tau_2 = \log_e 6.818 = 1.9196$

$$\therefore \quad \tau_2 = \frac{1.9196}{0.01073} = 178.89 \text{ sec}$$

Thus, the total time required for cooling is

$$\tau = \tau_1 + \tau_2$$
$$= 8.65 + 179.89 = 187.54 \text{ sec}$$

EXAMPLE 6.10

A die cast component has a mass of 1.25 kg and density 7250 kg/m³ with surface area 0.08 m². The component comes out of machine at 350°C and is exposed to air at 25°C with convective heat transfer coefficient 60 W/m²-deg. Make calculations for the followings :

(a) temperature of the component after 5 minutes,

(b) time constant,

(c) value of convective heat transfer coefficient upto which lumped parameter analysis is valid, and

(d) value of volume/area ratio upto which the lumped parameter model can be used.

For the material of the component take, thermal conductivity = 100 W/m-deg and specific heat = 400 J/kg K.

Solution : The characteristic linear dimension defined as the ratio of the volume of component to its surface area is worked out to be

$$l = \frac{\text{volume}}{\text{surface area}}$$
$$= \left(\frac{\text{mass}}{\text{density}}\right) \times \frac{1}{\text{surface area}}$$
$$= \frac{1.25}{7250} \times \frac{1}{0.08} = 0.00216 \text{ m}$$

Biot number B_i
$$= \frac{hl}{k} = \frac{60 \times 0.00216}{100}$$
$$= 0.001296 < 0.1$$

Since $B_i < 0.1$, the lumped parameter model can be adopted. Therefore,

$$\frac{t-t_a}{t_i-t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

Now, $\frac{hA}{\rho Vc}\tau = \frac{h}{\rho c}\left(\frac{A}{V}\right)\tau$

$$= \frac{60}{7250 \times 400} \times \frac{1}{0.00216} \times (5 \times 60)$$
$$= 2.8735$$

$$\therefore \frac{t-25}{350-25} = \exp[-2.8735]$$
$$= \frac{1}{\exp[2.8735]} = \frac{1}{17.7}$$
$$t = 25 + \frac{350-25}{17.7} = 43.36°\text{C}$$

(b) The parameter $\frac{\rho Vc}{hA}$ has units of time and is called time constant.

$\therefore$ Time constant τ^*

$$= \frac{\rho Vc}{hA} = \frac{\rho c}{h}\left(\frac{V}{A}\right) = \frac{\rho c}{h} l$$
$$= \frac{7250 \times 400}{60} \times 0.00216$$
$$= 104.4 \text{ sec}$$

(c) For the validity of lumped parameter analysis to hold good, Biot number should be less than 0.1. In the limit

$$B_i = \frac{hl}{k} = 0.1$$

$$\therefore \quad h = \frac{0.1k}{l} = \frac{0.1 \times 100}{0.00216}$$
$$= 4630 \text{ W/m}^2\text{-deg}$$

The lumped model can be adopted for any value of h less than 4630 W/m²-deg.

(d) $B_i = \frac{hl}{k} = 0.1$

$$\therefore l\left(=\frac{V}{A}\right) = \frac{0.1k}{h} = \frac{0.1 \times 100}{60} = 0.167 \text{ m}$$

The lumped model can be adopted for any value of volume/area ratio less than 0.167 m.

EXAMPLE 6.11

A 12 mm diameter mild steel sphere (k = 42.5 W/mK) is exposed to cooling airflow at 27°C resulting in the convective coefficient h = 114 W/m²K. Determine (i) time required to cool the sphere from 540°C to 95°C (ii) instantaneous heat transfer rate 2 minutes after the start of cooling and (iii) total energy transferred from the sphere during the first 2 minutes. The relevant properties of mild steel are :

density ρ = 7850 kg/m³

specific heat c = 475 J/kg K and

thermal diffusivity α = 0.043 m²/hr

Solution : The characterisitc linear dimension defined as the ratio of the volume of sphere to its surface area works out to be :

$$l = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{6}{3} = 2 \text{ mm}$$

Biot number, B_i
$$= \frac{hl}{k} = \frac{114 \times (2 \times 10^3)}{42.5}$$
$$= 5.36 \times 10^{-3}$$

Since the Biot number is less than 0.1, the internal temperature gradients are small. Consideration of the sphere as a lumped system would be quite accurate : it will introduce an error of no more than 5%. The lumped-parameter solution for transient conduction is stated as,

$$\frac{t-t_a}{t_i-t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

Now, $\frac{A}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r} = \frac{3}{6} = 0.5 \text{ mm}^{-1}$

$$\frac{hA}{\rho Vc} = \frac{k}{\rho c} \times \frac{h}{k} \times \frac{A}{V} = \alpha \times \frac{h}{k} \times \frac{A}{V}$$
$$= \left(\frac{0.043}{3600}\right) \times \left(\frac{114}{42.5}\right) \times (0.5 \times 1000)$$
$$= 1.602 \times 10^{-2}$$

$$\therefore \frac{95-27}{540-27} = \exp[-1.602 \times 10^{-2}\,\tau]$$

or $\exp[1.602 \times 10^{-2}\,\tau] = \frac{540-27}{95-27} = 7.544$

or $1.602 \times 10^{-2}\,\tau = \log_e 7.544 = 2.021$

$$\therefore \quad \tau = \frac{2.021}{1.602 \times 10^{-2}}$$
$$= 126.15 \text{ s} = 2.10 \text{ min.}$$

(ii) The instantaneous heat transfer rate 2 minutes (120 seconds) after the start of cooling may be computed from the relation.

$$Q = -hA\,(t_i - t_a)\exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

Now, $\exp\left(-\frac{hA}{\rho Vc}\tau\right)$
$$= \exp[-1.602 \times 10^{-2} \times 120]$$
$$= 0.1462$$

$$\therefore \quad Q = -114 \times (4\pi \times 0.006^2) \times (540-27) \times 0.1462$$
$$= 3.867 \text{ W}$$

(iii) The total energy transferred is

$$Q_t = \rho Vc\,(t_i - t_a)\left[\exp\left(-\frac{hA}{\rho Vc}\tau\right) - 1\right]$$
$$= 7850 \times \left(\frac{4}{3}\pi \times 0.006^3\right) \times 475 \times (540-27)\,(0.1462 - 1)$$
$$= 1467.93 \text{ W}$$

EXAMPLE 6.12.

The steel ball bearings of 40 mm diameter and initially at uniform temperature of 600°C are quenched in an oil bath maintained at 50°C temperature. The heat transfer coefficient between the ball bearings and oil is 325 W/m²K and the thermodynamic properties of bearings can be taken as :

Thermal conductivity $k = 45$ W/mK and thermal diffusivity $\alpha = 1.25 \times 10^{-5}$ m²/s

Determine :

(a) the time duration for which bearings must remain in oil to attain 225°C temperature,

(b) the amount of heat removed from bearings during this time, and

(c) the instantaneous heat transfer rate from the bearings when they are first immersed in oil and when they reach 225°C.

Solution : The characteristic linear dimension defined as the ratio of volume of bearing to its surface area is worked out to be

$$l = \frac{\text{volume}}{\text{surface area}}$$

$$= \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$= \frac{0.04/2}{3} = 6.67 \times 10^{-3}\ \text{m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{325 \times 6.67 \times 10^{-3}}{45}$$

$$= 0.0482$$

Since $B_i < 0.1$, internal resistance of the bearings is negligible and the lumped parameter model can be adopted. Therefore

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

$$\text{Now,}\quad \frac{hA}{\rho Vc} = \frac{h}{\rho c}\left(\frac{A}{V}\right)$$

$$= \frac{\alpha}{k}\, h \left(\frac{3}{r}\right) \qquad \left(\alpha = \frac{k}{\rho c}\right)$$

$$= \frac{1.25 \times 10^{-5}}{45} \times 325 \times \frac{3}{0.04/2}$$

$$= 0.01354$$

$$\therefore \quad \frac{225 - 50}{600 - 50} = \exp(-0.01354\,\tau)$$

$$\text{or } \exp(0.01354\,\tau) = \frac{600 - 50}{225 - 50} = 3.143$$

$$\text{or } 0.01354\,\tau = \log_e 3.143 = 1.1452$$

That gives :

$$\tau = \frac{1.1452}{0.01354} = 84.58\ \text{s}$$

(*b*) The total energy transferred is

$$Q_t = \rho Vc\,(t_i - t_a)\left[\exp\left(-\frac{hA}{\rho Vc}\tau\right) - 1\right]$$

$$\text{Now,}\quad \rho c = \frac{k}{\alpha} = \frac{45}{1.25 \times 10^{-5}}$$

$$= 36 \times 10^5 \quad \text{and}$$

$$\exp\left(-\frac{hA}{\rho Vc}\tau\right) = \exp(-0.01354 \times 84.58)$$

$$= 0.3182$$

$$\therefore \quad Q_t = 36 \times 10^5 \times \frac{4}{3}\pi\left(\frac{0.04}{2}\right)^3 \times (600 - 50)\,(0.3182 - 1)$$

$$= 45281\ \text{J/s} \quad \text{or} \quad 45.28\ \text{kW}$$

(*c*) Instantaneous heat flow rate is computed from the expression,

$$Q_i = -hA\,(t_i - t_a)\exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

$$= -325 \times 4\pi\left(\frac{0.04}{2}\right)^2 (600 - 50)\,(0.3182)$$

$$= 285.76\ \text{J/s} = 285.76\ \text{W}$$

6.2. TIME CONSTANT AND RESPONSE OF A THERMOCOUPLE

When two dissimilar metals are joined together at two points to form a closed loop and a temperature difference exists between the junctions, an electrical potential is set up between the junctions. Such an arrangement

is known as thermocouple and is frequently used for the measurement of temperature. Measurement of temperature by a thermocouple is an important application of the lumped parameter analysis.

The *response of a thermocouple* is defined as the time required for the thermocouple to reach the source temperature when it is exposed to it.

Referring to the lumped-parameter solution for transient heat conduction;

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right) \qquad \text{...(6.8)}$$

It is evident that larger the parameter $hA/\rho Vc$, the faster the exponential term will reach zero or more rapid will be the response of the thermocouple. A larger value of $hA/\rho Vc$ can be obtained either by increasing the value of convective coefficient, or by decreasing the wire diameter, density and specific heat.

The *sensitivity* of the thermocouple is defined as the time required by the thermocouple to reach 63.2% of its steady state value. When such a condition is attained, equation 6.2 can be written as

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho Vc}\tau\right)$$

$$= 1 - 0.632 = 0.368 = \exp\,[-1]$$

$$\therefore \frac{hA}{\rho Vc}\tau = 1 \quad \text{or} \quad \tau = \frac{\rho Vc}{hA}$$

The parameter $\rho Vc/hA$ has units of time and is called *time constant* of the system and is denoted by τ^*. Thus

$$\tau^* = \frac{\rho Vc}{hA} = \frac{k}{\alpha h}\frac{V}{A}$$

Using time constant, the temperature distribution in the solids can be expressed as

$$\frac{\theta}{\theta_i} = \frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{\tau}{\tau^*}\right)$$

The time constant is indicative of the speed of response, *i.e.*, how fast the thermocouple tends to reach the steady state value. A large time constant corresponds to a slow system response, and a small constant represents a fast response. A low value of time constant can be achieved for a thermocouple by

(i) decreasing the wire diameter

(ii) using light metals of low density and low specific heat

(iii) increasing the heat transfer coefficient

Depending upon the type of fluid used, the response times for different sizes and materials of thermocouple wires usually lie between 0.04 to 2.5 seconds.

EXAMPLE 6.13.

A mercury thermometer with bulb idealized as a sphere of 1 mm radius is used for measuring the temperature of fluid whose temperature is varying at a fast rate. For mercury;

$$k = 10\ \text{W/mK}; \quad \alpha = 0.00005\ \text{m}^2/\text{s}$$

$$\text{and} \quad h = 10\ \text{W/m}^2\text{K}$$

If the time for the temperature change of the fluid is 3 s, specify whether or not the thermometer is able to read the temperature faithfully ? If not, what should be the radius of a thermocouple to read the temperature of the fluid ?

For the thermocouple material,

$$k = 100\ \text{W/mK}, \quad \alpha = 0.0012\ \text{m}^2/\text{s}$$

$$\text{and} \quad h = 8\ \text{Wm}^2\text{K}$$

Solution : The characteristic linear dimension defined as the ratio of the volume of bulb to its surface area works out as

$$l = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{0.001}{3}\ \text{m}$$

The time constant for the thermometer is

$$\tau = \frac{k}{\alpha h}l$$

$$= \left(\frac{10}{0.00005 \times 10}\right) \times \frac{0.001}{3} = 6.67\ \text{s}$$

Since the time constant (6.67 s) is more than the time for the temperature change of the fluid (3 s), the thermometer will not give a faithful record of the time varying temperature of the fluid.

(b) The diameter of the thermocouple with the given properties can be worked out from the correlation for time constant. That is

$$\tau = \frac{k}{\alpha h} l = \frac{k}{\alpha h}\left(\frac{r}{3}\right)$$

$\therefore$ Radius of the thermocouple bead,

$$r = \frac{3\alpha h\tau}{k} = \frac{3 \times 0.0012 \times 8 \times 3}{100}$$

$$= 8.64 \times 10^{-4} \text{ m} = 0.864 \text{ mm}$$

EXAMPLE 6.14

A thermocouple junction of spherical form is to be used to measure the temperature of a gas stream. The junction is initially at 20°C and is placed in gas stream which is at 200°C. Make calculations for (a) junction diameter needed for the thermocouple to have thermal time constant of one second and (b) time required for the thermocouple to reach 197°C temperature. Assume the thermo-physical properties as given below :

$k = 20$ W/m-deg ; $h = 350$ W/m²-deg

$c = 0.4$ kJ/kg K and $\rho = 8000$ kg/m³

Solution : The time constant for a thermocouple is given by

$$\tau = \frac{\rho V c}{hA} = \frac{\rho(4/3\pi r^3)c}{h(4\pi r^2)} = \frac{\rho r c}{3h}$$

$$r = \frac{3h\tau}{\rho c} = \frac{3 \times 350 \times 1}{8000 \times 400}$$

$$= 0.000328 \text{ m}$$

$$= 0.328 \text{ m}$$

$\therefore$ Junction diameter of the thermocouple,

$$D = 2 \times 0.328 = 0.656 \text{ mm}$$

(b) The temperature variation with respect to time is

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{\rho V c}{hA}\tau\right)$$

$$\frac{\rho V c}{hA} = \frac{\rho r c}{3h}$$

$$= \frac{8000 \times 0.000328 \times 400}{3 \times 350} = 1$$

$$\therefore \frac{197 - 200}{20 - 200} = \exp[-\tau]$$

or $\exp[\tau] = \frac{200 - 20}{200 - 197} = \frac{180}{3} = 60$

$\therefore \quad \tau = \log_e 60 = 4.094$ sec.

EXAMPLE 6.15

The following data pertains to the junction of a thermocouple wire used to measure the temperature of a gas stream

Density $\rho = 8500$ kg/m³ ; specific heat C = 325 J/kgK; thermal conductivity k = 40 W/mK and the heat transfer coefficient between the junction and gas h = 215 W/m²K.

If thermocouple junction can be approximated as 1 mm diameter sphere, determine how long it will take for the thermocouple to read 99 percent of the initial temperature difference?

Solution : The characteristic linear dimension defined as the ratio of volume of sphere to its surface area is worked out to be

$$l = \frac{\text{volume}}{\text{surface area}}$$

$$= \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$= \frac{0.001/2}{3} = 1.667 \times 10^{-4} \text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{215 \times (1.667 \times 10^{-4})}{40}$$

$$= 0.000896$$

Since $B_i < 0.1$, the lumped parameter model can be adopted.

Therefore,

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

$$\text{Now, } \frac{hA}{\rho V c} = \frac{h}{\rho c}\left(\frac{A}{V}\right)\tau$$

$$= \frac{h}{\rho c}\left(\frac{3}{r}\right)$$

$$= \frac{215}{8500 \times 325}\left(\frac{3}{0.001/2}\right) = 0.467$$

$\therefore \quad 0.01 = \exp(-0.046\,\tau)$

or $\exp(0.467\,\tau) = \frac{1}{0.01} = 100$

or $0.467\,\tau = \log_e 100 = 4.605$

That gives

$$\tau = \frac{4.605}{0.467} = 9.86 \text{ s}$$

The solution indicates that the operator must wait at least 10 s for the temperature of thermocouple function to read within 1 percent of the initial temperature difference between the junction and gas temperature.

EXAMPLE 6.16

A thermocouple junction in the form of 4 mm radius sphere is to be used to measure the temperature of a gas stream. The junction is initially at 35°C and is placed in a gas stream which is at 300°C. The thermocouple is removed from the hot gas stream after 10 seconds and kept in still air at 25°C with convective coefficient 10 W/m²-deg. Make calculations for (a) time constant of the thermocouple and (b) temperature attained by the thermocouple junction 20 seconds after removal from the hot gas stream. Assume the thermo-physical properties as given below :

$h = 37.5$ W/m²-deg ; $\rho = 7500$ kg/m³

and $c = 400$ J/kg-deg

Solution: The time constant for a thermocouple is given by

$$\tau = \frac{\rho V c}{hA} = \frac{\rho(4/3\pi r^3)c}{h(4\pi r^2)} = \frac{\rho r c}{3h}$$

$$= \frac{7500 \times 0.004 \times 400}{3 \times 37.5} = 106.67 \text{ sec}$$

(b) The temperature variation with respect to time during heating (when placed in gas stream) is :

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

Given :

$t_a = 300$°C ;

$t_i = 35$°C and $\tau = 10$ seconds

$$\therefore \frac{t - 300}{35 - 300} = \exp\left[-\frac{10}{106.67}\right] \quad \left(\because \frac{\rho V c}{hA} = 106.67 \text{ sec}\right)$$

$$= \exp[-0.0937]$$

$$= \frac{1}{\exp[0.0937]} = \frac{1}{1.0982}$$

$$\therefore \quad t = 300 + \frac{35 - 300}{1.0982} = 58.7°\text{C}$$

The temperature variation with respect to time during cooling (when exposed to air) is :

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

Given :

$t_i = 58.7$°C ; $t_a = 25$°C

$h = 10$ W/m²-deg and $\tau = 20$ sec

$$\frac{hA}{\rho V c} = \frac{\rho r c}{3h}$$

$$= \frac{7500 \times 0.004 \times 400}{3 \times 10} = 400$$

$$\therefore \frac{t - 25}{58.7 - 25} = \exp\left[-\frac{20}{400}\right]$$

$$= \exp\left[-\frac{1}{0.05}\right] = \frac{1}{1.05127}$$

$$\therefore \quad t = 25 + \frac{58.7 - 25}{1.05127} = 57.056°\text{C}$$

EXAMPLE 6.17

The tip of a copper constanton thermocouple probe may be idealised as a sphere 2.5 mm in diameter. The probe is to measure the temperature of air at atmospheric pressure flowing with a velocity of 3 m/s. Initially the probe and air are at a temperature of 20°C. The air temperature suddenly changes to and is maintained at 225°C. Workout the time required for the thermocouple to indicate a temperature of 200°C. Also determine the thermal time constant and temperature indicated by the thermocouple at that instant.

(b) Comment upon the suitability of this thermocouple to measure unsteady temperature

(temperature changing with time) of a fluid. The temperature variations in the fluid has a time period of about 3.5 seconds.

The thermocouple bead properties are :

density ρ = 8960 kg/m^3

specific heat c = 0.366 kJ/kg-deg

thermal conductivity k = 100 kJ/hr-m-deg

and convective coefficient h = 520 kJ/m^2-hr-deg

Solution : The characteristic length defined as the ratio of the volume of probe tip up to its surface area works out to be :

$$l = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$= \frac{1.25}{3} = 0.4167 \text{ mm}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{520 \times (0.4167 \times 10^{-3})}{100}$$

$$= 2.167 \times 10^{-3}$$

Since the Biot number is less than 0.1, the internal temperature gradients are small. Consideration of the thermocouple tip as a lumped system will be quite appropriate. The lumped analysis will provide a good representation of the transient temperature; it will introduce an error of no more than 5%.

The lumped parameter solution for transient conduction is stated as :

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{hA}{\rho V c}\tau\right)$$

$$\text{Now, } \frac{A}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r}$$

$$= \frac{3}{0.00125} = 2400 \text{ m}^{-1}$$

$$\frac{h}{\rho c}\left(\frac{A}{V}\right) = \frac{(520 \times 1000/3600)}{8960 \times (0.366 \times 1000)} \times 2400$$

$$= 0.1057 \text{ s}^{-1}$$

$$\therefore \frac{200 - 225}{20 - 225} = \exp[-0.01057\ \tau]$$

$$\text{or } \exp[0.01057\ \tau] = \frac{20 - 225}{200 - 225} = 8.2$$

$$\text{or } 0.1057\ \tau = \log_e 8.2 = 2.104$$

$$\therefore \tau = \frac{2.104}{0.1057} = 19.91 \text{ seconds}$$

Thus the thermo-couple requires 19.91 seconds to indicate a temperature of 200°C. The actual time requirement will, however, be greater because of radiation from the probe and conduction along the thermocouple lead wires.

The time constant is defined as the time required to yield a value of unity for the exponent term in the transient relation.

$$\frac{h}{\rho c}\left(\frac{A}{V}\right)\tau = 1$$

$$0.1057\ \tau = 1 ;$$

$$\tau = \frac{1}{0.1057} = 9.46 \text{ seconds}$$

At 9.46 seconds the thermocouple would indicate a temperature t such that :

$$\frac{t - t_a}{t_i - t_a} = e^{-1} \quad \text{or} \quad \frac{t - 225}{20 - 225} = e^{-1}$$

$$t = 225 + (20 - 225)\, e^{-1} = 300.4°\text{C}$$

At thermal time constant, the thermocouple attains $(1 - e^{-1})$ or 63.2% of sudden temperature change.

(*b*) As the characteristic time (9.46 seconds) is more than the time of temperature variations (3.5 seconds), the thermocouple will not give a faithful record of the time varying temperature of the fluid.

6.3. TRANSIENT HEAT CONDUCTION IN SOLIDS WITH FINITE CONDUCTION AND CONVECTIVE RESISTANCE ($0 < B_i < 100$)

Consider the heating or cooling of a plane wall of thickness $l = 2\delta$ and extending to infinity in the y and z directions. Initially the wall is at uniform temperature t_i, and suddenly both surfaces ($x = \pm\,\delta$) are exposed to and maintained at the ambient temperature t_a (Fig. 6.7). The controlling differential

equation for the transient heat conduction is :

$$\frac{d^2 t}{dx^2} = \frac{1}{\alpha}\frac{dt}{d\tau} \qquad ...(6.9)$$

and the appropriate boundary conditions are :

(*i*) $t = t_i$ at $\tau = 0$; initially the wall is at uniform temperature t_i

(*ii*) $dt/dx = 0$ at $x = 0$; symmetrical nature of the temperature profile within the plane wall; symmetry in conduction occurs at the mid plane ($x = 0$) of the wall.

(*iii*) $kA\ (dt/dx) = hA\ (t - t_a)$ at $x = \pm\,\delta$. This condition stems from the fact that conduction heat transfer equals the convective heat transfer at the wall surface.

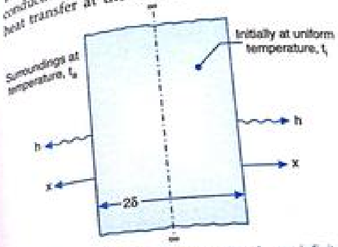


Fig. 6.7. Transient heat conduction in an infinite plane wall

The solution of the controlling differential equation in conjunction with initial boundary conditions would give an expression for temperature variation both with time and position. The solutions obtained after rigorous mathematical analysis indicate that

$$\frac{t - t_a}{t_i - t_a} = f\left(\frac{x}{l}, \frac{hl}{k}, \frac{\alpha\tau}{l^2}\right) \qquad ...(6.10)$$

Obviously when conduction resistance is not negligible, the temperature history becomes a function of Biot number hl/k, Fourier number $\alpha\tau/l^2$ and the dimensionless parameter x/l which indicates the location of point within the plate where temperature is to be obtained. In case of cylinders and spheres x/l is replaced by r/R.

Graphical charts have been prepared for the equation 6.10 in a variety of forms. The Heisler charts given in Figs. 6.8 to 6.10 depict the dimensionless temperature $(t_0 - t_a)/(t_i - t_a)$ versus F_0 for various values of $1/B_i$ for solids of different geometrical shapes such as plates, cylinders and spheres. These charts give the temperature history of the solid at its mid plane, $x = 0$. Temperatures at other locations are worked out by multiplying the mid-plane temperature by the correction factors read from charts given in Fig. 6.11 to 6.13. Use is made of the following relationship.

$$\frac{t - t_a}{t_i - t_a} = \left(\frac{t_0 - t_a}{t_i - t_a}\right) \times \left(\frac{t - t_a}{t_0 - t_a}\right)$$

The values of Biot number and Fourier number, as used in the Heisler charts, are evaluated on the basis of a characteristic parameter s which is the semi-thickness in case of plates and the surface radius in case of cylinders and spheres.

The Heisler charts are extensively used to determine the temperature distribution and heat flow rate when both conduction and convection resistances are almost of equal importance.

EXAMPLE 6.18

A large steel plate 50 mm thick is initially at a uniform temperature of 425°C. It is suddenly exposed on both sides to an environment with convective coefficient 285 W/m^2 K and temperature 65°C. Determine the centre line temperature, and the temperature inside the plate 12.5 mm from the mid plane after 3 minutes.

For Steel :

Thermal conductivity k = 42.5 W/mK

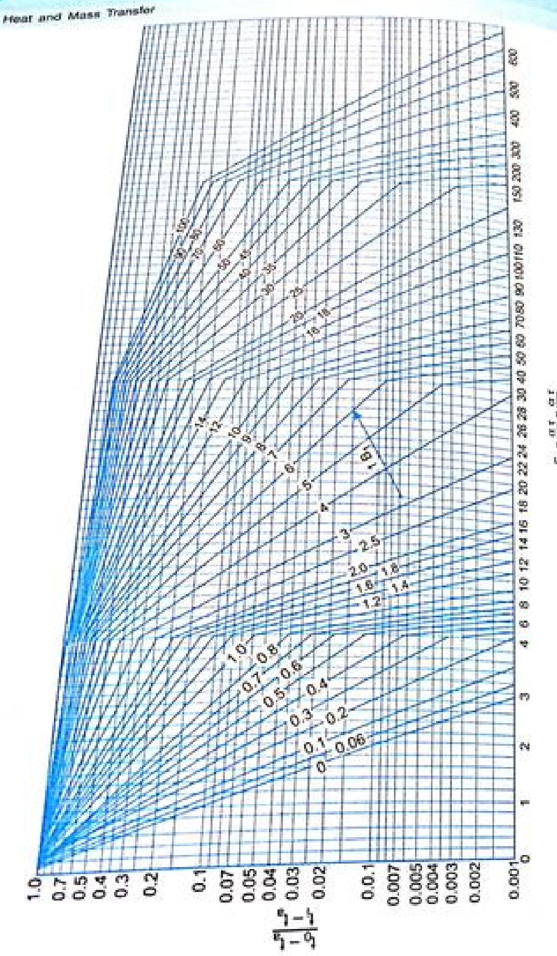
Thermal diffusivity α = 0.043 m^2/hr

Solution : The characteristic linear dimension for a flat plate equals half the plate thickness, *i.e.*,

$$l = \frac{50}{2} = 25 \text{ mm}$$

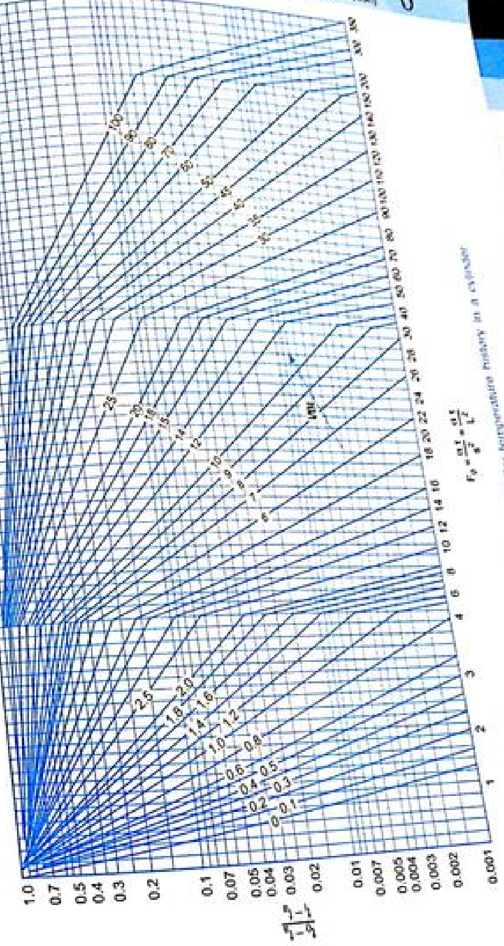
Fourier number F_0

$$= \frac{\alpha\tau}{l^2} = \frac{0.043 \times (3/60)}{(25 \times 10^{-3})^2} = 3.4$$



$F_0 = \frac{\alpha t}{s^2} = \frac{\alpha t}{L^2}$

Fig. 6.8. Heisler chart for temperature history at the centre of a plane



$F_0 = \frac{\alpha t}{s^2} = \frac{\alpha t}{L^2}$

Fig. 6.9. Heisler chart for temperature history in a cylinder

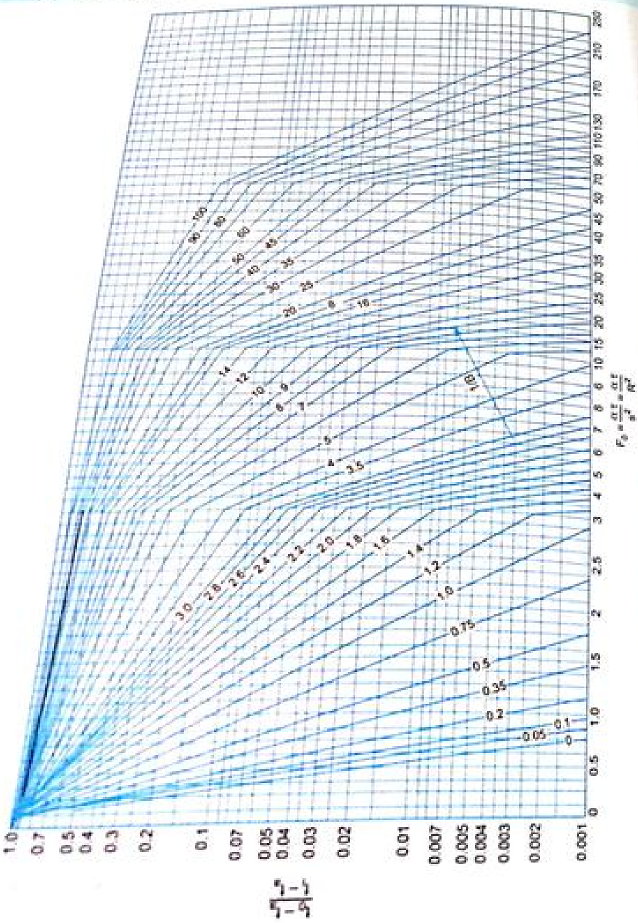


Fig. 6.10. Heisler chart for central temperature history in a sphere

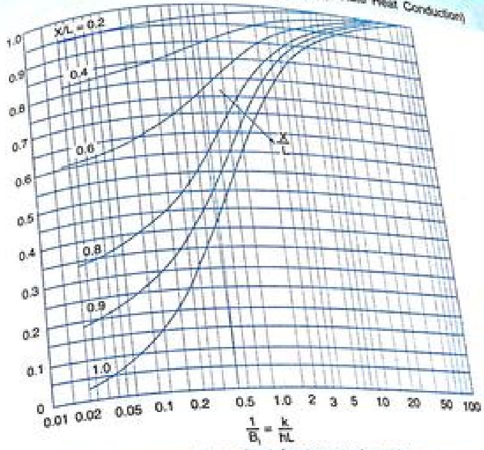


Fig. 6.11. Correction factor chart for temperature history in a plate

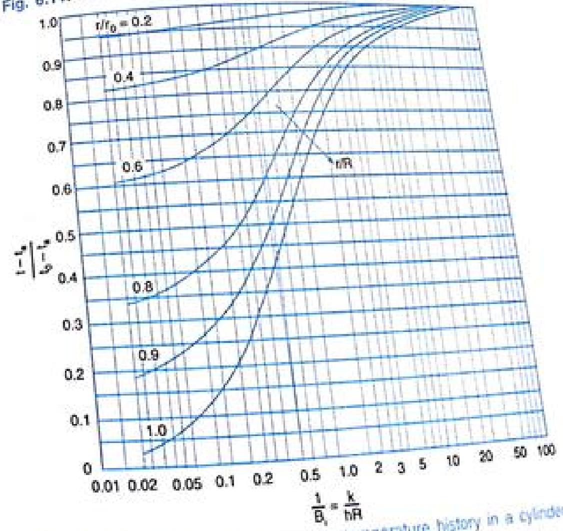


Fig. 6.12. Correction factor chart for temperature history in a cylinder

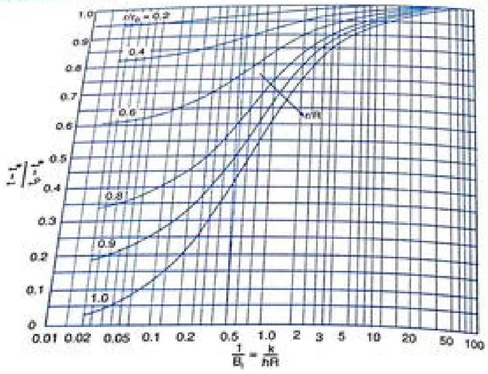


Fig. 6.13. Correction factor chart for temperature history in a sphere

Biot number B_i

$$= \frac{hl}{k} = \frac{285 \times (25 \times 10^{-3})}{42.5} = 0.167$$

Since Biot number is greater than 0.1, the internal temperature gradients are not small and so the internal resistance cannot be neglected. Consideration of the plate as a lumped system would then be inappropriate. Further, the Biot number is less than 100, and accordingly the transient solution can be obtained by employing Heisler charts.

(i) The following parametric values apply :

$$F_0 = 3.44 \;;$$

$$\frac{1}{B_i} = 5.988$$

and $\frac{x}{l} = 0$ *(mid plane)*

Reading from the charts for an infinite ...e (Fig. 6.8)

$$\frac{t_0 - t_a}{t_i - t_a} = 0.6$$

Therefore the temperature at the mid plane (centre line) of the plate is

$$t_0 = t_a + 0.6\,(t_i - t_a)$$
$$= 65 + 0.6\,(425 - 65) = 281°C$$

(ii) The distance 12.5 mm from the mid plane implies that

$$\frac{x}{l} = \frac{25 - 12.5}{25} = 0.5$$

From Fig. 6.11 at $x/l = 0.5$ and $1/B_i = 5.988$

$$\frac{t - t_a}{t_0 - t_a} = 0.96$$

$$\therefore \quad t = t_a + 0.96\,(t_0 - t_a)$$
$$= 65 + 0.96\,(281 - 65) = 272.36°C.$$

EXAMPLE 6.19

The nose section of a missile is formed of a 6 mm thick stainless steel plate and is held initially at ... temperature of 88°C. The missile enters ... layers of the atmosphere at a very high velocity. The effective temperature of air surrounding the nose region attains the value 2200°C and the surface convective coefficient is estimated at 3405 W/m²K. Make calculations for the maximum permissible time in these surroundings if the maximum metal temperature is not to exceed 1095°C. Also work out the inside surface temperature under these conditions.

The constant values for steel properties are :

density $\rho = 7800$ kg/m³

specific heat $c = 465$ J/kgK

and thermal conductivity $k = 54$ W/mK

Solution : The nose section may be idealised as a flat plate for which the characteristic linear dimension equals half the plate thickness, i.e.,

$l = 6/2 = 3$ mm

Biot number B_i

$$= \frac{hl}{k} = \frac{3505 \times 3 \times 10^{-3}}{54} = 0.189$$

Since Biot number is greater than 0.1, the lumped analysis would be inappropriate. Further $B_i < 100$ and accordingly the transient solution can be obtained by employing Heisler charts.

For the flat plate, the following parametric values apply :

$1/B_i = 5.29$

and $x/l = 1$ (outside surface of nose section)

Reading from the chart for an infinite plane (Fig. 6.11)

$$\frac{t - t_a}{t_0 - t_a} = 0.925$$

Then from the correlation,

$$\frac{t - t_a}{t_i - t_a} = \left(\frac{t_0 - t_a}{t_i - t_a}\right) \times \left(\frac{t - t_a}{t_0 - t_a}\right)$$

$$\frac{1095 - 2200}{38 - 2200} = \left(\frac{t_0 - t_a}{t_i - t_a}\right) \times 0.925$$

$$\text{or} \quad \frac{t_0 - t_a}{t_i - t_a} = \frac{1}{0.925}\left(\frac{1095 - 2200}{38 - 2200}\right)$$
$$= 0.508$$

Now, from Fig. 6.4, for the above dimensionless temperature and $1/B_i = 5.29$, we find the value of Fourier number $F_0 = 3.0$.

$$\therefore \quad \frac{\alpha\tau}{l^2} = 3.0 \text{ or } \left(\frac{k}{\rho c}\right)\frac{\tau}{l^2} = 3.0$$

$$\text{or} \quad \left(\frac{54}{7800 \times 465}\right)\frac{\tau}{(3 \times 10^{-3})^2} = 3.0$$

$$\therefore \quad \tau = 1.813 \text{ seconds}$$

The temperature t_0 at the inside surface $(x = 0)$ is given by

$$\frac{t_0 - t_a}{t_i - t_a} = 0.508$$

$$t_0 = t_a + 0.508\,(t_i - t_a)$$
$$= 2200 + 0.508\,(38 - 2200)$$
$$= 1101.7°C$$

EXAMPLE 6.20

During the manufacture of plastic sheets 10 cm thick, the sheets are brought to a uniform temperature of 175°C and then allowed to cool to a surface temperature of 52°C in air at 38°C before further processing. How long a cooling period will be required if natural convection cooling is employed with average surface coefficient of 39.2 kJ/m²-hr-deg ? Also determine the temperature at the centre of plastic sheet when the surface temperature has reached 52°C.

Properties of plastic material are :

density $\rho = 1280$ kg/m³

specific heat $c = 1.6$ kJ/kgK, *and*

thermal conductivity $k = 0.98$ kJ/m hr K

Solution : The characteristic linear dimension for a flat plate equals half the plate thickness, i.e.,

$l = 10/2 = 5$ cm

Biot number B_i

$$= \frac{hl}{k} = \frac{39.2 \times (5 \times 10^{-2})}{0.98} = 2.0$$

Since Biot number is greater than 0.1, the lumped analysis would be inappropriate. Further $B_i < 100$ and accordingly the transient solution can be obtained by using Heisler charts.

For the flat plate, the following parametric values apply :

$1/B_i = 0.5$

and $x/l = 1$ (outside surface of the sheet)

Reading from the chart for an infinite plane (Fig. 6.11)

$$\frac{t-t_a}{t_0-t_a} = 0.5$$

Then from the correlation,

$$\frac{t-t_a}{t_i-t_a} = \left(\frac{t_0-t_a}{t_i-t_a}\right)\times\left(\frac{t-t_a}{t_0-t_a}\right)$$

$$\frac{52-38}{175-38} = \left(\frac{t_0-t_a}{t_i-t_a}\right)\times 0.5$$

$$\text{or } \frac{t_0-t_a}{t_i-t_a} = \frac{1}{0.5}\left(\frac{52-38}{175-38}\right) = 0.204$$

Now from Fig. 6.8, for the above dimensionless temperature and $1/B_i = 0.5$, we read the value of Fourier number $F_0 = 1.5$

$$\therefore \quad \frac{\alpha\tau}{l^2} = 1.5\ ; \quad \left(\frac{k}{\rho c}\right)\frac{\tau}{l^2} = 1.5$$

$$\text{or } \left(\frac{0.98}{1280\times 1.6}\right)\frac{\tau}{(5\times 10^{-2})^2} = 1.5\ ;$$

$$\tau = 7.837 \text{ hr}$$

The temperature t_0 at the centre of sheet is given by :

$$\frac{t_0-t_a}{t_i-t_a} = 0.204$$

$$t_0 = t_a + 0.204\,(t_i - t_a)$$
$$= 38 + 0.204\,(175 - 38)$$
$$= 65.95°C$$

EXAMPLE 6.21

A flat wall of fire clay brick, 50 cm thick and initially at 25°C, has one of its faces suddenly exposed to a hot gas at 950°C. If the heat transfer coefficient on the hot side is 7.5 W/m²K and the other face of the wall is insulated so that no heat passes out of that face, determine the time necessary to raise the centre of the wall to 350°C. Also calculate the temperature of the insulated wall face at that instant. For fire clay brick :

thermal conductivity $k = 1.12$ W/mK and thermal diffusivity $\alpha = 5.16 \times 10^{-7}$ m²/s

Solution : The insulated face limits the energy transfer into the conducting medium to only one direction. This is equivalent to heat transfer from a wall of twice the thickness (100 cm in this case). The insulated face then becomes the line of symmetry or the central plane and any distance x is measured from it. Solution to this transient conduction problem would now be arrived at by considering the wall to be 100 cm thick.

The characteristic linear dimension for a plane wall equals half the wall thickness, *i.e.*,

$$l = 100/2 = 50 \text{ cm},$$

Biot number B_i is equal to

$$\frac{hl}{k} = \frac{7.5\times 0.5}{1.12} = 3.348$$

Since Biot number is greater than 0.1, lumped analysis would be inappropriate. Further $B_i < 100$ and accordingly the transient solution can be obtained by employing Heisler charts.

(*i*) For the flat plate, the following parametric values apply :

$1/B_i = 0.2987$

and $x/l = 0.5$ (centre of the given wall)

Reading from the chart for an infinite plane

$$\frac{t-t_a}{t_0-t_a} = 0.86$$

Then from the correlation

$$\frac{t-t_a}{t_i-t_a} = \left(\frac{t_0-t_a}{t_i-t_a}\right)\times\left(\frac{t-t_a}{t_0-t_a}\right)$$

$$\frac{350-950}{25-950} = \left(\frac{t_0-t_a}{t_i-t_a}\right)\times 0.86$$

$$\text{or } \frac{t_0-t_a}{t_i-t_a} = \frac{1}{0.86}\left(\frac{350-950}{25-950}\right) = 0.753$$

Now, from Fig. 6.8, for the above dimensionless temperature and $1/B_i = 0.2987$, we read the Fourier number $F_0 = 0.32$

$$\therefore \quad \frac{\alpha\tau}{l^2} = 0.32$$

$$\tau = \frac{0.32\times l^2}{\alpha} = \frac{0.32\times 0.5^2}{5.16\times 10^{-7}}$$
$$= 1.55\times 10^5 \text{ seconds} = 43.07 \text{ hrs}$$

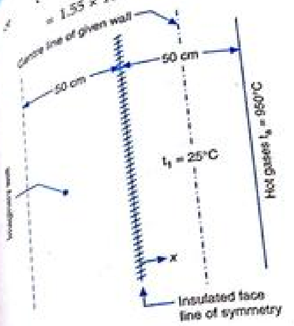


Fig. 6.14. Energy transfer through a wall with one face insulated

(*ii*) The temperature t_0 at the insulated face ($x = 0$) is computed from

$$\frac{t_0-t_a}{t_i-t_a} = 0.753$$

$$t_0 = t_a + 0.753\,(t_i - t_a)$$
$$= 950 + 0.753\,(25 - 950)$$
$$= 253.47°C$$

EXAMPLE 6.22

A long cylindrical shaft of radius 7.5 cm comes out of an oven at 815°C throughout and is cooled by quenching it in a large bath of 38°C coolant. If the surface coefficient of heat transfer between the bar surface and the coolant is 175 W/m²-deg, calculate the time it takes for the shaft centre to reach 116°C.

Assume that k = 17.5 W/m-deg and

$$\alpha = 0.0185 \text{ m}^2\text{/hr}$$

* *(ii) What would be the surface temperature of the shaft when its centre temperature is 116°C. Also calculate the temperature gradient at the outside surface at the same instant of time.*

Solution: The characteristic linear dimension l defined as the ratio of the volume of the cylindrical shaft to its surface area works out as

$$l = \frac{\pi R^2 L}{2\pi R L} = \frac{R}{2} = \frac{7.5}{2} = 3.75 \text{ cm}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{175\times 0.0375}{17.5} = 0.375$$

Since the Biot number is greater than 0.1, the lumped parameter solution is invalid. Further, Biot number is less than 100 and accordingly the transient solution can be obtained by using Heisler charts.

For a cylindrical solid, the following parametric values apply :

$$\frac{1}{B_i} = \frac{k}{hR}$$
$$= \frac{17.5}{175\times(7.5\times 10^{-2})} = 1.33$$

$$\frac{t-t_a}{t_i-t_a} = \frac{116-38}{815-38} = 0.1$$

$$\frac{x}{l} = 0 \text{ (centre of the bar)}$$

From the chart for an infinite cylinder (Fig. 6.9), we read the Fourier number $F_0 = 1.92$

$$\therefore \quad \frac{\alpha\tau}{R^2} = 1.92\ ; \quad \frac{0.0185\times\tau}{(7.5\times 10^{-2})^2} = 1.92$$

$$\tau = \frac{1.92\times(7.5\times 10^{-2})^2}{0.0185}$$
$$= 0.584 \text{ hr} = 2102 \text{ sec}$$

(ii) At the cylinder surface : $r/R = 1$ and $1/B_i = 1.33$.

From the chart (Fig. 6.12) for an infinite cylinder, we read

$$\frac{t-t_a}{t_0-t_a} = 0.7$$

The temperature t at the surface then works out to be

$$t = t_a + 0.7\,(t_0 - t_a)$$
$$= 38 + 0.7\,(116 - 38) = 92.6°C$$

The temperature gradient at the outside surface is determined by the boundary condition at $r = R$ which equates the rate at which energy is conducted to the fluid-solid interface from within the solid to the rate at which it is convected away into the fluid. That is

$$k\,(2\pi RL)\frac{\partial t}{\partial r} = h\,(2\pi RL)\,(t_i - t_f)$$

$$\frac{\partial t}{\partial r} = \frac{h}{k}\,(t_i - t_f)$$

$$= \frac{175}{17.5}\,(92.6 - 38) = \mathbf{546°C/m}$$

EXAMPLE 6.23

A 12 cm diameter cylindrical bar, initially at a uniform temperature of 40°C, is placed in a medium at 650°C with a convective coefficient of 22 W/m²K. Determine the time required for the centre to reach 255°C. Also workout the temperature of the surface at this instant. For the material of the bar:

Thermal conductivity $k = 20$ W/mK

density $\rho = 580$ kg/m³

specific heat $c = 1050$ J/kgK

Solution : The characteristic linear dimension l defined as the ratio of the volume of the cylinder to its surface area works out to be :

$$l = \frac{\pi R^2 L}{2\pi RL} = \frac{R}{2} = \frac{6}{2} = 3\text{ cm}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{22 \times 3 \times 10^{-2}}{0.20} = 3.3$$

Since Biot number is greater than 0.1, a lumped-parameter solution is invalid. Further Biot number is less than 100 and accordingly the transient solution can be obtained by using Heisler charts.

For a cylindrical solid, the following parameter values apply :

$$\frac{1}{B_i} = \frac{k}{hR} = \frac{0.20}{22 \times (6 \times 10^{-2})} = 0.1515$$

$$\frac{t - t_a}{t_i - t_a} = \frac{255 - 650}{40 - 650} = 0.647$$

$$\frac{x}{l} = 0 \text{ (centre of the bar)}$$

From the chart for an infinite cylinder (Fig. 6.9) we read Fourier number $F_0 = 0.18$.

$$\therefore \quad \frac{\alpha\tau}{R^2} = 0.18\ ; \quad \left(\frac{k}{\rho c}\right)\frac{\tau}{R^2} = 0.18$$

$$\text{or} \quad \left(\frac{0.20}{580 \times 1050}\right)\frac{\tau}{(6 \times 10^{-2})^2} = 0.18$$

$$\tau = \mathbf{1973.16\ seconds}$$

(*ii*) At the cylinder surface :

$r/R = 1$ and $1/B_i = 0.1515$

From the chart (Fig. 6.12) for an infinite cylinder, we read :

$$\frac{t - t_a}{t_0 - t_a} = 0.18$$

The temperature t at the surface then works out to be :

$$t = t_a + 0.18\,(t_0 - t_a)$$
$$= 650 + 0.18\,(255 - 650)$$
$$= \mathbf{578.9°C}$$

EXAMPLE 6.24

A 10 cm diameter apple, approximately spherical in shape, is taken from a 20°C environment and placed in a refrigerator where temperature is 5°C and average convective heat transfer coefficient over the surface of apple is 6 W/m²K. Calculate the temperature at the centre of the apple after a period of 1 hour.

Thermo-physical properties of apple are :

$\rho = 998$ kg/m³ ; $c = 4180$ J/kgK

and $k = 0.6$ W/mK

Solution : The characteristic linear dimension defined as the ratio of the volume of the apple to its surface area works out to be :

$$l = \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3} = \frac{5}{3} = 1.67\text{ cm}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{6 \times (1.67 \times 10^{-2})}{0.6} = 0.167$$

Since the Biot number is greater than 0.1, a lumped-capacity approach is not suitable.

Further $B_i < 100$ and accordingly the transient solution can be obtained by using Heisler charts.

For a spherical solid, the dimensionless parameters for Heisler charts are :

$$\frac{1}{B_i} = \frac{k}{hR} = \frac{0.6}{6 \times (5 \times 10^{-2})} = 2$$

$$F_0 = \frac{\alpha\tau}{R^2} = \left(\frac{k}{\rho c}\right)\frac{\tau}{R^2}$$

$$= \frac{0.6 \times (1 \times 3600)}{998 \times 4180 \times (5 \times 10^{-2})} = 0.207$$

$$\frac{r}{R} = 0$$

(*mid plane or centre of the apple*)

Reading from the charts for sphere (Fig. 6.10).

$$\frac{t_0 - t_a}{t_i - t_a} = 0.85$$

Therefore, the temperature at the mid plane (centre) of the apple is

$$t_0 = t_a + 0.85\,(t_i - t_a)$$
$$= 5 + 0.85\,(20 - 5) = \mathbf{17.75°C}$$

EXAMPLE 6.25

A 3.6 cm diameter egg, approximately spherical in shape, is initially at 25°C temperature. To boil it to the consumer's taste, it needs to be placed for 225 seconds in a saucepan of boiling water at 100°C. For how long should a similar egg for the same consumer be boiled when taken from a refrigerator at a temperature of 5°C. Thermo-physical properties of egg are :

$k = 2.5$ W/mK ; $\rho = 1250$ kg/m³

and $c = 2200$ J/kgK

and the heat transfer coefficient for the shell and shell-water interface may be taken at 280 W/m² K (b) Compare the centre temperature attained with that computed by treating the egg as a lumped-heat-capacity system.

Solution : For a spherical solid, the dimensionless parameters for Heisler charts are :

$$\frac{1}{B_i} = \frac{k}{hR} = \frac{2.5}{280 \times 0.018} = 0.496$$

$$F_0 = \frac{\alpha\tau}{R^2} = \left(\frac{k}{\rho c}\right)\frac{\tau}{R^2}$$

$$= \left(\frac{2.5}{1250 \times 2200}\right) \times \frac{225}{(0.018)^2}$$

$$= 0.63$$

$$\frac{r}{R} = 0 \quad \text{(mid plane or centre of egg)}$$

Reading from charts for sphere (Fig. 6.10)

$$\frac{t_0 - t_a}{t_i - t_a} = 0.22$$

Therefore, the temperature at the mid plane (centre) of the egg is

$$t_0 = t_a + 0.22\,(t_i - t_a)$$
$$= 100 + 0.22\,(25 - 100) = 83.5°C$$

When the egg is taken from the refrigerator, the dimensionless parameter for Heisler charts take the values,

$$\frac{t_0 - t_a}{t_i - t_a} = \frac{83.5 - 100}{5 - 100} = 0.174$$

$$\frac{1}{B_i} = \frac{k}{hR} = 0.496$$

As before, from the Heisler charts (Fig. 6.10), we read the Fourier number $F_0 = 0.71$

$$\therefore \quad \frac{\alpha\tau}{R^2} = 0.71 \quad \text{or} \quad \left(\frac{k}{\rho c}\right)\frac{\tau}{R^2} = 0.71$$

$$\text{or} \quad \left(\frac{2.5}{1250 \times 2200}\right)\frac{\tau}{(0.018)^2} = 0.71$$

$\therefore$ New value of time $\tau = 253$ seconds

(b) The characteristic length defined as the ratio of the volume of egg to its surface area works out as:

$$l = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

$$= \frac{0.018}{3} = 0.006\text{ m}$$

Biot number B_i

$$= \frac{hl}{k} = \frac{280 \times 0.006}{2.5} = 0.67$$

Fourier number F_0

$$= \frac{\alpha\tau}{l^2} = \left(\frac{k}{\rho c}\right)\times\frac{\tau}{l^2}$$

$$= \left(\frac{2.5}{1250\times 2200}\right)\times\frac{225}{(0.006)^2}$$

$$= 5.68$$

The lumped parameter solution for transient conduction is stated as :

$$\frac{t_0 - t_a}{t_i - t_0} = \exp[-B_i F_0]$$

$$\frac{t_0 - 100}{25 - 100} = \exp[-0.672 \times 5.68] = 0.0219$$

$$t_0 = 100 - 0.0219\ (100 - 25)$$

$$= 98.3°C$$

The lumped-heat capacity method is based on the assumption of uniform temperature throughout the egg. The difference between the values of temperature at the centre of the egg (83.5°C evaluated by Heisler charts and 98.3°C obtained by the application of lumped heat capacity method) is mainly due to this assumption.

6.4. TRANSIENT HEAT CONDUCTION IN INFINITE THICK SOLIDS ($B_i \to \infty$)

*An **infinite solid** is one which extends itself infinitely in all directions of space. If an infinite solid is split in the middle by a plane, each half is known as **semi-infinite solid**.*

Consider the semi-infinite plate ; a plate bounded by a plane $x = 0$ and extending to infinity in the positive x-direction. The body is initially at the uniform temperature t_i including the surface at $x = 0$. The surface temperature at $x = 0$ is instantaneously changed to and held at t_a for all times greater than $\tau = 0$. Thus the boundary conditions are :

$$t\ (x, 0) = t_i$$

$$t\ (0, \tau) = t_a \quad \text{for} \quad \tau > 0$$

$$t\ (\infty, \tau) = t_i \quad \text{for} \quad \tau > 0$$

The solution of the differential equation transient heat conduction

$$\frac{d^2t}{dx^2} = \frac{1}{\alpha}\frac{dt}{d\tau}$$

with these bounding conditions would give the following solution for temperature distribution at any time τ at a plane parallel to and at a distance x from the surface.

$$\frac{t(x, \tau) - t_a}{t_i - t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right) \quad ...(6.11)$$

where the Gaussian error function is defined by

$$erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{\alpha\tau}} e^{-\eta^2}\,d\eta \quad ...(6.12)$$

Suitable values of the error functions may be obtained from Fig. 6.16 or Table 6.1.

Inserting the definition of the error function is equations 6.11, we obtain

$$t\ (x, \tau) = t_a + (t_i - t_a)\frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{\alpha\tau}} e^{-\eta^2}\,d\eta$$

which may be differentiated to yield

$$\frac{\partial t}{\partial x} = \frac{t_i - t_a}{\sqrt{\pi\alpha\tau}}\exp\left[\frac{-x^2}{4\alpha\tau}\right]$$

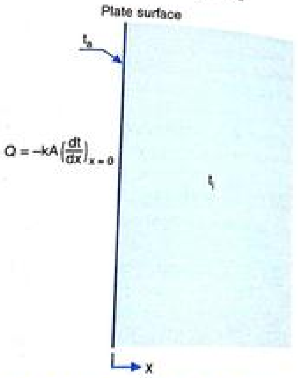


Fig. 6.15. Transient heat flow in a semi-infinite plate

Table 6.1. The Error Functions

$$erf\ \phi = \frac{2}{\sqrt{\pi}}\int_0^{\phi} e^{-\eta^2}\,d\eta$$

ϕ	$erf\ \phi$	ϕ	$erf\ \phi$	ϕ	$erf\ \phi$
0.00	0.0	0.42	0.4475	1.35	0.9431
0.02	0.0225	0.44	0.4662	1.40	0.9523
0.04	0.0451	0.46	0.4847	1.45	0.9592
0.06	0.0676	0.48	0.5027	1.50	0.9661
0.08	0.0901	0.50	0.5205	1.55	0.9712
0.10	0.1125	0.55	0.5633	1.60	0.9763
0.12	0.1348	0.60	0.6039	1.65	0.9800
0.14	0.1569	0.65	0.6420	1.70	0.9838
0.16	0.1709	0.70	0.6778	1.75	0.9864
0.18	0.2009	0.75	0.7112	1.80	0.9891
0.20	0.2227	0.80	0.7421	1.85	0.9909
0.22	0.2443	0.85	0.7707	1.90	0.9928
0.24	0.2657	0.90	0.7970	1.95	0.9940
0.26	0.2869	0.95	0.8270	2.0	0.9953
0.28	0.3079	1.0	0.8427	2.10	0.9967
0.30	0.3286	1.05	0.8614	2.20	0.9981
0.32	0.3491	1.10	0.8802	2.30	0.9987
0.34	0.3694	1.15	0.8952	2.40	0.9993
0.36	0.3893	1.20	0.9103	2.50	0.9995
0.38	0.4090	1.25	0.9221	2.60	0.9998
0.40	0.4284	1.30	0.9340	2.80	0.9999

Substituting the gradient in Fourier's law, the instanteneous heat flow rate at a given x-location within the semi-infinite body at a specified time is

$$Q_i = -k\,A\,(t_i - t_a)\frac{\exp\left[-x^2/4\alpha\tau\right]}{\sqrt{\pi\alpha\tau}} \quad ...(6.13)$$

At the surface ($x = 0$) the heat flow is

$$Q_{surface} = -\frac{kA(t_i - t_a)}{\sqrt{\pi\alpha\tau}} \quad ...(6.14)$$

and this heat flux clearly diminishes with time.

The total heat flow can be worked out by integrating equation 6.14 over the time interval

$$\tau = 0 \quad \text{to} \quad \tau = \tau$$

$$Q_i = -\frac{kA\,(t_i - t_a)}{\sqrt{\pi\alpha}}\int_0^{\tau}\frac{1}{\sqrt{\tau}}\,d\tau$$

$$= -kA\,(t_i - t_a)\,2\sqrt{\frac{\tau}{\pi\alpha}}$$

$$= -1.13\,kA\,(t_i - t_a)\sqrt{\frac{\tau}{\alpha}} \quad ...(6.15)$$

The general criterion for the infinite solution to apply to a body of finite thickness (slab) subjected to one dimensional heat transfer is :

$$\frac{\delta}{2\sqrt{\alpha\tau}} \geq 0.5$$

where δ is the thickness of the body.

Under similar conditions of heating cooling, the temperature at the centre cylinder or sphere of radius R is given

$$\frac{t - t_a}{t_i - t_a} = erf\left[\frac{\alpha\tau}{R^2}\right] \quad ...$$

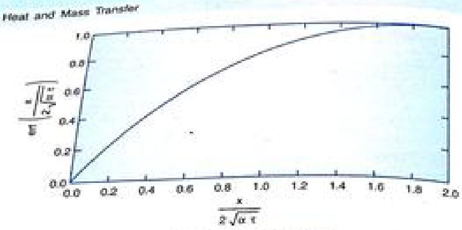


Fig. 6.16. Gauss's error integral

The values of function $erf\,(\alpha\tau/R^2)$ for the cylinderical and spherical surfaces can be obtained from Fig. 6.17.

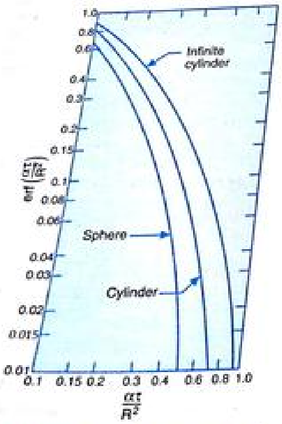


Fig. 6.17. Error integral for cylinders and spheres

Penetration depth and penetration time

Penetration depth refers to the location of a point where the temperature change is within 1% of the change in the surface temperature, *i.e.*,

$$\frac{t-t_a}{t_i-t_a} = 0.9$$

From the table for Gaussian error integral, this corresponds to $x/2\sqrt{\alpha\tau} = 1.8$.

Thus the temperature perturbation at all the surface has penetrated to the depth $d = 3.6\sqrt{\alpha\tau}$

Penetration time at a given depth indicates the time taken for a surface perturbation to be felt at that depth. Hence at the penetration depth d, there will be 1% perturbation at a time τ_p given by

$$\frac{d}{2\sqrt{\alpha\tau_p}} = 1.8\ ; \quad \tau_p = \frac{d^2}{13\alpha}$$

Thus the perturbation time varies as $\frac{d^2}{\alpha}$.

EXAMPLE 6.26

A large steel ingot, which has been uniformly heated to 750°C, is hardened by quenching it in an oil bath that is maintained at 25°C. What length of time is required for the temperature to reach 600°C at a depth of 1 cm ? Thermal diffusivity for the steel ingot is 1.21×10^{-5} m²/s. The ingot may be approximated as a flat plate.

Solution : The temperature distribution at any time τ at a plane parallel to and at a distance x from the surface is

$$\frac{t-t_a}{t_i-t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\frac{600-25}{750-25} = 0.793 = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\therefore \frac{x}{2\sqrt{\alpha\tau}} = 0.90 \quad \text{(Table 6.1 or Fig. 6.16)}$$

Thus
$$\tau = \frac{x^2}{4\alpha(0.9)^2} = \frac{(1\times10^{-2})^2}{4\times(1.21\times10^{-5})\times(0.9)^2}$$
$$= \mathbf{2.55\ seconds}$$

EXAMPLE 6.27

Water pipes are to be burried underground in a wet soil ($\alpha = 2.78 \times 10^{-3}$ m²/hr) which is initially at 4.5°C. The soil surface temperature suddenly drops to (– 5°C) and remains at this value for 10 hours. Calculate the minimum depth at which the pipes be laid if the surrounding soil temperature is to remain above 0°C (no freezing of water). The soil may be considered as semi-infinite solid.

Solution : At the critical depth, the temperature will just reach 0°C after 10 hours. Thus

$$\frac{t-t_a}{t_i-t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\frac{0-(-5)}{4.5-(-5)} = 0.526 = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\therefore \frac{x}{2\sqrt{\alpha\tau}} = 0.50 \quad \text{(Table 6.1 or Fig. 6.16)}$$

Thus
$$x = 2 \times 0.50\sqrt{\alpha\tau}$$
$$= 2 \times 0.50\sqrt{2.78\times10^{-3}\times10}$$
$$= \mathbf{0.167\ m}$$

EXAMPLE 6.28

A water line is buried underground in dry soil that has an assumed initial temperature of 4.5°C. The pipe may have no flow through it for long period of time, yet it will not be drained in order that no freezing occurs, the pipe must be kept at a temperature not lower than 0°C.

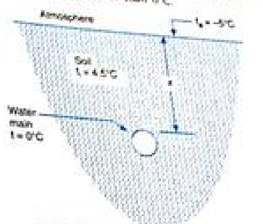


Fig. 6.18. Water pipe buried underground

The pipe is to be designed for a 36-hr period at the beginning of which the soil surface temperature suddenly drops to (– 17.8°C). Workout the minimum earth covering needed above the water pipe so as to prevent the possibility of freezing during the 36-hr cold spell. The soil in which the pipe is buried has the following properties :

density = 640 kg/m³

specific heat = 1843 J/kg-deg

thermal conductivity = 0.345 W/m-deg

Solution : At the critical depth, the temperature will just reach 0°C after 36-hour. Thus

$$\frac{t-t_a}{t_i-t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\frac{0-(-17.8)}{4.5-(-17.8)} = 0.798 = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\therefore \frac{x}{2\sqrt{\alpha\tau}} = 0.90 \quad \text{(Table 6.1 or Fig. 6.16)}$$

Thermal diffusivity, from the given soil properties, works out as

$$\alpha = \frac{k}{\rho c} = \frac{0.345}{640\times1843} = 2.92 \times 10^{-7}\ \text{m}^2/\text{s}$$

Earth covering for time τ

$= 36$ hr $= 36 \times 3600$ sec is

$$x = 0.90 \times \left[2\sqrt{2.92\times10^{-7}\times36\times3600}\right]$$

$$= 0.35 \text{ m}$$

EXAMPLE 6.29

A mild steel plate 5 cm thick and initially at 40°C temperature is suddenly exposed on one side to a fluid which causes the surface temperature to increase to and remain at 90°C. Calculate (i) maximum time that the slab be treated as a semi-infinite body (ii) temperature at the centre of the slab one minute after the change in surface temperature. For steel $\alpha = 1.25 \times 10^{-5}$ m²/s.

Solution : The general criterion for the infinite solution to apply to a body of finite thickness subjected to one dimensional heat transfer is,

$$\frac{\delta}{2\sqrt{\alpha\tau}} \geq 0.5$$

where δ is the thickness of the body.

$$\therefore \quad \tau_{max} = \frac{-\delta^2}{4\alpha(0.5)^2}$$

$$= \frac{(5\times10^{-2})^2}{4\times1.25\times10^{-5}\times(0.5)^2}$$

$$= 200 \text{ seconds}$$

(ii) At the centre of slab, $x = 2.5$ cm and at $\tau = 60$ seconds

$$\frac{t-t_a}{t_i-t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$= erf\left(\frac{2.5\times10^{-2}}{2\sqrt{1.25\times10^{-5}\times60}}\right)$$

$$= erf\,(0.456) = 0.48 \quad \text{(Table 6.1)}$$

$\therefore$ Temperature at the centre of slab works out to be

$$t = t_a + 0.48\,(t_i - t_a)$$

$$= 90 + 0.48\,(40 - 90) = \mathbf{66°C}$$

EXAMPLE 6.30

A thick concrete wall of a jet engine test cell is initially at a constant temperature of 21°C. When there occurs a combination of exhaust gases from the turbojet and the spray of cooling water, the surface temperature of the wall suddenly rises to 315°C. Calculate the temperature at a point 7.5 cm from the surface after 7.5 hours. Also workout the instantaneous heat flow rate at the specified plane and at the surface itself at this instant. Use the solution for semi-infinite solid and take thermal diffusivity $\alpha = 1.58 \times 10^{-3}$ m²/h, thermal conductivity $k = 0.937$ W/m-deg for the concrete.

Solution : The temperature distribution at any time τ at a plane wall parallel to and at a distance x from the surface is

$$\frac{t-t_a}{t_i-t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$= erf\left(\frac{0.075}{2\sqrt{1.58\times10^{-3}\times7.5}}\right)$$

$$= erf\,(0.3447)$$

$$= 0.32 \quad \text{(Table 6.1 or Fig. 6.16)}$$

The temperature at the specific plane then works out as

$$t = 0.32\,(t_i - t_a) + t_a$$

$$= 0.32\,(21 - 315) + 315 = 220.92°C$$

(b) The instantaneous heat flow rate at the specified plane is

$$Q_i = -kA\,(t_i - t_a)\,\frac{\exp\left[-x^2/4\alpha\tau\right]}{\sqrt{\pi\alpha\tau}}$$

$$= -0.935 \times 1 \times (21 - 315)$$

$$\frac{\exp\left[-(0.075)^2/4\times1.58\times10^{-3}\times7.5\right]}{\sqrt{\pi\times1.58\times10^{-3}\times7.5}}$$

$$= \mathbf{1265.6 \text{ W per m}^2 \text{ of wall area}}$$

At the surface ($x = 0$), the heat flow is

$$Q_{surface} = -\frac{kA(t_i-t_a)}{\sqrt{\pi\alpha\tau}}$$

$$= -\frac{0.935\times1\times(21-315)}{\sqrt{\pi\times1.58\times10^{-3}\times7.5}}$$

$$= \mathbf{1425 \text{ W per m}^2 \text{ of wall area}}$$

EXAMPLE 6.31

A thick steel slab is initially at a uniform temperature of 25°C. When the slab is exposed to hot flue gases, the surface temperature suddenly changes to 450°C. Make calculations for the temperature in a plane 250 mm from the slab surface 5 hours after the operation of change in surface temperature. Find also the heat flowing into 2 square metres of this plane and the total energy flowing through the surface during the 5 hours period. It may be presumed that for steel thermal conductivity k = 160 kJ/m-hr-deg, density ρ = 8000 kg/m³ and the specific heat c = 0.48 kJ/kg-deg.

Solution : Thermal diffusivity

$$\alpha = \frac{k}{\rho c} = \frac{160}{8000\times0.48}$$

$$= 0.0417 \text{ m}^2/\text{hr}$$

The temperature distribution at any time τ at a plane parallel to and at a distance x from the surface is :

$$\frac{t-t_a}{t_i-t_a} = erf\left(\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$= erf\left(\frac{0.25}{2\sqrt{0.0417\times5}}\right)$$

$$= erf\,(0.274) = 0.30$$

(from Table 6.1)

$\therefore$ Temperature at the specified plane works out to be :

$$t = t_a + 0.30\,(t_i - t_a)$$

$$= 450 + 0.30\,(25 - 450) = 322.5°C$$

(ii) The instantaneous heat flow rate is worked out from the relation :

$$Q_i = -kA\,(t_i - t_a)\,\frac{\exp\left[-x^2/4\alpha\tau\right]}{\sqrt{\pi\alpha\tau}}$$

$$= -160 \times 2\,(25 - 450)$$

$$\times \frac{\exp\left[-\left(0.25^2/4\times0.0417\times5\right)\right]}{\sqrt{11\times0.0417\times5}}$$

$$= 42022 \times \exp\,[-0.07494]$$

$$= \mathbf{1.56 \times 10^5 \text{ kJ/hr}}$$

(iii) The total heat flow is computed from the expression,

$$Q_i = -1.13\,kA\,(t_i - t_a)\sqrt{\frac{\tau}{\alpha}}$$

$$= -1.13 \times 160 \times 2\,(25 - 450)\sqrt{\frac{5}{0.0417}}$$

$$= \mathbf{1.683 \times 10^6 \text{ kJ}}$$

EXAMPLE 6.32

A large mass of a material, initially at a uniform temperature of 100°C, has its surface suddenly exposed to and held permanently at 5°C. If thermal diffusivity of the material is 0.41 m²/hr, calculate the time required for the temperature gradient at the surface to reach 3.5°C/m.

Solution : Heat flow at a surface ($x = 0$) is prescribed by the relation,

$$Q = \frac{-kA(t_i-t_a)}{\sqrt{\pi\alpha\tau}}$$

$$\text{or} \quad -kA\left(\frac{\partial t}{\partial x}\right)_{x=0} = \frac{-kA(t_i-t_a)}{\sqrt{\pi\alpha\tau}}$$

$\therefore$ Temperature gradient at the surface,

$$\left(\frac{\partial t}{\partial x}\right)_{x=0} = \frac{t_i-t_a}{\sqrt{\pi\alpha\tau}}$$

$$\text{Given :} \left(\frac{\partial t}{\partial x}\right)_{x=0} = 3.5°C/cm$$

$$= 3.5 \times 100°C/m$$

$$\therefore 3.5 \times 100 = \frac{100-5}{\sqrt{\pi\times0.41\tau}}$$

$$\text{or} \quad \tau = \frac{1}{0.41\pi}\left(\frac{95}{3.5\times100}\right)^2$$

$$= 0.0572 \text{ hour} = \mathbf{206 \text{ sec}}$$

EXAMPLE 6.33

A motor car travelling at 60 km/hr is brought to rest within a 6 second period when the brakes are applied. The braking system consists of 4 brakes with each brake band of 300 cm² area; these press against the steel drums of equivalent area. The brake lining and the drum surfaces are at the same temperature, and the heat generated during the stoppage action is dissipated by flowing across the surface of the drums. Treating the drum surface as semi-infinite plane, workout the maximum temperature rise. The car weighs 1500 kg and the thermo-physical properties of drum surface are

Thermal conductivity $k = 55$ W/mK and thermal diffusivity $\alpha = 1.24 \times 10^{-5}$ m²/s

Solution : The heat dissipation equals the kinetic energy of the moving car

$$\text{Kinetic energy} = \frac{1}{2} mV^2$$

$$= \frac{1}{2} \times 1500 \times \left(\frac{60 \times 1000}{3600}\right)^2$$

$$= 208 \times 10^3 \text{ J in 6 seconds}$$

∴ Heat flow rate is equal to

$$\frac{208 \times 10^3}{6} = 34.67 \times 10^3 \text{ J/s}$$

This equals the instantaneous heat flow rate at the surface ($x = 0$). Thus

$$\frac{-kA(t_i - t_a)}{\sqrt{\pi\alpha\tau}} = 34.67 \times 10^3$$

∴ Temperature rise is equal to

$$(t_i - t_a) = \frac{34.67 \times 10^3 \sqrt{\pi\alpha\tau}}{kA}$$

$$= \frac{34.67 \times 10^3 \sqrt{\pi \times 1.24 \times 10^{-6} \times 6}}{55 \times (4 \times 300^{-4})}$$

$$= 80.61°\text{C}$$

EXAMPLE 6.34

A copper cylinder 60 cm diameter and 75 cm long, is initially at a uniform temperature of 25°C. When the cylinder is exposed to hot flue gases, the surface temperature suddenly changes to 450°C. Make calculations for the temperature at the centre of cylinder 3 minutes after the operation of change in surface temperature. Also calculate the time required for the temperature at the centre to attain the value 375°C. For copper :

thermal diffusivity $\alpha = 1.12 \times 10^{-4}$ m²/s

Solution : The temperature at the centre of cylinder or sphere is given by,

$$\frac{t - t_a}{t_i - t_a} = erf\left(\frac{\alpha\tau}{R^2}\right)$$

$$= erf\left[\frac{1.12 \times 10^{-4} \times 3 \times 60}{(0.30)^2}\right]$$

$$= erf\,(0.224) = 0.35 \quad \text{(Fig. 6.17)}$$

Then the temperature at the centre of cylinder works out,

$$t = t_a + 0.35\,(t_i - t_a)$$

$$= 450 + 0.35\,(25 - 450)$$

$$= 301.25°\text{C}$$

(*ii*) Temperature at the centre of cylinder is given to be 375°C

$$\therefore \quad \frac{375 - 450}{25 - 450} = erf\left(\frac{\alpha\tau}{R^2}\right)$$

$$0.176 = erf\left(\frac{\alpha\tau}{R^2}\right)$$

$$\therefore \quad \frac{\alpha\tau}{R^2} = 0.29 \quad \text{(Fig. 6.17)}$$

$$\text{or} \quad \tau = \frac{0.29 R^2}{\alpha}$$

$$= \frac{0.29 \times 0.3^2}{1.12 \times 10^{-4}} = 2335 \text{ seconds}$$

EXAMPLE 6.35

Two infinite bodies of thermal conductivities k_1 and k_2, thermal diffusivities α_1 and α_2 are initially at temperature t_1 and t_2 respectively. Each body has single plane surface and these surfaces are placed in contact with each other. Determine the conditions under which the contact surface remains at constant temperature t_s where $t_1 > t_s > t_2$.

Solution : Heat flow at a surface ($x = 0$) is given by,

$$Q = -\frac{kA\Delta t}{\sqrt{\pi\alpha\tau}}$$

Therefore heat received by each unit area of contact surface from the body at temperature t_1 is

$$Q_1 = \frac{-k_1(t_1 - t_s)}{\sqrt{\pi\alpha_1\tau}}$$

Heat lost by each unit area of contact surface to the body at temperature t_2 is,

$$Q_2 = \frac{-k_2(t_s - t_2)}{\sqrt{\pi\alpha_2\tau}}$$

The contact surface will remain at a constant temperature if the heat received by it (from the body at t_1) equals the heat loss from it (to the body at t_2). Thus,

$$\frac{-k_1(t_1 - t_s)}{\sqrt{\pi\alpha_1\tau}} = \frac{-k_2(t_s - t_2)}{\sqrt{\pi\alpha_2\tau}}$$

$$\text{or} \quad \frac{k_1(t_1 - t_s)}{\sqrt{\alpha_1}} = \frac{k_2(t_s - t_2)}{\sqrt{\alpha_2}}$$

Simplification gives :

$$t_s = \frac{(k_1 t_1/\sqrt{\alpha_1}) + (k_2 t_2/\sqrt{\alpha_2})}{(k_1/\sqrt{\alpha_1}) + (k_2/\sqrt{\alpha_2})}$$

6.5. PERIODIC VARIATION

When the surface temperature variations inside a solid are periodic in nature, the profile of temperature variation with time may assume a triangular, rectangular or a simple harmonic waveform. Any type of waveform can be analysed and resolved into an infinite number of sine or cosine waves or a combination of these wave patterns.

Consider one-dimensional case of a thick plane wall whose surface temperature changes according to a sine function depicted in Fig. 6.19. The surface temperature $t_{s,\tau}$ of the wall has been indicated by the curve in full line. The temperature $t_{x,\tau}$ within the wall at depth x is also a sine wave and this curve has been represented by the dotted line.

The surface temperature oscillates about the mean temperature level t_m in accordance with the relation,

$$\theta_{s,\tau} = \theta_{s,a} \sin(2\pi n\tau) \quad ...(6.16a)$$

where, $\theta_{s,\tau}$ is excess over the mean temperature ;

$$\theta_{s,\tau} = t_{s,\tau} - t_m$$

$t_{s,a}$ is the amplitude of temperature excess, *i.e.*, the maximum temperature excess at the surface n is the frequency of temperature wave, *i.e.*, the number of complete changes per revolution.

The temperature excess at any depth x and time τ can be expressed by the relation

$$\theta_{x,\tau} = \theta_{s,a} \exp\left[-x\sqrt{\pi n/\alpha}\right] \times \sin\left(2\pi n\tau - x\sqrt{\frac{\pi n}{\alpha}}\right) \quad ...(6.17)$$

At the surface ($x = 0$) the temperature excess becomes zero at $\tau = 0$. However at any depth, $x > 0$, a time $\tau = \frac{x}{2}\left(\frac{1}{\sqrt{\alpha\pi n}}\right)$ would elapse before the temperature excess $\theta_{x,\tau}$ becomes zero. The time interval between the two instants is called the **time lag**.

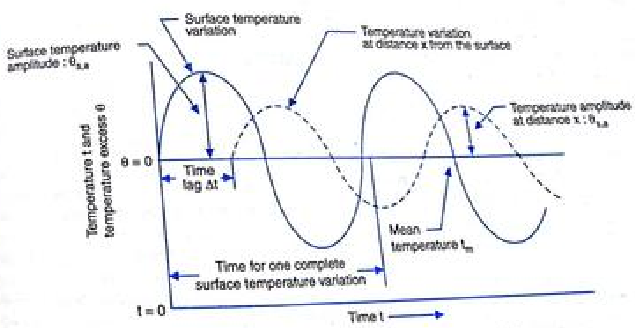


Fig. 6.19. Temperature curves for periodic variation

Time lag $\Delta\tau$ is equal to

$$\frac{x}{2}\left(\frac{1}{\sqrt{\alpha\pi n}}\right) \qquad ...(6.18)$$

Temperature amplitude at depth x is,

$$\theta_{x,a} = \theta_{s,a} \exp\left[-x\sqrt{\frac{\pi n}{\alpha}}\right] \qquad ...(6.19)$$

The relations given above indicate that :

(i) Time lag and the amplitude reduce with increasing value of frequency

(ii) At any depth, $x > 0$, the amplitude (maximum value) occurs late and is smaller than that at the surface ($x = 0$)

(iii) Increase in diffusivity α decreases the time lag but keeps the amplitude large.

(iv) With increasing depth, the amplitude of temperature oscillation decreases. Therefore, at a particular depth inside the solid the amplitude becomes negligibly small. Consequently a solid thicker than this particular depth is of not any importance as far as variation in temperature is concerned.

(v) The temperature amplitude depends both upon the depth x and the factor $\sqrt{n/\alpha}$. Evidently if $\sqrt{n/\alpha}$ is large, equation 6.19 is equally valid for thin solid rods.

EXAMPLE 6.36

The temperature variation of a thick brick wall during periodic heating or cooling follows a sinusoidal waveform. During a period of 24 hours, the surface temperature ranges from 25°C to 75°C. Workout the time lag of the temperature wave corresponding to a point located at 25 cm from the wall surface. Thermo-physical properties of the wall material are; thermal conductivity $k = 0.62$ W/mK; specific heat $c = 450$ J/kgK and density $\rho = 1620$ kg/m³.

Solution : The time lag of the temperature wave is prescribed by the relation :

$$\Delta\tau = \frac{x}{2}\sqrt{\frac{1}{\alpha\pi n}}$$

where x = depth = 0.25 m

n = frequency

$= 1/24 = 0.04167$ cycles/hr

$$\alpha = \frac{k}{\rho c} = \frac{0.62}{450 \times 1620}$$

$= 8.505 \times 10^{-7}$ m²/s

$= 3.062 \times 10^{-3}$ m²/hr

$$\therefore \quad \Delta\tau = \frac{0.25}{2}\sqrt{\frac{1}{3.062 \times 10^{-3} \times \pi \times 0.04167}}$$

$= 6.245$ hr

EXAMPLE 6.37

A single cylinder 2-stroke engine operates at 1500 rpm. Calculate the depth where the temperature wave due to variation in cylinder temperature is damped to 1% of its surface value. For the cylinder material, thermal diffusivity $\alpha = 0.042$ m²/hr.

Solution : At any depth x, the amplitude of temperature excess is prescribed by the relation,

$$\theta_{x,a} = \theta_{s,a} \exp\left[-x\sqrt{\frac{\pi n}{\alpha}}\right]$$

where frequency $n = 1500 \times 60$

$= 9 \times 10^4$ cycles/hr

$$\therefore \quad 0.01 = \exp\left[-x\sqrt{\frac{\pi \times 9 \times 10^4}{0.042}}\right]$$

$$x = -\frac{\log_e 0.01}{\sqrt{(\pi \times 9 \times 10^4)/0.042}}$$

$= 0.1775 \times 10^{-2}$ m $= 0.1775$ cm

EXAMPLE 6.38

A cold wave of 2 weeks duration causes a temperature drop of 25°C at the surface and the temperature variation follows a sinusoidal waveform. Wokout the drop in temperature at a depth of 1.2 m, and the time lag for a soil having thermal diffusivity $\alpha = 0.0018$ m²/hr.

(b) If the base temperature is 6.5°C, calculate the minimum burial depth you would recommend in laying water mains to avoid freezing of water.

Solution : At any depth x, the amplitude of temperature excess is prescribed by the relation:

$$\theta_{x,a} = \theta_{s,a} \exp\left[-x\sqrt{\frac{\pi n}{\alpha}}\right] \qquad ...(i)$$

where x = depth = 1.2 m

$\theta_{s,a}$ = amplitude at the surface

$= 25/2 = 12.5$°C

α = thermal diffusivity

$= 0.0018$ m²/hr

and frequency $n = \dfrac{1}{2 \times 7 \times 24}$

$= 0.00297$ cycles/hr

$$\therefore \quad \theta_{x,a} = 12.5\exp\left[-1.2\sqrt{\frac{\pi \times 0.00297}{0.0018}}\right]$$

$= 0.810$°C

$$\text{Time lag } \Delta\tau = \frac{x}{2}\sqrt{\frac{1}{\alpha\pi n}}$$

$$= \frac{1.20}{2}\sqrt{\frac{1}{0.0018 \times \pi \times 0.00297}}$$

$= 146.4$ hrs

(b) Amplitude of temperature excess above freezing point of water = 6.5°C. The depth x for the water mains is then computed from the relation (i),

$$6.5 = 12.5 \exp\left[-x\sqrt{\frac{\pi \times 0.00297}{0.0018}}\right]$$

$= 12.5 \exp[-2.276\, x]$

or $\quad x = 0.287$ m $= 28.7$ cm

EXAMPLE 6.39

The thermal diffusivity of soil is to be estimated at a given location by noting the prepagation characteristics of the diurnal (period of 24 hours) temperature variations of the soil. This variation is assumed to be due to sinusoidal variation in surface temperature. Thermocouple junctions are buried at depths of 150 mm and 300 mm. If the temperature amplitudes at these depths are 5°C and 2.5°C respectively, compute the thermal diffusivity of the soil.

Solution : The amplitude of temperature excess at any depth is prescribed by the relation

$$\theta_{x,a} = \theta_{s,a} \exp\left[-x\sqrt{\frac{\pi n}{\alpha}}\right]$$

where $\theta_{x,a}$ is the temperature, n is the frequency and α is the thermal diffusivity of the soil. Since the temperature changes are taking place in 24 hours,

$$n = \frac{1}{24} = 0.0417 \text{ cycles/hr}$$

Inserting the data appropriate to the two locations,

$$5 = \theta_{s,a} \exp\left[-0.15\sqrt{\frac{\pi \times 0.0417}{\alpha}}\right]$$

$$\text{and} \quad 2.5 = \theta_{s,a} \exp\left[-0.3\sqrt{\frac{\pi \times 0.0417}{\alpha}}\right]$$

Upon division, we obtain

$$2 = \exp\left[0.15\sqrt{\frac{\pi \times 0.0417}{\alpha}}\right]$$

$$\text{or} \quad 0.15\sqrt{\frac{\pi \times 0.0417}{\alpha}} = \log_e 2 = 0.693$$

Solution gives :

Thermal diffusivity $\alpha = 0.00613$ m²/hr

6.6. TRANSIENT CONDUCTION WITH GIVEN TEMPERATURE DISTRIBUTION

Quite often, the temperature distribution at some instant of time is known for the one-dimensional unsteady heat conduction through a solid. The prescribed temperature distribution is combined with the relevant conduction equation to evaluate either the rate of heat flow or the rate of temperature variation at different points within the solid.

EXAMPLE 6.40

The temperature distribution at a certain time instant through a 50 cm thick wall is prescribed by the relation

$$t = 300 - 500x + 100x^2 + 140x^3$$

where temperature t is in degree celsius and the distance x in metres has been measured from th

hot surface. If thermal conductivity of the wall material is 20 kJ/m-hr-deg, calculate the heat energy stored per unit area of the wall.

Solution : The temperature distribution is given as :

$$t = 300 - 500x + 100x^2 + 140x^3$$

$$\therefore \quad \frac{dt}{dx} = -500 + 200x + 420x^2$$

Heat entering the wall from the face being heated ($x = 0$)

$$Q_{in} = -kA\left(\frac{dt}{dx}\right)_{x=0}$$

$$= -20 \times 1\,(-500) = 10000 \text{ kJ/hr}$$

Heat leaving the wall face at x = 50 cm = 0.5 m

$$Q_{out} = -kA\left(\frac{dt}{dx}\right)_{x=0.5}$$

$$= -20 \times 1\,[-500 + 200 \times 0.5 + 420 \times (0.5)^2]$$

$$= 5900 \text{ kJ/hr}$$

$\therefore$ Heat storage rate

$$= Q_{in} - Q_{out}$$

$$= 10000 - 5900 = \mathbf{4100 \text{ kJ/hr}}$$

EXAMPLE 6.41

A large plane wall, 40 cm thick and 8 m^2 area, is heated from one side and temperature distribution at a certain time instant is approximately prescribed by the relation

$$t = 80 - 60x + 12x^2 + 25x^3 - 20x^4$$

where temperature t is in degree celsius and the distance x is in metres. Make calculations for the:

(i) heat energy stored in the wall in unit time

(ii) rate of temperature change at 20 cm distance from the side being heated and

(iii) location where the rate of heating or cooling is maximum.

For the wall material :

thermal conductivity k = 6 W/mK and thermal diffusivity α = 0.02 m^2/hr.

Solution : The temperature distribution is given as :

$$t = 80 - 60x + 12x^2 + 25x^3 - 20x^4$$

$$\therefore \quad \frac{dt}{dx} = -60 + 24x + 75x^2 - 80x^3$$

$$\frac{d^2t}{dx^2} = 24 + 150x - 240x^2$$

$$\frac{d^3t}{dx^3} = 150 - 480x$$

(i) heat entering the wall from the face being heated ($x = 0$)

$$Q_{in} = -kA\left(\frac{dt}{dx}\right)_{x=0}$$

$$= 6 \times 8\,(-60) = 2880 \text{ W}$$

Heat leaving the face at x = 40 cm = 0.4 m

$$Q_{out} = -kA\left(\frac{dt}{dx}\right)_{x=0.4}$$

$$= -6 \times 8\,[-60 + 24 \times 0.4 + 75 \times 0.4^2 - 80 \times 0.4^3]$$

$$= 2088.96 \text{ W}$$

$\therefore$ Heat storage rate

$$= Q_{in} - Q_{out}$$

$$= 2880 - 2088.96 = \mathbf{791.04 \text{ W}}$$

(ii) Rate of temperature change is given by

$$\frac{dt}{d\tau} = \alpha\,\frac{d^2t}{dx^2}$$

$$\therefore \quad \left(\frac{dt}{d\tau}\right)_{x=0.2} = \alpha\left(\frac{d^2t}{dx^2}\right)_{x=0.2}$$

$$= 0.02\,[24 + 150 \times 0.2 - 240 \times (0.2)^2]$$

$$= \mathbf{0.888°C/hr}$$

(iii) The rate of heating or cooling would be maximum at a location where

$$\frac{d}{dx}\left(\frac{dt}{d\tau}\right) = 0$$

$$\text{or} \quad \frac{d}{dx}\left(\alpha\frac{d^2t}{dx^2}\right) = 0$$

$$\text{or} \quad \frac{d^3t}{dx^3} = 0$$

or $150 - 480x = 0$; x = **0.3125 m**

EXAMPLE 6.42

At a certain time instant, the temperature distribution in a long cylindrical fire tube can be represented approximately by the relation

$$t = 650 + 800r - 4250r^2$$

where temperature t is in degree celsius and radius r is in metre.

Make calculations for rate of heat flow and rate of change of temperature at the inside and outside surface of the tube, and the heat energy stored inside. The tube measures : inside radius 25 cm, outside radius 40 cm and length 1.5 m. For the tube material

$$k = 5.5 \text{ W/mK},$$

$$\alpha = 0.004 \text{ m}^2\text{/hr}$$

Solution : The temperature distribution is given as

$$t = 650 + 800r - 4250r^2$$

$$\therefore \quad \frac{dt}{dr} = 800 - 8500r$$

$$\frac{d^2t}{dr^2} = -8500$$

(a) Heat flow rates at the two surfaces can be obtained by invoking Fourier law of heat conduction:

$$Q = -kA\left(\frac{dt}{dr}\right)$$

$\therefore$ At the outside surface (r = 0.40 m)

$$Q_{out} = -kA\left(\frac{dt}{dr}\right)_{r=0.40}$$

$$= -55 \times (2\pi \times 0.40 \times 1.5)(800 - 8500 \times 0.4)$$

$$= 5.388 \times 10^5 \text{ W}$$

At the inside surface (r = 0.25 m)

$$Q_{in} = -kA\left(\frac{dt}{dr}\right)_{r=0.25}$$

$$= -55 \times (2\pi \times 0.25 \times 1.5)(800 - 8500 \times 0.25)$$

$$= 1.716 \times 10^5 \text{ W}$$

Rate of heat storage

$$= Q_{in} - Q_{out}$$

$$= 1.716 \times 10^5 - 5.388 \times 10^5$$

$$= -3.672 \times 10^5 \text{ W}$$

The negative sign indicates that there is depletion of heat energy, *i.e.*, the heat is being lost by the fire tube.

(b) For a cylindrical object, the rate of temperature change is

$$\frac{dt}{d\tau} = \alpha\left[\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr}\right]$$

At the inside surface, (r = 0.025 m)

$$\frac{dt}{d\tau} = 0.004\left[-8500 + \frac{1}{0.25}(800 - 8500 \times 0.25)\right]$$

$$= -55.2° \text{ C/hr}$$

At the outside surface (r = 0.040 m)

$$\frac{dt}{d\tau} = 0.004\left[-8500 + \frac{1}{0.40}(800 - 8500 \times 0.40)\right]$$

$$= -60°\text{C/hr}$$

SALIENT POINTS

1. Conduction of heat in unsteady state refers to the transient conditions wherein the heat flow and the temperature distribution at any point of the system vary continuously with time.
2. Transient conduction occurs in
   - heating or cooling of metal billets
   - cooling of IC engine cylinder
   - heat treatment of metals by quenching
   - brick burning and vulcanisation of rubber
   - starting and stopping of various heat exchange units in power installations

   Change in temperature during unsteady state may follow a periodic or non-periodic variation.
3. Lumped heat capacity analysis is employed where the internal resistance for conduction is very small compared to the external resistance for convection.

   In such situations, the temperature-time history of the heating or cooling of a body is given

$$\frac{t - t_a}{t_i - t_a} = \exp\left[-\frac{hA}{\rho Vc}\tau\right]$$

where the body has density ρ, volume V, specific heat c, surface area A, initial temperature t_i and suddenly exposed to an environment at t_a. The parameter t represents the temperature of the body at time τ.

The dimensionless argument of the exponential can be arranged as

$$\frac{hA}{\rho Vc}\tau = \frac{hV}{kA}\left(\frac{A^2k}{\rho V^2c}\tau\right)$$

$$= \left(\frac{hl}{k}\right)\left(\frac{\alpha\tau}{l^2}\right)$$

Here $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity of the solid and l is the characteristic length equal to the ratio of volume of solid to its surface area.

The non-dimensional factor $\frac{hl}{k}$ is called the Biot number B_i, and it gives an indication of the ratio of internal conduction resistance to the surface convection resistance

$$B_i = \frac{\text{internal conduction resistance}}{\text{external convection resistance}}$$

$$= \frac{l/kA}{1/hA} = \frac{hl}{k}$$

The non-dimensional factor $\frac{\alpha\tau}{l^2}$ is called the Fourier modulus F_0. It signifies the degree of penetration of heating or cooling effect through the solid.

The temperature-time history of the body can be then expressed as

$$\frac{t - t_a}{t_i - t_a} = \exp(-B_i F_0)$$

Further,

Instantaneous heat flow rate:

$$Q_i = -hA(t_i - t_a)\exp(-B_i F_o)$$

Total or cummulative heat transfer

$$Q = \rho Vc(t_i - t_a)[\exp(-B_i F_o) - 1]$$

4. The response of a thermocouple is defined as the time required by the thermocouple to reach the source temperature to which it is exposed. The time required by a thermocouple to reach to 63.2% of the value of initial temperature difference is called its sensitivity. That is

$$\frac{t - t_a}{t_i - t_a} = \exp\left[-\frac{hA}{\rho Vc}\tau\right]$$

$$= 1 - 0.632 = 0.368 = \exp(-1)$$

The parameter $\frac{\rho Vc}{hA}$ has thus units of time and is called time constant of the system.

5. For semi-infinite solid originally at temperature t_i and suddenly changed to temperature t_a at time $t = 0$, the temperature distribution at any time τ at a plane parallel to and at distance x from the surface is given by

$$\frac{t - t_a}{t_i - t_a} = \text{erf}\left[\frac{x}{2\sqrt{\alpha\tau}}\right]$$

The term on the right hand side is called the Gaussian error function.

The instantaneous heat flow rate at a given x-location within the semi-infinite body at a specified time is

$$Q_i = -kA(t_i - t_a)\frac{\exp\left[-\frac{x^2}{4\alpha\tau}\right]}{\sqrt{\pi\alpha\tau}}$$

At the surface ($x = 0$), the heat flow is

$$Q_{surface} = -\frac{kA(t_i - t_a)}{\sqrt{\pi\alpha\tau}}$$

The total heat flow is

$$Q = -1.13\,kA(t_i - t_a)\sqrt{\frac{\tau}{\alpha}}$$

6. The solutions to the transient heat flow in infinite flat plates, infinite cylinders and spheres with finite conduction and convection resistances are available in the form of Heisler charts. These charts are valid if $F_o > 0.2$ and

$$0 < \frac{1}{B_i} < 100.$$

The different types of Heisler plots are :

(i) $\frac{t_o - t_a}{t_i - t_a}$ at the centre versus F_o with $\frac{1}{B_i}$ as a parameter. Here t_0 is the temperature at $x = 0$ or $r = 0$.

These plots are used to find the temperature at the centre of solid from known values of F_o and B_i.

(ii) $\frac{t - t_a}{t_o - t_a}$ versus $\frac{1}{B_i}$ with $\frac{x}{l}$ or $\frac{r}{R}$ as a parameters.

These charts along with the plots mentioned at (i) above help to determine temperature t at any location from known values of B_i, F_o and $\frac{x}{l}$ or $\frac{r}{R}$. Use is made of the following relationship

$$\frac{t - t_a}{t_i - t_a} = \left[\frac{t_o - t_a}{t_i - t_a}\right] \times \left[\frac{t - t_a}{t_o - t_a}\right]$$

7. For systems with periodic variation of surface temperature,

$$\theta_{x,\tau} = \theta_{s,a}\exp\left[-x\sqrt{\frac{\pi n}{\alpha}}\right]$$

$$\sin\left[2\pi n\tau - x\sqrt{\frac{\pi n}{\alpha}}\right]$$

where $\theta_{s,a}$ is amplitude of temperature excess, n is frequency of temperature wave.

At the surface ($x = 0$), the temperature excess becomes zero at $\tau = 0$. However at any depth $x > 0$, a time $\tau = \frac{x}{2}\left[\frac{1}{\sqrt{\alpha\pi n}}\right]$ would elapse before the temperature excess $\theta_{x,\tau}$ becomes zero. The time interval between the two instants is called the time lag.

$$\text{Time lag, } \Delta\tau = \frac{x}{2}\left[\frac{1}{\sqrt{\alpha\pi n}}\right], \text{ and}$$

Temperature amplitude at depth x is

$$\theta_{x,a} = \theta_{s,a}\exp\left[-x\sqrt{\frac{\pi n}{\alpha}}\right]$$

When putting the above expressions, it has been presumed that surface temperature oscillates about the mean temperature level t_m in accordance with the relation

$$\theta_{s,\tau} = \theta_{s,a}\sin(2\pi n\tau)$$

where $\theta_{s,\tau}$ is excess over the mean temperature

$$\theta_{s,\tau} = t_{s,\tau} - t_m$$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. What is meant by transient heat transfer? Mention some of the situations where transient conduction occurs.
2. State the difference between periodic and non-periodic variation in temperature during unsteady state. Give noteable examples of both the variations.
3. What is lumped system analysis? Under what conditions it is applicable?
4. (i) Define Biot and Fourier numbers, and point out their physical significance.
   (ii) What is the limiting value of Biot number for making an assumption that the body is at uniform temperature?
5. Draw the simple electrical circuit which is analogous to transient heat transfer from a body at uniform temperature.
6. Define the term thermal time constant in the analysis of heat transfer by lumped capacity method.
7. Set up expressions for instantaneous and total heat flow rate for transient heat conduction in solids with infinite thermal conductivity.
8. What is meant by semi-infinite medium? Give examples of solid bodies that can be treated as semi-infinite mediums for heat transfer purposes.
9. What is an error function? How it is applied for analysing transient heat conduction in a semi-infinite body?
10. Explain a method to obtain solutions to the problems of heat conduction in infinitely thick solids.
11. What are Heisler charts? How these charts are used to obtain temperature distribution when both conduction and convection resistances are almost of equal importance.
12. Explain a method to obtain solution to the problem of heat conduction when the surface temperature variations are periodic in nature.
13. A 7.5 cm diameter orange, subjected to a cold air environment, may be idealised as a sphere. Determine the suitability of a lumped analysis for predicting the temperature of the orange during cooling. The orange properties are : thermal conductivity $k = 0.597$ W/m-deg and connective coefficient $h = 11.35$ W/m^2-deg.

(Ans. B_i = 0.2376; lumped parametric analysis will not be accurate

14. Prove that the temperature of a body at any time t during newtonian heating or cooling is given by the relation

$$\frac{t-t_a}{t_i-t_a} = \exp\,[-B_i\,F_0]$$

where B_i and F_0 are the Biot and Fourier modulus respectively; t_a is the ambient temperature and t_i is the initial temperature of the body.

A solid aluminium (k = 210 W/mK) cube at 100°C has been exposed to a convective flow resulting in convective coefficient

$$h = 25 \text{ W/m}^2\text{K}$$

Calculate the maximum edge dimension of the cube if the lumped analysis is to be accurate within 5%, *i.e.*, limiting value of Biot number is 0.1.

(Ans. 5.04 m)

15. Prove that for a body whose thermal resistance is zero, the temperature required for cooling or heating can be obtained from the relation

$$\frac{t-t_a}{t_i-t_a} = \exp\left[-\frac{hA}{Wc_p}\right]$$

where t is the temperature of the body at any time τ, t_i is the initial temperature of the body, t_a is the atmospheric temperature, A is the surface area and W is the weight of the body, c_p is the specific heat of the body material, and h is the heat transfer coefficient on the surface of the body.

16. Estimate the time required to cook a carrot in boiling water at atmospheric pressure. The carrot is initially at the room temperature 25°C and the cooking requirement stipulates that a minimum temperature of 90°C is reached at the centre of carrot. Treat the carrot as a long cylinder of 20 mm diameter and having the following thermo-physical properties :

$$\rho = 1025 \text{ kg/m}^3,\quad c_p = 4000 \text{ J/kgK}$$

and $k = 0.48$ W/mK

and the convective heat transfer coefficient

$$h = 2000 \text{ W/m}^2\text{K}.$$

A steel plate of thickness 20 mm is heated in the furnace to a temperature of 500°C and then exposed to ambient air at a temperature of 20°C. Determine the interval of time after the expiry of which the steel plate acquires a temperature differing from the ambient temperature by not more than 1%. The relevant thermo-physical properties are :

$$k = 45.5 \text{ W/mK},\quad \rho = 7900 \text{ kg/m}^3$$

and $c_p = 0.46$ kJ/kgK

and the local coefficient of heat transfer from the surface of plate to the surrounding air is

$$h = 35 \text{ W/m}^2\text{K}$$

(Ans. 2.25 hrs)

18. The bulb of a mercury thermometer may be idealised as a sphere 1 mm radius. It is intended to use this thermometer for measuring unsteady temperature (a temperature changing with time) of a fluid. Indicate the suitability of this thermometer for the task if temperature variations in the fluid have a time period less than 3.5 seconds. For mercury:

$$k = 10 \text{ W/m-deg};\quad h = 10 \text{ W/m}^2\text{-deg}$$

and $\alpha = 5 \times 10^{-5}$ m²/s

Also workout the diameter of a thermocouple which would record the temperature variations faithfully. For the thermocouple material

$$k = 95 \text{ W/mK},\quad \alpha = 11 \times 10^{-5} \text{ m}^2/\text{s}$$

and $h = 8$ W/m²K

(Ans. Not suitable, 0.0972 mm)

19. Long cylindrical billets of brass 100 mm in diameter are heated to 600°C preparatory to extrusion process. If the minimum metal temperature permissible for extrusion is 480°C, how long these metal billets be exposed to ambient conditions at 27°C temperature and 85 W/m² K film conductance ? What would be the temperature at the cylinder axis at that time ? For brass :

$$k = 107 \text{ W/mK and } \alpha = 0.0344 \text{ m}^2/\text{hr}$$

20. A long steel shaft of 12 cm diameter is initially at a temperature of 25°C. Determine the time required for the temperature at the axis of the shaft to reach 750°C when the shaft is placed into a furnace at 800°C temperature. Also calculate the surface temperature of the shaft at the end of heating. Thermo-physical properties of steel shaft are :

Thermal conductivity k = 20 W/mK; thermal diffusivity α = 0.0212 m²/hr and the local coefficient of heat transfer at the surface of shaft in the furnace h = 135 W/m²K.

21. A large disc of 15 cm thickness is initially held at 200°C, and then suddenly exposed to ambient conditions at 20°C temperature. What would be the temperature at the centre of disc 10 minutes after this change ? The following properties are given :

$$k = 48.7 \text{ W/mK};\quad \rho = 1600 \text{ kg/m}^3$$
$$c_p = 1046 \text{ J/kgK};\quad h = 23.5 \text{ W/m}^2\text{K}$$

(Ans. 173°C)

22. A 10 cm diameter cylindrical bar, heated in the furnace to a uniform temperature of 200°C, is allowed to cool in an environment with convective coefficient 150 W/m²K and temperature 40°C. Determine (*i*) temperature required to cool the centre of bar to 50°C, (*ii*) temperature of the surface at this instant. For the material of the bar : thermal conductivity k = 50 W/m K and thermal diffusivity $\alpha = 2.0 \times 10^{-5}$ m²/s.

(Ans. 17.92 minutes, 49.3°C)

23. The soil, initially at a uniform temperature of 20°C, has its surface temperature suddenly raised to and maintained at 850°C. Workout the temperature at a depth of 25 cm after a period of 5 hours has elapsed at this surface condition. For the given soil

$$\alpha = 5 \times 10^{-7} \text{ m}^2/\text{s}$$

(Ans. 73.95°C)

24. A thick wall (k = 0.95 W/mK) is initially at a uniform temperature of 20°C. Suddenly the surface temperature changes to 800°C and remains constant thereafter. Make calculations for the temperature in a place 20 cm from the surface and the rate of heat flow into 1 m² of this plane 10 hours after the operation of change in surface temperature. Also find the total energy taken up by the wall during the 10 hours period. Take thermal diffusivity

$$\alpha = 2.75 \times 10^{-2} \text{ m}^2/\text{hr}$$

25. A thick concrete retaining wall is in contact with air on its exposed side. During a particular season, the daily variation in temperature of the air is sinusoidal over the range 10 to 25°C, and the expected convection coefficient is 11.35 W/m²K. Find the variation in temperature at the surface of the wall and at a point 50 mm inside the wall.

$$\alpha = 1.486 \times 10^{-3} \text{ m}^2/\text{hr}$$

and $k = 1.73$ W/mK.

26. The surface temperature of earth at a certain locality is measured over 24 hours period and found to range from 20°C to 35°C. Find the amplitude of temperature variation and time lag of temperature wave at a depth of 15 cm. Assume that a sinusoidal variaton exists at the surface and that the earth has an average thermal diffusivity of 1.39×10^{-3} m²/hr.

27. A thermocouple junction is in the form of a sphere of radius 1.2 mm and is initially at 25°C. The thermo-physical properties of the thermocouple material are

$$\rho = 7850 \text{ kg/m}^3$$

and $c = 0.4$ kJ/kg-deg.

The thermocouple is kept in hot water stream at 75°C. If the surface heat transfer coefficient is 145 W/m²-deg, calculate (*a*) the time constant of the thermocouple, and (*b*) the temperature attained by the thermocouple often one minute.

(Ans. 8.66 sec ; 74.97°C)

B. Fill in the blanks with appropriate word/words :

1. The term .... state designates a phenomenon which is time dependent.
2. The heating of an ingot in a furnance represents .... variation in temperature.
3. In .... analysis of heating or cooling process, the temperature is considered to be uniform at a given time.
4. The non-dimensional parameter which gives an indication of the ratio of internal conduction resistance to surface convection resistance is called ....
5. The .... is a non-dimensional factor and it signifies the degree of penetration of heating or cooling effect through a solid.
6. The characteristic length used in calculating the Biot and Fourier numbers is equal to the ratio of .... of the solid to its ....
7. The time required by a thermocouple to reach 63.2% of the initial temperature difference is called the .... of the thermocouple.
8. When Biot number exceeds 0.1 but is less than 100, use is made of .... for the solution of transient heat conduction.
9. A solid which extends itself infinitely in all directions of space is termed as an ....
10. .... refers to the location of a point where the temperature change is within 1 percent of the change in surface temperature.

Answers : 1. transient; 2. non-periodic; 3. Lumped parameter; 4. Biot number; 5. Fourier number; 6. volume, surface area; 7. sensitivity; 8. Heisler charts; 9. infinite solid; 10. penetration depth.

C. Multiple choice questions :

1. Transient conduction means
   (*a*) very little heat transfer
   (*b*) heat transfer for a short time
   (*c*) heat transfer with a very small temperature difference
   (*d*) conduction when the temperature at a point varies with time

2. Which of the following statements is not correct?
   In a transient flow process
   (*a*) the rates of inflow and outflow of mass are different
   (*b*) the state of matter inside the control volume varies with time
   (*c*) there can be work and heat interactions across the control volume
   (*d*) there is no accumulation of energy inside the control volume

3. Which of the following changes in temperature during unsteady state follow a periodic variation?
   (*a*) temperature variations in the cycle of an internal combustion engine
   (*b*) heating of an ingot in a furnace
   (*c*) cooling of bars, blanks and metal billets in steel works
   (*d*) variation of temperature of a building during a full day period of 24 hours
   (*e*) heat processing of regenerators whose packings are alternately heated by flue gases and cooled by air

Lumped parameter analysis of transient heat conduction in solid stipulates
1. infinite thermal conductivity
2. negligible temperature gradient, *i.e.*, practically uniform temperature within the solid
3. small conduction resistance
4. predominance of convective resistance

Which of the statements made above are correct?

(*a*) 1, 3 and 4 (*b*) 2 and 3
(*c*) 1 and 2 (*d*) 1, 2, 3 and 4

5. In the lumped parameter model, the temperature variation with time is
   (*a*) linear (*b*) cubic
   (*c*) exponential (*d*) sinusoidal

6. The temperature of a body at any time during newtonian heating or cooling is stated as
   (*a*) $\dfrac{t - t_a}{t_i - t_a} = \exp(-B_i F_o)$
   (*b*) $\dfrac{t - t_a}{t_i - t_a} = \exp[-B_i F_o/2]$
   (*c*) $\dfrac{t - t_a}{t_i - t_a} = \exp\left[-\sqrt{B_i F_o}\right]$
   (*d*) $\dfrac{t - t_a}{t_i - t_a} = \exp[-(B_i F_o)^2]$

   where t_i is the body temperature at the commencement of heating or cooling process, t_a is the temperature of the surroundings, B_i and F_o are the non-dimensional Biot number and Fourier number respectively

7. In transient heat conduction, the two significant dimensionless parameters are
   (*a*) Reynolds number and Prandtl number
   (*b*) Biot number and Fourier number
   (*c*) Reynolds number and Biot number
   (*d*) Fourier number and Reynolds number

8. Which of the following dimensionless number gives an indication of the ratio of internal (conduction) resistance to the surface (convection) resistance ?
   (*a*) Fourier number (*b*) Biot number
   (*c*) Nusselt number (*d*) Stanton number

9. Lumped parameter analysis for transient heat conduction is essentially valid for
   (*a*) $B_i < 0.1$ (*b*) $0.1 < B_i < 0.5$
   (*c*) $1 < B_i < 10$ (*d*) $B_i \to \infty$
   where B_i is the non-dimensional Biot number.

10. In the non-dimensional Biot number, the characteristic length is the ratio of
    (*a*) volume of solid to its surface area
    (*b*) surface area to volume of solid
    (*c*) surface area to perimeter of solid
    (*d*) perimeter to surface area of solid

11. Which of the followings is the wrong value of characteristic length l which appears in the Biot number hl/k and the Fourier number $\alpha\tau/l^2$?
    (*a*) $l = R/3$ in case of a sphere of radius R
    (*b*) $l = R/2$ in case of a cylinder of radius R and length L
    (*c*) $l = L/6$ in case of a cube with each side of length L
    (*d*) $l = b/2$ for a flat plate of thickness δ, breadth b and height h

12. The temperature distribution during transient heat conduction in a solid does not depend upon
    (*a*) location of point within the solid
    (*b*) Biot number $h\,l/k$
    (*c*) Prandtl number $\mu\, c_p/k$
    (*d*) Fourier number $\alpha\tau\ /l^2$

13. The curve for unsteady state cooling or heating of bodies is
    (*a*) parabolic curve asymptotic to time axis
    (*b*) exponential curve asymptotic to time axis
    (*c*) exponential curve asymptotic both to time and temperature axes
    (*d*) hyperbolic curve asymptotic both to time and temperature axes

14. The time constant of a thermocouple is the time taken to
    (*a*) attain the final value to the measured
    (*b*) attain 63.2% of the value of initial temperature difference
    (*c*) attain 50% of initial temperature difference
    (*d*) minimum time taken to record a temperature reading

15. Heisler charts are used to determine the transient heat flow rate and temperature distribution when
    (*a*) solids possess infinitely large thermal conductivity
    (*b*) internal conduction resistance is small and the convective resistance is large
    (*c*) internal conduction resistance is large and the convective resistance is small
    (*d*) both conduction and convection resistances are almost of equal importance.

16. A large concrete slab 1 m thick has one-dimensional temperature distribution precribed as

    $$T = 4 - 10x + 20x^2 + 10x^3$$

    where T is temperature and x is distance from one face towards other face of the wall. If the slab material has thermal diffusity of

    $$2 \times 10^{-3}\ \text{m}^2/\text{hr},$$

    what is the rate of change of temperature at the other face of the wall?
    (*a*) 0.1°C/hr (*b*) 0.2°C/hr
    (*c*) 0.3°C/hr (*d*) 0.4°C/hr

Answers :

1. (*d*)	2. (*d*)	3. (*a*)	4. (*d*)	5. (*c*)
6. (*a*)	7. (*b*)	8. (*b*)	9. (*a*)	10. (*a*)
11. (*d*)	12. (*c*)	13. (*b*)	14. (*b*)	15. (*d*)
16. (*b*)				

HINTS AND COMMENTS

5(*c*):

In lumped parameter model, the temperature variation with time follows the exponential curve prescribed by the relation

$$\frac{t - t_a}{t_i - t_a} = \exp\left[-\frac{hA}{\rho V c}\tau\right]$$

11(*d*):

The characteristic length l equals the ratio of the volume of solid to its surface area. For a flat plate, the heat transfer occurs from both sides, *i.e.*, area exposed for heat transfer is 2 bh

$$\therefore \quad l = \frac{\delta\, bh}{2\, bh} = \frac{\delta}{2}$$

i.e., half the plate thickness.

13(b):

Refer to the relation

$$\frac{\theta}{\theta_o} = e^{-\frac{hA\tau}{\rho c_p V}}$$

which represents an exponential curve asymptotic to time axis τ.

14(b):

The time constant of a thermocouple represents the time required to attain 63.2% of the value of initial temperature difference.

15(b):

$$T = 4 + 10\,x + 20\,x^2 - 10\,x^3$$

$$\frac{dT}{dx} = 10 + 40\,x - 30\,x^2$$

$$\frac{d^2T}{dx^2} = 40 - 60\,x$$

For one-dimensional unsteady heat flow with no internal heat generation.

$$\left.\frac{d^2T}{dx^2}\right|_{x=1} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\text{or} \quad 40 - 60 \times 1 = \frac{1}{2 \times 10^{-3}} \times \frac{\partial T}{\partial \tau}$$

$\therefore$ Rate of change of temperature with time

$$\frac{\partial T}{\partial \tau} = 2 \times 10^{-3} \times 100$$

$$= 0.2^\circ C/hr$$

❑❑❑

Chapter

7

Radiation : Processes and Properties

Learning objectives : Attention has been focussed in this chapter to enable the reader to understand the concepts of

- *salient features and characteristics of radiation*
- *absorptivity, reflectivity and transmissivity*
- *wavelength distribution of black body radiation; Planck's law*
- *total emissive power; Stefan Boltzman law and Wien's displacement law*
- *Kirchoff's law; emissivity; gray body*
- *intensity of radiation and Lambert's cosine law*

Thermal radiation is the transmission of thermal energy without any physical contact between the bodies involved. Unlike heat transfer by conduction and convection, transport of thermal radiation does not necessarily affect the material medium between the heat source and the receiver. An intervening medium is not even necessary and the radiation can be affected through vacuum or a space devoid of any matter. Radiation exchange, in fact, occurs most effectively in vacuum. A material present between the heat source and the receiver would either reduce or eliminate entirely the propagation of radiant energy.

Energy released by a radiating surface is not continuous but is in the form of successive and separate (discrete) packets or quanta of energy called **photons**. The photons are propagated through space as rays; the movement of swarm of photons is described as the electromagenetic waves. The photons (as carriers of energy) travel with unchanged frequency in straight paths and with speed equal to that of light. For propagation in vacuum $c = 3 \times 10^8$ m/s. When the photons approach the receving surface, there occurs reconversion of wave motion into thermal energy which is partly absorbed, reflected or transmitted through the receiving surface. The magnitude of each fraction depends upon the nature of the surface that receives the thermal radiation.

Attention would be restricted in this chapter to the radiation processes which occur at a single surface.

7.1. SALIENT FEATURES AND CHARACTERISTICS OF RADIATION

Some salient features and characteristics of radiation are enumerated below :

(i) The electromagnetic waves are emitted as a result of vibrational and rotational movements of the molecular, atomic or sub-atomic particles comprising the matter. The emission occurs when the body is excited by an oscillating electrical signal, electronic c

neutronic bombardment, chemical reactions etc. The emission of thermal radiations is associated with thermally excited conditions which depend upon the nature of surface and its absolute temperature.

(*ii*) The distinction between one form of radiation and another lies only in its frequency and wavelength which are related by

$$c \text{ (speed of light)} = \lambda \text{ (wavelength)} \times f \text{ (frequency)}$$

Consequently longer wavelengths correspond to lower frequencies and a shorter wavelengths to higher frequencies. Again, a high temperature body will have a high frequency quantum and so shorter wavelengths. Each photon can be thought of as a particle (just like the molecule of a gas) having mass m, energy e and momentum (= mass × velocity)

$$e = mc^2 = hf$$

or $$m = \frac{hf}{c^2}$$

and momentum $= mc = \dfrac{hf}{c}$

where the Planck's constant h has a value (6.625×10^{-34} J/s). The wavelength λ may be expressed in metres, micron (μ or μm) or in angstrom (Å).

$$1\mu = 10^{-6} \text{ m} = 10^4 \text{ Å}$$

(*iii*) The general phenomenon of radiation covers the propagation of electromagnetic waves of all the wavelengths, from short wavelength gamma rays, X-rays and ultravoilet radiation to the long wavelength microwaves and radio waves. Thermal radiation is limited to range of wavelength between 0.1 and 100 μm; it thus includes the entire visible and infrared, and a part of the ultraviolet spectrum (Fig. 7.1).

The sun with an effective surface temperature of 5600°C emits most of its radiations at the extreme lower end of the spectrum 0.1 to 4 μm. The radiations from a lamp filament are in the frequency range 1 to 10 μm. Most solids and liquids have a continuous spectrum; they emit radiations of all the wavelengths. Gases and vapours radiate energy only at certain bands of wavelength and hence are called the **selective emitters**. The emission of thermal radiation depends upon the nature, temperature and state of the emitting surface . However with gases the dependence is also upon the thickness of emitting layer and the gas pressure.

The common electromagnetic waves and their wavelength bands have been indicated below:

1. Cosmic rays upto 4×10^{-7} μm
2. Gamma rays 4×10^{-7} to 1.4×10^{-4} μm
3. X-rays 1×10^{-5} to 2×10^{-2} μm
4. Ultraviolet rays 1×10^{-2} to 3.9×10^{-1} μm
5. Visible light 3.9×10^{-1} to 7.8×10^{-1} μm
6. Solar radiation 1×10^{-1} to 3.0 μm
7. Infrared rays 7.8×10^{-1} to 1×10^{8} μm
8. Thermal radiation 1×10^{-1} to 1×10^{2} μm
9. Microwave, radar, TV and radio waves 1×10^{3} to 2×10^{10} μm

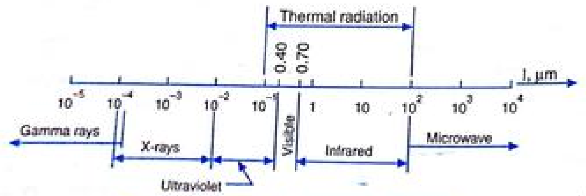


Fig. 7.1. Spectrum of electromagnetic radiation

Note : 1 μm = 10^{-6} m ; 1 μm is called the micron.

(*iv*) Thermal radiations exhibit characteristics similar to those of visible light, and follow optical laws. They can be reflected, refracted, and are subject to scattering and absorption when they pass through a media. They get polarised and weakened in strength with the inverse square of radial distance from the radiating surface.

(*v*) All matter emits radiant energy and bodies at high temperature emit at a greater rate than bodies at low temperature. Normally a body radiating heat is simultaneously receiving heat from other bodies as incident radiation. The net exchange of heat between two radiating surfaces is due to the fact that one at higher temperature radiates more and receives less energy for its absorption. An isolated body which remains at constant temperature emits just as much energy by radiation as it receives. The entire system is then in a state of mobile thermal equilibrium.

(*vi*) Heat transfer by conduction and convection depends primarily on the temperature difference gradient and is little affected by temperature level. With other factors (k, h and A) remaining constant, the heat transfer due to conduction and convection from a hot source at 1000°C to the surroundings at 200°C would practically remain same if the hot source and the surroundings take up the temperature values as 900°C and 100°C respectively. However, even with the same temperature difference, the heat exchange by radiation gets enhanced at elevated temperature of the source and the surroundings.

The most vivid evidence of radiation transfer is that represented by the solar energy which passes through interstellar space (conditions close to that for perfect vacuum) on its way to the earth surface. Solar radiation plays an important part in the design of heating and ventilating systems. Heat transfer by radiation is encountered in boiler furnaces, billet reheating furnaces and other types of heat exchangers. The design and construction of engines, gas turbines, nuclear reactors and solar collectors is also significantly influenced by the radiation heat transfer.

EXAMPLE 7.1

Calculate the frequency and wavelength in vacuum of a photon having an energy of 0.25 pico-joules (pJ)

Solution : The energy e of a photon is given by

$$e = hf = \frac{hc}{\lambda}$$

where λ is the wavelength, f is the frequency, c is the speed of light (3×10^8 m/s) and h is Planck's constant 6.626×10^{-34} Js

$$\therefore \quad f = \frac{e}{h} = \frac{0.25 \times 10^{-12}\text{ J}}{6.626 \times 10^{-34}\text{ Js}}$$

$$= 3.77 \times 10^{20} \text{ per second}$$

$$= 3.77 \times 10^{20} \text{ Hz}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m/s}}{3.77 \times 10^{20} \text{ /s}}$$

$$= 7.96 \times 10^{-13} \text{ m}$$

$$= 0.796 \text{ pm (pico meters)}$$

7.2. ABSORPTIVITY, REFLECTIVITY AND TRANSMISSIVITY

The total radiant energy (Q_0) impinging upon a body would be partially or totally absorbed by it (Q_a), reflected from its surface (Q_r), or transmitted through it (Q_t) in accordance with the characteristics of the body (Fig. 7.2). By the conservation of energy principle, the total sum must be equal to the incident radiation. That is :

$$Q_a + Q_r + Q_t = Q_0$$

Dividing throughout by Q_0, we have

$$\frac{Q_a}{Q_0} + \frac{Q_r}{Q_0} + \frac{Q_t}{Q_0} = \frac{Q_0}{Q_0}$$

$$\alpha + \rho + \tau = 1 \qquad \text{...(7.1)}$$

where α = absorptivity or fraction of total energy absorbed by the body

ρ = reflectivity or fraction of total energy reflected from the body

τ = transmissivity or fraction of total energy transmitted through the body

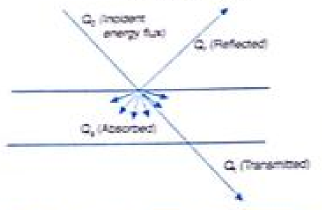


Fig. 7.2. Absorption, reflection and transmission of radiation

The factors α, ρ and τ are dimensionless and vary from 0 to 1. The value depends upon the nature of the surface of the body, its temperature and wavelength of incident rays. The response of the body to incident radiations is, however, completely independent of and unaffected by the simulatneous emission from the body.

Black surfaces are effective absorbers of radiation in the wavelengths that are encountered in heat transfer. Accordingly the name **black body** is assigned to a perfect absorber of radiation. The thermal radiations impinging upon a black body are totally absorbed by it; the radiations are neither reflected from the surface nor transmitted through it. For a black body $\alpha = 1$ and $\rho = \tau = 0$. Incidently this implies that a black body is a perfectly non-reflecting and non transmitting surface. Snow, with its absorptivity 0.985, is nearly black to thermal radiations. The absorptivity of surfaces can be increased to 90-95% by coating their surfaces with lamp black or a dark rough paint. In actual practice there does not exist a perfectly black body which will absorb all the incident radiations. The absorptivity of a surface depends upon the direction of incident radiation, temperature of the surface, composition and structure of the irradiated surface and the spectral distribution of incident radiation. When a surface absorbs a certain fixed percentage of impinging radiations, the surface is called the ***gray body***. The absorptivity of a gray body is necessarily below unity, but it remains constant over the entire range of temperature and wavelength of incident radiation. This condition of constant absorptivity too is not satisfied by the real materials and as such even a gray body remains a hypothetical concept like the black body.

A body that reflects all the incident thermal radiations is called a ***specular body*** (if the reflection is regular) or an ***absolutely white body*** (if the reflection is diffused). For such bodies $\rho = 1$, and $\alpha = \tau = 0$. The specular and diffused type of reflections have been indicated in Fig. 7.3.

Regular (specular) reflection implies that angle between the reflected beam and the

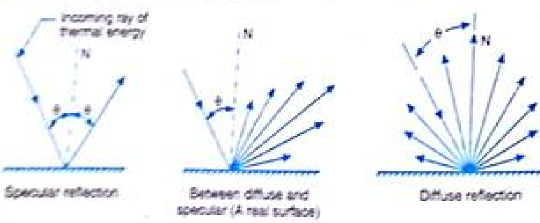


Fig. 7.3. Specular and diffused reflections

normal to the surface equals the angle made by the incident radiation with the same normal. Reflection from highly polished and smooth surfaces approaches specular characteristics. In a diffused reflection, the incident beam is reflected in all directions, *i.e.*, there is directional independence of the reflected beam. Most of the engineering materials have rough surfaces, and these rough surfaces give diffused reflections. Diffused reflection is sometimes likened to the situation in which incident energy is absorbed near the surface and then re-emitted.

A body that allows all the incident radiations to pass through it is called **transparent** or **diathermaneous**. For such bodies $\tau = 1$, and $\alpha = \rho = 0$. Transmissivity varies with wavelength of incident radiation. A material may be non-transparent for a certain wavelength band and transparent for another. A thin glass plate transmits most of the thermal radiations from the sun, but absorbs in equally great measure the thermal radiations emitted from the low temperature interior of a building.

The absorption of a radiation is a surface phenomenon; it occurs in a very thin layer (approx 1 μm thick) of material near the surface. Since most of the solids and liquids encountered in engineering are thick enough to cover this layer, they can be considered non-transparent (opaque, diathermanous) to thermal radiations. Exceptions are few solid substances like glass, quartz, rock salt and most liquids in the visible and near infrared range. For opaque bodies $\tau = 0$ and so $\alpha + \rho = 1$. This result does suggest that good absorbs are bad reflectors and vice versa. However gases have relatively high transmissivity, they transmit an appreciable portion of radiation even in layers of fairly large thickness. Further, the gases are known to reflect very little of the radiation impinging on their interface. Therefore, for gases reflectivity can be neglected and so $\alpha + \tau = 1$. In a transparent medium, the absorption occurs throughout the material.

Consider a large hollow sphere or cylinder provided with only one small opening and let it be maintained at a uniform temperature. The inner surface of the cylinder is coated with lamp black which absorbs about 95% of the incident radiation. A beam of thermal radiation entering the hole strikes the inner surface. Since the surface has a high absorptivity, the major protion of the radiation is absorbed and only a small fraction is reflected. The weak reflected ray does not find any way out and again strikes the inner surface. Here it is again partly absorbed and partly reflected. Likewise the reflected radiation is successively absorbed and finally when it escapes out, it has only a negligible amount of energy associated with it.

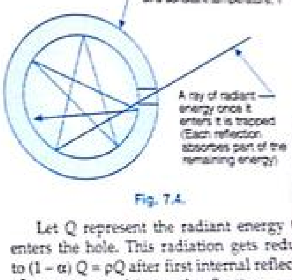


Fig. 7.4.

Let Q represent the radiant energy that enters the hole. This radiation gets reduced to $(1 - \alpha) Q = \rho Q$ after first internal reflection, $\rho^2 Q$ after second internal reflection, ... $\rho^n Q$ after the *n*th reflection and this approaches zero at $n \to \infty$. The few rays that emerge from the hole will have suffered many reflections; the emergent flux then becomes essentially zero and that give absorptivity $\alpha = 1$ for the hole. A small hole leading into a cavity (Hohlraum) thus acts very nearly as a black body because all the radiant energy entering through it gets absorbed. The smaller the opening, better the approximation to black body behaviour. For most experimentation, a

hole of 2.5 cm diameter in the end of a hollow cylinder 25 cm long and 7.5 cm diameter would suffice. Isothermal furnaces, with small apertures, approximate a black body and are frequently used to calibrate heat flux gauges, thermometers and other radiometric devices.

The values pertaining to black body are commonly designated by the suffix '*b*'.

7.3. SPECTRAL AND SPATIAL ENERGY DISTRIBUTION

The distribution of radiant energy is non-uniform with respect to both wavelength and direction.

(*i*) **Spectral distribution** : The radiation emitted by a surface consists of electromagnetic waves of various wavelengths, and the term spectral refers to the variation in thermal radiations with wavelength. Magnitude of the radiation at any wavelength (monochromatic) and the spectral distribution are found to vary with the nature and temperature of the emitting surface.

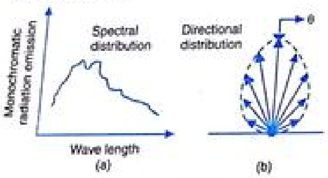


Fig. 7.5. Spectral and spatial (directional) energy distribution

(*ii*) **Spatial or directional distribution** : A surface element emits radiation in all directions; the intensity of radiation is however different in different directions. The surface may emit preferentially in certain directions creating a directional distribution of the emitting radiations.

EXAMPLE 7.2

What happens to the radiant energy when it is incident upon a surface ? Give the general equation that relates absorptivity, reflectivity and transmissivity.

Of the radiant energy 350 W/m² incident upon a surface 250 W/m² is absorbed, 60 W/m² is reflected and the remainder is transmitted through the surface. Workout the values for absorptivity, reflectivity and transmissivity for the surface material.

Solution : Absorptivity α = fractional of total energy absorbed by the surface

$$= \frac{250}{350} = 0.714$$

Reflectivity ρ = fraction of total energy reflected from the surface

$$= \frac{60}{350} = 0.171$$

Transmissivity τ = fraction of total energy transmitted through the surface

$$= \frac{350-(250+60)}{350}$$

$$= 0.115$$

EXAMPLE 7.3

Why does a cavity with a small hole behave as a black body ?

Thermal radiation strikes a surface which has a reflectivity of 0.55 and a transmissivity of 0.032. The absorbed flux as measured indirectly by heating effect works out to be 95 W/m². Determine the rate of incident flux.

Solution : From an energy balance,

$$\alpha + \rho + \tau = 1$$

or $$\frac{Q_a}{Q_0} + \rho + \tau = 1$$

or $$\frac{Q_a}{Q_0} + 0.032 + 0.55 = 1$$

$\therefore$ Incident flux Q_0

$$= \frac{Q_a}{1-0.032-0.55}$$

$$= \frac{95}{0.418} = 227.27 \text{ W/m}^2$$

EXAMPLE 7.4

What is transmissivity of an opaque solid ?

Radiant energy with an intensity of 800 W/m² strikes a flat plate normally. The absorptivity is twice the transmissivity and thrice the reflectivity. Determine the rate of absorption, transmission and reflection of energy.

Solution : From an energy balance,

$$\alpha + \rho + \tau = 1$$

or $$\alpha + \frac{\alpha}{2} + \frac{\alpha}{3} = 1 \; ; \quad \alpha = 0.5455$$

$\therefore$ Absorption Q_a

$$= \alpha \, Q_0$$
$$= 0.5455 \times 800$$
$$= 436.40 \text{ W/m}^2$$

Transmission Q_t

$$= \tau \, Q_0$$
$$= \frac{0.5455}{3} \times 800$$
$$= 145.47 \text{ W/m}^2$$

Reflection Q_r

$$= \rho \, Q_0$$
$$= \frac{0.5455}{2} \times 800$$
$$= 218.20 \text{ W/m}^2$$

EXAMPLE 7.5

A thin metal plate of 4 cm diameter is suspended in atmospheric air whose temperature is 290 K. The plate attains a temperature of 295 K when one of its face receives radiant energy from a heat source at the rate of 2 W. If heat transfer coefficient on both surfaces of the plate is stated to be 87.5 W/m²-deg, workout the reflectivity of the plates.

Solution : Heat lost by convection from both sides of the plate

$$= 2h \, A \, \Delta t$$

The factors 2 accounts for two sides of the plate

$$= 2 \times 87.5 \times \left\{\frac{\pi}{4}(0.04)^2\right\} \times (295 - 290)$$

$$= 1.1 \text{ W}$$

For most of solids, the transmissivity is zero.

$\therefore$ Energy lost by reflection

$$= 2.0 - 1.1$$
$$= 0.9 \text{ W}$$

Reflectivity $$\rho = \frac{Q_r}{Q}$$

$$= \frac{0.9}{2.0} = 0.45.$$

EXAMPLE 7.6

On clear nights there is radiation from earths surface to the space. On such a night, the water particles on the plant leaves radiate to the sky whose temperature may be taken as 200 K. The water particles receive heat by convection from the surrounding air; the convective heat transfer coefficient has a value of 30 W/m²-deg. If the water should not freeze, make calculations for the air temperature.

Solution : For water just to freeze, its temperature has to be 0°C or 273 K.

Assuming water surface to be black, the heat balance gives

heat radiated to sky

= heat received by convection

$$5.67 \times 10^{-8} \times A \times (273^4 - 200^4)$$
$$= 30 \times A \times (T - 273)$$
$$224.22 = 30\,(T - 273)$$

$\therefore$ Temperature of air T

$$= \frac{224.22}{30} + 273$$

$$= 280.474 \text{ K} \quad \text{or} \quad 7.474°\text{C}$$

Any temperature of air lower than this value will cause frost or freezing on the leave surfaces.

7.4. WAVELENGTH DISTRIBUTION OF BLACK BODY RADIATION : PLANCK'S LAW

The energy emitted by a black surface varies in accordance with wavelength, temperature and surface characteristics of the body. For a

prescribed wavelength, the body radiates much more energy at elevated temperatures. Likewise the amount of emitted radiation is strongly influenced by the wavelength even if temperature of the body remains at a constant fixed value.

The laws governing the distribution of radiant energy over wavelength for a black body at a fixed temperature were formulated by Planck. Based upon extensive experimental evidence, Planck suggested the following law for the spectral distribution of emissive power :

$$(E_\lambda)_b = 2\pi c^2 h \frac{\lambda^{-5}}{\exp[ch/k\lambda T]-1} \quad ...(7.2)$$

The symbols used have the following meanings :

h = Planck constant, 6.6236×10^{-34} Js

c = Velocity of light in vacuum 2.998×10^8 m/s

k = Boltzman constant, 13.802×10^{-4} J/K

λ = Wavelength of radiation waves, m

T = Absolute temperature of the black body, K

Quite often the above expression is written as

$$(E_\lambda)_b = \frac{C_1\lambda^{-5}}{\exp\left[C_2/\lambda T\right]-1} \quad ...(7.3)$$

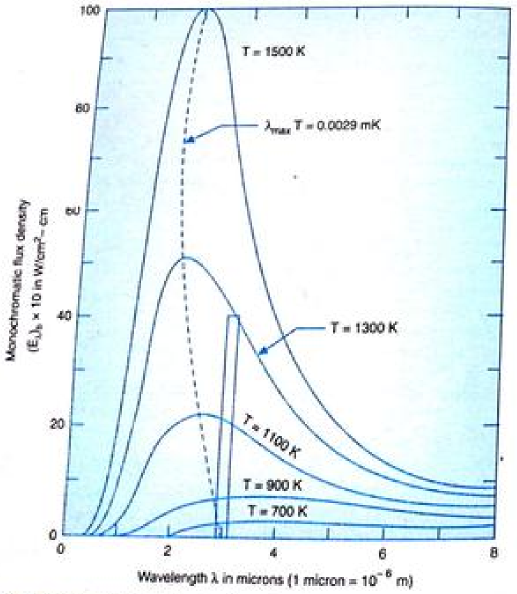


Fig. 7.6. Radiation of black body as a function of wavelength and temperature

where $C_1 = 2\pi c^2 h = 0.374 \times 10^{-15}$ Jm²/s

$C_2 = \frac{ch}{k} = 1.4385 \times 10^{-2}$ mK

The quantity $(E_\lambda)_b$ denotes the monochromatic (single wavelength) emissive power, and is defined as the energy emitted by the black surface (in all directions) at a given wavelength λ per unit wavelength interval around λ. That is, the rate of energy emission in the interval $d\lambda$ is equal to $(E_\lambda)_b\, d\lambda$.

The variation of distribution of the monochromatic emissive power with wavelength is called the **spectral energy distribution**, and this has been depicted in Fig. 7.6 for a number of selected temperatures.

The following important features can be noted from this plot :

(*i*) The monochromatic emissive power varies across the wavelength spectrum; the distribution is continuous but non-uniform. The emitted radiation is practically zero at the zero wavelength. With increase in wavelength, the monochromatic emissive power increases and attains a certain maximum value. With further increase in wavelength, the emissive power diminishes and drops again to almost zero value at infinite wavelength.

(*ii*) At any wavelength the magnitude of the emitted radiation increases with increasing temperature.

(*iii*) The wavelength at which the monochromatic emissive power is maximum shifts in the direction of shorter wavelengths as the temperature increases. This shift signifies that at elevated temperature, much of the energy is emitted in a narrow band ranging on both sides of wavelength at which the monochromatic emissive power is maximum. For example, the sun with its surface temperature of about 5600°C emits 90% of its radiations between 0.1 and 3µ.

(*iv*) At any temperature, the area under the monochromatic emissive power versus wavelength gives the rate of radiant energy emitted within the wavelength interval $d\lambda$.

Thus $dE_b = (E_\lambda)_b\, d\lambda$. Upon integration over the entire range of wavelength.

$$E_b = \int_{\lambda=0}^{\lambda=\infty} (E_\lambda)_b\, d\lambda \quad ...(7.4)$$

The integral measures the total area under the monochromatic emissive power versus wavelength curve for the black body, and it represents the total emissive power per unit area (radiant energy flux density) radiated from a black body.

(*a*) For shorter wavelengths, the factor $C_2/\lambda T$ becomes very large. In that case

$$\exp\left[\frac{C_2}{\lambda T}\right] > 1$$

Obviously the term (– 1) appearing in the denominator of the Planck's distribution law can be neglected compared to this large value. The Planck's law then reduces to

$$(E_\lambda)_b = \frac{C_1\lambda^{-5}}{\exp\left[C_2/\lambda T\right]} \quad ...(7.5)$$

Equation 7.5 is called **Wien's law**, and it is accurate within 1 percent for λT less than 3000 µK.

(*b*) For longer wavelengths, the factor $C_2/\lambda T$ is small. In that case $\exp(C_2/\lambda T)$ can be expanded in series to give

$$\exp\left[C_2/\lambda T\right] = 1 + \frac{C_2}{\lambda T} + \frac{1}{2!}\left(\frac{C_2}{\lambda T}\right)^2 + ...$$

$$= 1 + \frac{C_2}{\lambda T}$$

The Planck's distribution law then becomes

$$(E_\lambda)_b = \frac{C_1\lambda^{-5}}{1 + C_2/\lambda T - 1} = \frac{C_1 T}{C_2 \lambda^4} \quad ...(7.6)$$

The above identity is known as **Rayleigh-Jean's Law**. It is accurate within 1 percent for $\lambda T > 8 \times 10^5$ µK. As black body emits over 99.9 percent of its energy at λT values below this limit; the Rayleigh-Jean's formula is apparently well outside the range of thermal radiation. The relation, however, is quite useful for analysing long wave radiations such as radio waves.

7.5. TOTAL EMISSIVE POWER : STEFAN-BOLTZMAN LAW

The *total emissive power* E of a surface is defined as the total radiant energy emitted by the surface in all directions over the entire wavelength range per unit surface area per unit time.

The basic rate equation for radiation transfer is based on Stefan-Boltzman law which states that the amount of radiant energy emitted per unit time from unit area of black surface is proportional to the fourth power of its absolute temperature.

$$E_b = \sigma_b T^4 \quad \text{...(7.7)}$$

where σ_b is the radiation coefficient of a black body. This rate equation can be set-up by the integration of monochromatic emissive power over the entire band width of wavelength for $\lambda = 0$ to $\lambda = \infty$.

$$E_b = \int_0^\infty (E_\lambda)_b \, d\lambda$$

$$= \int_0^\infty \frac{C_1 \lambda^{-5}}{\exp[C_2/\lambda T] - 1} d\lambda \quad \text{...(7.8)}$$

Let $\frac{C_2}{\lambda T} = y$; $\lambda = \frac{C_2}{yT}$; $d\lambda = \frac{-C_2}{y^2 T} dy$

With this substitution, the new integration limits are :

At $\lambda = 0$, $y = \infty$ and at $\lambda = \infty$, $y = 0$

$$\therefore \quad E_b = -C_1 \int_\infty^0 \frac{y^5 T^5 C_2}{C_2^5 [\exp(y) - 1] y^2 T} dy$$

$$= \frac{C_1 T^4}{C_2^4} \int_0^\infty y^3 [\exp(y) - 1]^{-1} dy$$

Expanding $[\exp(y) - 1]^{-1}$ by a series, we obtain

$$E_b = \frac{C_1 T^4}{C_2^4} \int_0^\infty y^3 [\exp(-y) + \exp(-2y) + \exp(-3y) + ...] dy$$

The integral is of the form

$$\int_0^\infty y^n [\exp(-ay)] dy = \frac{n!}{a^{n+1}}$$

$$\therefore \quad E_b = \frac{C_1 T^4}{C_2^4} \left[\frac{3!}{1^4} + \frac{3!}{2^4} + \frac{3!}{3^4} + ... \right]$$

$$= \frac{C_1 T^4}{C_2^4} \times (6.48)$$

Substituting the value for constants C_1 and C_2 which are :

$$C_1 = 0.374 \times 10^{-15} \text{ Jm}^2/\text{s} ;$$
$$C_2 = 1.4388 \times 10^{-2} \text{ mK}$$

we get :

$$E_b = \frac{0.374 \times 10^{-15}}{(1.4388 \times 10^{-2})} \times 6.48 \times T^4$$

$$= 5.67 \times 10^{-8} \, T^4 = \sigma_b \, T^4 \quad \text{...(7.9)}$$

where $\sigma_b = 5.67 \times 10^{-8}$ W/m^{2}K^4 is the ***radiation coefficient*** or the Stefan-Boltzman constant.

Undoubtedly, the Stefan Boltzman law helps us to determine the amount of radiations emitted in all the directions and over the entire wavelength spectrum from a simple knowledge of the temperature of the black body.

Normally a body radiating heat is simultaneously receiving heat from other bodies as radiation. Consider that surface 1 at temperature T_1 is completely enclosed by another black surface at temperature T_2. The net radiant heat flux is then given by

$$Q_{net} = \sigma_b (T_1^4 - T_2^4) \quad \text{...(7.10)}$$

Often it is desirable to calculate the fraction of emissive power which lies in the wavelength interval λ_1 and λ_2 at a given surface temperature T. Towards that end, we determine the emissive power of a black body within a given range of wavelength, λ_1 to λ_2, by evaluating the integral

$$(E_{\lambda_1 - \lambda_2})_b = \int_{\lambda_1}^{\lambda_2} (E_\lambda)_b \, d\lambda$$

$$= \int_0^{\lambda_2} (E_\lambda)_b \, d\lambda - \int_0^{\lambda_1} (E_\lambda)_b \, d\lambda$$

Expressing it as fraction of the total emissive power,

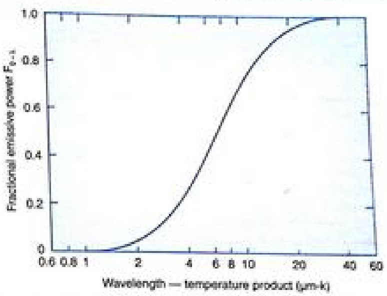


Fig. 7.7. Fraction of black body emission occuring in the range 0 to λT

$$\frac{(E_{\lambda_1 - \lambda_2})_b}{\int_0^\infty (E_\lambda)_b \, d\lambda} = \frac{\int_0^{\lambda_2} (E_\lambda)_b \, d\lambda}{\int_0^\infty (E_\lambda)_b \, d\lambda} - \frac{\int_0^{\lambda_1} (E_\lambda)_b \, d\lambda}{\int_0^\infty (E_\lambda)_b \, d\lambda}$$

$$\text{or} \quad \frac{(E_{\lambda_1 - \lambda_2})_b}{\sigma_b T^4} = \frac{\int_0^{\lambda_2} (E_\lambda)_b \, d\lambda}{\sigma_b T^4} - \frac{\int_0^{\lambda_1} (E_\lambda)_b \, d\lambda}{\sigma_b T^4}$$

$$\text{or} \quad F_{\lambda_1 - \lambda_2} = F_{0-\lambda_2} - F_{0-\lambda_1} \quad \text{...(7.11)}$$

The fraction $F_{0-\lambda} = \frac{\int_0^\lambda (E_\lambda)_b \, d\lambda}{\sigma_b T^4}$ can be expressed as

$$\int_0^{\lambda T} \frac{(E_\lambda)_b \, d(\lambda T)}{\sigma_b T^5} = f(\lambda T) \quad \text{...(7.12)}$$

Thus the integrand $(E_\lambda)_b / \sigma_b T^5$ is exclusively a function of the wavelength-temperature product. The values of the fraction $F_{0-\lambda}$ versus λT have been compiled and tabulated (Table 7.1) or plotted (Fig. 7.7). These values specify how much emission occurs in a specific portion of the total wavelength spectrum.

EXAMPLE 7.7

List the salient features of a black body radiation. Calculate the radiant flux density from a black body at 400°C ? If the emitted radiant energy is to be doubled, to what temperature surface of the black body needs to be raised ?

Solution : A black body is an ideal or hypothetical surface having the following radiation heat transfer characteristics :

(*i*) A black body absorbs all the incident radiation regardless of wavelength and direction.

(*ii*) A black body neither reflects nor transmits any amount of incident radiation.

(*iii*) For a prescribed wavelength a black body radiates the maximum energy possible at the temperature of the body.

(*iv*) The black body is a diffused emitter. This implies that the radiation emitted by a black surface is a function of wavelength and temperature but is independent of direction.

From Stefan-Boltzman law, the rate of energy transmission from a black body is

$$E = \sigma_b T^4$$
$$= 5.67 \times 10^{-8} \times (400 + 273)^4$$
$$= 11631.7 \text{ W/m}^2$$

Table 7.1. Radiation Functions

λT (μm-K)	$F_{0-\lambda}$	λT (μm-K)	$F_{0-\lambda}$
400	0.0000	7200	0.8192
600	0.0000	7400	0.8295
800	0.000016	7600	0.8480
1000	0.00032	7800	0.8480
1200	0.00213	8000	0.8563
1400	0.0078	8500	0.8746
1600	0.0197	9000	0.8900
1800	0.0393	9500	0.9031
2000	0.0667	10000	0.9142
2200	0.1009	10500	0.9237
2400	0.1403	11000	0.9319
2600	0.1831	11500	0.9399
2800	0.2279	12000	0.9451
3000	0.2732	12500	0.9505
3200	0.3181	13000	0.9551
3400	0.3617	13500	0.9592
3600	0.4036	14000	0.9628
3800	0.4434	14500	0.9661
4000	0.4809	15000	0.9689
4200	0.5160	16000	0.9738
4400	0.5488	17000	0.9776
4600	0.5793	18000	0.9808
4800	0.6075	19000	0.9834
5000	0.6337	20000	0.9855
5200	0.6590	25000	0.9922
5400	0.6804	30000	0.9953
5600	0.7010	35000	0.9969
5800	0.7201	40000	0.9979
6000	0.7378	45000	0.9985
6200	0.7541	50000	0.9989
6400	0.7962	75000	0.9997
6600	0.7832	100000	0.9999
6800	0.7961		
7000	0.8081	∞	1.0000

Let T be the absolute temperature at which the radiant flux gets doubled. Then

$$2\,(11631.7) = 5.67 \times 10^{-8}\, T^4$$

$$T^4 = \frac{2 \times 11631.7}{5.67 \times 10^{-8}}\,;$$

$$T = 800.3\text{°K or } 527.3\text{°C}$$

EXAMPLE 7.8

A furnace having inside temperature of 2250 K has a glass circular viewing of 6 cm diameter. If the transmissivity of glass is 0.08, make calculations for the heat loss from the glass window due to radiation.

Solution : The radiation heat loss from the glass window is given by

$$Q = \sigma_b\, A\, T^4 \times \tau$$

where τ is the transmissivity of glass

$$Q = 5.67 \times 10^{-8} \times \frac{\pi}{4}\,(0.06)^2 \times 2250^4 \times 0.08$$

$$= 328.53 \text{ W}$$

EXAMPLE 7.9

Measurements were made of the monochromatic absorptivity and monochromatic hemispherical irradiation incident on an opaque surface, and the variation of these parameters with wavelength may be approximated by the results shown below. Determine the absorbed radiant flux, the total hemispherical absorptivity and the total reflectivity of the surface.

Solution : Incident flux

$$= 800\,(8 - 2) = 4800 \text{ W/m}^2$$

Absorbed radiant flux

$$= \int_0^\infty \alpha_\lambda E_\lambda\, d\lambda$$

$$= \int_2^4 (1 \times 800)\, d\lambda + \int_4^8 (0.5 \times 800)\, d\lambda$$

$$= 800\,(4 - 2) + 400\,(8 - 4)$$

$$= 3200 \text{ W/m}^2$$

$\therefore$ Absorptivity α

$$= 3200/4800 = 0.667$$

The requirement that all the radiant energy striking any surface may be accounted for is

$$\sigma + \tau + \rho = 1$$

Here, $\tau = 0$ as the surface is opaque and therefore reflectivity of the surface is

$$\rho = 1 - \alpha$$

$$= 1 - 0.667 = 0.333$$

EXAMPLE 7.10

A black body of total area 0.045 m² is completely enclosed in a space bounded by 5 cm thick walls. The walls have a surface area 0.5 m² and thermal conductivity 1.07 W/m-deg. If the inner surface of the enveloping wall is to be maintained at 215°C and the outer wall surface is at 30°C, calculate the temperature of the black body. Neglect the difference between inner and outer surface areas of enveloping material.

Solution : Net heat radiated by the black body to the enclosing wall,

$$Q_r = \sigma_b\, A\, (T_b^4 - T_w^4)$$

$$= 5.67 \times 10^{-8} \times 0.045\,(T_b^4 - 488^4)$$

where T_b is the temperature of the black body in degree kelvin.

Heat conducted through the wall,

$$Q_c = \frac{kA\,\Delta t}{\delta} = \frac{1.07 \times 0.5 \times (215 - 30)}{0.05}$$

$$= 1979.5 \text{ W}$$

Under steady state conditions, the heat conducted through the wall must equal the net radiation loss from the black body. Thus

$$5.67 \times 10^{-8} \times 0.045(T_b^4 - 488^4) = 1979.5$$

$$T_b^4 = \frac{1979.5}{5.67 \times 10^{-8} \times 0.045} + 488^4$$

$$= 8349.47 \times 10^8$$

$\therefore$ Temperature of the black body, T_b

$$= 955.9 \text{ K}$$

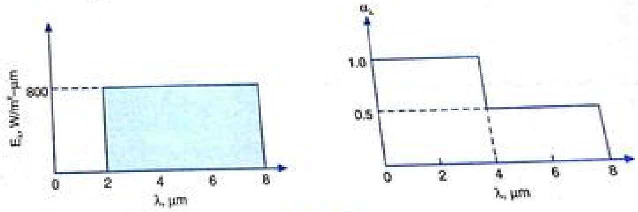


Fig. 7.8.

EXAMPLE 7.11

A radiation measuring instrument detects all emissions occuring between 0.6 and 4.5 μ but is unaffected by frequencies outside this range. What fraction of total emission from a black surface will be detected for emitting surface temperature of 2500 K. The pertinent data as taken from the radiation tables is listed below :

λT (μK)	1400	1600	...	11000	11500
$F_{0-\lambda}$	0.0078	0.0197	...	0.932	0.940

Solution : $\lambda_1 T = 0.6 \times 2500 = 1500$ μm-K

$\lambda_2 T = 4.5 \times 2500 = 11250$ μm-K

Corresponding to these wavelength-temperature products, the fractional emissive powers $F_{0-\lambda_1}$ and $F_{0-\lambda_2}$ can be estimated from the given data.

$$F_{0-\lambda_1} = \frac{0.0078+0.0197}{2} = 0.0137$$

$$F_{0-\lambda_2} = \frac{0.932+0.940}{2} = 0.936$$

$$\therefore F_{\lambda_1-\lambda_2} = F_{0-\lambda_2} - F_{0-\lambda_1} = 0.936 - 0.0137 = 0.9223$$

Thus at the surface temperature of 2500 K, approximately 92% of the total emission will be detected by the measuring instrument and its absolute value will be

$$= 0.9223\ \sigma_b\ T^4$$
$$= 0.9223 \times (5.67 \times 10^{-8}) \times (2500)^4$$
$$= \mathbf{20.42 \times 10^5\ W/m^2}$$

EXAMPLE 7.12

A glass plate has a transmissivity of 0.95 for wavelengths between 0.4 μm and 3 μm and is opaque to all other wavelengths. Make calculations for the percent of incident solar energy transmitted through the glass. Take surface temperature of sun to be 5600 K. The pertinent data as taken from the radiation tables is listed below :

λT (μm-K)	2200	2400	...1600	17000
$F_{0-\lambda}$	0.1009	0.1403	...0.9738	0.9776

Solution : $\lambda_1 T = 0.4 \times 5600 = 2240$ μm-K

$\lambda_2 T = 3 \times 5600 = 16800$ μm-K

Corresponding to these wavelength-temperature products, the fractional emissive powers $F_{0-\lambda_1}$ and $F_{0-\lambda_2}$ can be estimated from the given data

$$F_{0-\lambda_1} = 0.1009 + \frac{0.1403 - 0.1009}{2400 - 2200} \times 40 = 0.10878$$

$$F_{0-\lambda_2} = 0.9738 + \frac{0.9776 - 0.9738}{17000 - 16000} \times 800 = 0.97684$$

$$\therefore F_{\lambda_1-\lambda_2} = F_{0-\lambda_2} - F_{0-\lambda_1} = 0.97684 - 0.10878 = 0.8681$$

This implies that 86.81 percent of the incident solar energy lies in the wavelength band from 0.4 μm to 3 μm. Since the transmissivity is 0.95, the energy transmitted is

$$0.95 \times 0.8681 = 0.783 \text{ or } 78.3 \text{ percent}$$

of the solar energy which is incident on the glass plate

EXAMPLE 7.13

Consider a tungsten filament light bulb whose filament is at a temperature of 2860 K. If the filament is considered to be gray, what fraction of the total energy emitted by the bulb is in the visible wavelength spectrum from 0.35 μm to 0.7 μm. Comment on its effectivness as a light source. If the filament is a rectangle of size 5 mm × 2 mm and consumes 60 W, determine the efficiency of the bulb.

Solution : $\lambda_1 T = 0.35 \times 2860 = 1001$ μm-K

$\lambda_2 T = 0.7 \times 2800 = 2002$ μm-K

Corresponding to the these wavelength-temperature products, the fractional emissive powers $F_{0-\lambda_1}$ and $F_{0-\lambda_2}$ can be read from Table 7.1

$$F_{0-\lambda_1} = 0.00032 \text{ and } F_{0-\lambda_2} = 0.0667$$

$$\therefore F_{\lambda_1-\lambda_2} = F_{0-\lambda_2} - F_{0-\lambda_1} = 0.0667 - 0.00032 = 0.06638$$

Only 6.638 percent of energy is emitted in the visible wavelength range. The remaining 93.362 percent goes into heating the room. Obviously tungsten filament bulbs are highly inefficient as sources of light

Area of the bulb filament from two sides

$$= 2 \times (0.005 \times 0.002) = 20 \times 10^{-6}\ m^2$$

Energy in the visible region

$$= 0.06638 \times 5.67 \times 10^{-8} \times 20 \times 10^{-6} \times (2860)^4 = 5.036\ W$$

$$\therefore \text{Efficiency} = \frac{5.036}{60} = \mathbf{0.0839 \text{ or } 8.39\ \%}$$

7.6. WIEN'S DISPLACEMENT LAW

From the spectral distribution of black body emissive power, it is apparent that the wavelength associated with maximum rate of emission depends upon the absolute temperature of the radiating surface. The nature of this dependence can be obtained by differentiating the Planck's expression

$$E_\lambda = \frac{C_1 \lambda^{-5}}{\exp\left[\frac{C_2}{\lambda T}\right] - 1}$$

with respect to λ and setting the derivative equal to zero. That is

$$\frac{d}{d\lambda}(E_\lambda) = \frac{d}{d\lambda}\left(\frac{C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right) - 1}\right) = 0$$

$$\therefore \frac{\left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right] C_1 (-5) \lambda^{-6} - C_1 \lambda^{-5}\left[\exp\left(\frac{C_2}{\lambda T}\right)\right] \times \left(\frac{C_2}{T}\right)(-1)\lambda^{-2}}{\left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right]^2} = 0$$

Since the denominator ≠ 0, we have

$$-5C_1 \left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right]\lambda^{-6} + \frac{C_1 C_2}{T}\left[\exp\left(\frac{C_2}{\lambda T}\right)\right]\lambda^{-7} = 0$$

Simplification gives the following transcendental equation

$$\frac{C_2}{5\lambda T} + \left[\exp\left(-\frac{C_2}{\lambda T}\right) - 1\right] = 0$$

whose solution by hit and trial method gives

$$\frac{C_2}{\lambda T} = 4.965$$

$$\therefore \lambda_{max}\ T = \frac{C_2}{4.965} = \frac{1.4388 \times 10^{-2}}{4.965} = 2.898 \times 10^{-3} = 0.0029 \text{ mK} \quad ...(7.13)$$

where λ_{max} denotes the wavelength at which emissive power is maximum. The Wien's displacement law may be stated as ***"the product of absolute temperature and the wavelength at which the emissive power is maximum, is constant"***. The law suggests that λ_{max} is inversely proportional to the absolute temperature and accordingly the maximum spectral intensity of radiation shifts towards the shorter wavelength with rising temperature. The locus of points described by Wien's law has been plotted as the dashed curve in Fig. 7.6.

A combination of Planck's law and the Wien's displacement law yields the correlation for maximum monochromatic emissive power for a black body.

$$(E_\lambda)_{max} = \frac{C_1 \lambda^{-5}}{\exp\left[\frac{C_2}{\lambda T}\right] - 1}$$

$$= \frac{0.374 \times 10^{-15} \times (2.898 \times 10^{3} / T)^{-5}}{\exp\left[\frac{1.4388 \times 10^{-2}}{2.898 \times 10^{-3}}\right] - 1}$$

$$= \frac{1.829 \times 10^{-3} T^5}{142.30}$$

$$= 1.285 \times 10^{-5}\ T^5\ W/m^2$$

per metre wavelength

Thus the magnitude of the maximum monochromatic emissive power varie

proportionally with the fifth power of the absolute temperature of the black surface.

Wien's displacement law holds true for more real substances; there is however some deviation in the case of a metallic radiator where the product (λ_{max} T) is found to vary with absolute temperature. The law finds application in the prediction of a very high temperature through measurement of wavelength.

EXAMPLE 7.14

What are the ranges of wavelength of electromagnetic waves covering ultra-violets visible, infrared and thermal radiation.

A small black body has a total emissive power of 4.5 kW/m². Determine its surface temperature and the wavelength of emission maximum. In which range of the spectrum does this wavelength fall ?

Solution : From Stefan Boltzman law, the rate of energy transmission from a black body is

$$E = \sigma_b T^4 ;$$

$$4.5 \times 1000 = 5.67 \times 10^{-8} T^4$$

$$\therefore \quad T = \left[\frac{4.5 \times 1000}{5.67 \times 10^{-8}}\right]^{\frac{1}{4}} = 530.77 \text{ K}$$

The wavelength of emission maximum is given by Wien's law. That is

$$\lambda_{max} T = 2.8908 \times 10^{-3}$$

$$\lambda_{max} = \frac{2.898 \times 10^{-3}}{530.77}$$

$$= 5.46 \times 10^{-6} \text{ m} = 5.46 \ \mu\text{m}$$

From Fig. 7.1, it may be seen that this wavelength falls in the infrared region of the spectrum.

EXAMPLE 7.15

Making use of Planck's law of distribution, establish the relation for the Wien's displacement law.

The sun emits maximum radiation at λ = 0.52 μm. Assuming the sun to be a black body, calculate the surface temperature of the sun and the emissive ability of the sun's surface at that temperature. Also determine the maximum monochromatic emissive power of the sun's surface.

Solution : From Wien's displacement law

$$T = \frac{2.898 \times 10^{-3}}{\lambda_{max}}$$

$$= \frac{2.898 \times 10^{-3}}{0.52 \times 10^{-6}} = \mathbf{5573 \ K}$$

From Stefan's Boltzman law,

$$E = \sigma_b T^4$$

$$= 5.67 \times 10^{-8} (5573)^4$$

$$= \mathbf{5.47 \times 10^7 \ W/m^2}$$

Maximum monochromatic emissive power can be worked out from the relation

$$(E_\lambda)_{max} = 1.285 \times 10^{-5} T^5$$

$$= 1.285 \times 10^{-5} (5573)^5$$

$$= \mathbf{6.908 \times 10^{13} \ W/m^2}$$

per metre wavelength

EXAMPLE 7.16

A furnace emits radiation at 2000 K. Treating it as a black body radiation, calculate the

(i) monochromatic radiant flux density at 1 μm wavelength.

(ii) wavelength at which emission is maximum and the corresponding radiant flux density

(iii) total emissive power, and

(iv) wavelength λ such that emission from 0 to λ is equal to the emission from λ to ∞.

Solution : (*a*) From Planck's law of distribution,

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp[C_2 / \lambda T] - 1}$$

$$= \frac{0.374 \times 10^{-15} \times (1 \times 10^{-6})^{-5}}{\exp[1.4388 \times 10^{-2} / 1 \times 10^{-6} \times 2000] - 1}$$

$$= \frac{0.374 \times 10^{15}}{1331.4 - 1}$$

$$= \mathbf{2.81 \times 10^7 \ W/m^2}$$

per meter wavelength

(*b*) From Wien's displacement law ;

$$\lambda_{max} T = 2.898 \times 10^{-3}$$

$$\therefore \quad \lambda_{max} = \frac{2.898 \times 10^{-3}}{2000} = \mathbf{1.449 \times 10^{-6} \ m}$$

Maximum radiant flux density,

$$1.285 \times 10^{-5} T^5 = 1.285 \times 10^{-5} \times (2000)^5$$

$$= \mathbf{4.11 \times 10^{11} \ W/m^2}$$

per metre wavelength

(*c*) From Stefan-Boltzman law,

$$E = \sigma_b T^4$$

$$= 5.67 \times 10^{-8} \times (2000)^4$$

$$= \mathbf{907200 \ W/m^2}$$

(*d*) The emission in the band width 0 to λ is half of the total emission from 0 to ∞. Therefore, fractional emissive power is 0.5. Corresponding to $F_{0-\lambda} = 0.5$, the wavelength temperature product as read from Table 7.1 is approximately 4100 μm-K. Thus $\lambda T = 4100$ and so

$$\lambda = \frac{4100}{2000} = 2.05 \ \mu\text{m}$$

EXAMPLE 7.17

Estimate the fraction of sun's emitted radiant energy which lies in the visible range of wavelength from 0.3 to 0.7 micron. Further proceed to calculate the wavelength at which the monochromatic emissive power is maximum. Consider the sun as a black surface at 5560 K.

The pertinent data as taken from the radiant tables is listed below :

λT	1600	1700	.....	3800	3900	μm-K
$F_{0-\lambda}$	0.0197	0.0285	....	0.4438	0.4623	

Solution : $\lambda_1 T = 0.3 \times 5560 = 1668$ μK

$\lambda_2 T = 0.7 \times 5560 = 3892$ μK

Corresponding to these wavelength-temperature products, the fractional emissive powers can be obtained from the given data.

$$F_{0-\lambda_1} = 0.0197 + \frac{68}{100}(0.0285 - 0.0197)$$

$$= 0.257$$

$$F_{0-\lambda_2} = 0.4438 + \frac{92}{100}(0.4624 - 0.4438)$$

$$= 0.4609$$

Therefore, the fraction of sun's emitted radiant energy that lies in the visible range

$$= 0.4609 - 0.0257 = 0.4352$$

Hence 43.52% of the sun's surface emission lies in the visible range and its absolute value will be

$$= 0.4352 \ \sigma_b T^4$$

$$= 0.4352 \times (5.67 \times 10^{-8}) \times (5560)^4$$

$$= 0.36 \times 10^7 \ \text{W/m}^2$$

From Wien's displacement law :

$$\lambda_{max} = 2.898 \times 10^{-3}$$

$$\therefore \quad \lambda_{max} = \frac{2.898 \times 10^{-3}}{5560}$$

$$= 0.5216 \times 10^{-6} \text{ m}$$

$$= \mathbf{0.5216 \ micron}$$

7.7. KIRCHOFF'S LAW

Consider two surfaces, one absolutely black at temperature T_b and the other non-black at temperature T (Fig. 7.9). The surfaces are arranged parallel to each other and so close that the radiation of one falls totally on the other. The radiant energy E emitted by the non-black surface impinges on the black surface and gets fully absorbed. Likewise the radiant energy E_b emitted by the black surface strikes the non-black surface. If the non-black surface has absorptivity α, it will absorb αE_b radiations and the remainder $(1 - \alpha) E_b$ will be reflected back for full absorption at the black surface. Radiant interchange for the non-black surface equals $(E - \alpha E_b)$. If both the surfaces are at the same temperature, $T = T_b$, then the conditions correspond to mobile thermal equilibrium for which the resultant interchange of heat is zero.

Under these conditions :

$$E - \alpha E_b = 0 \quad \text{or} \quad \frac{E}{\alpha} = E_b$$

This relationship can be extended by considering different surfaces in turn, Thus in general:

$$\frac{E_1}{\alpha_1} = \frac{E_2}{\alpha_2} = \frac{E_3}{\alpha_3} = \ldots\ldots$$

$$= \frac{E_b}{\alpha_b} = E_b = f(T) \qquad \ldots(7.14)$$

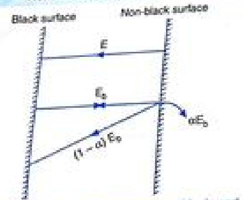


Fig. 7.9. Exchange of heat between a black and a non-black surface

This is because the absorptivity α_b of a black body equals unity.

Equations 7.14 shows that the ratio of the emissive power E to absorptivity α is same for all bodies, and is equal to the emissive power of a black body at the same temperature. This relationship is known as the Kirchoff's law.

The ratio of the emissive power of a certain non-black body E to the emissive power of a black body E_b, both bodies being at the same temperature, is called the **emissivity** of the body. Emissivity of a body is a function of its physical and chemical properties and the state of its surface-whether rough or smooth.

From equation 7.14 :

$$\frac{E}{E_b} = \alpha \quad \text{or} \quad \epsilon = \alpha \qquad ...(7.15)$$

where, $\epsilon = \frac{E}{E_b}$ is emissivity.

Kirchoff's law can also be stated as : "*The emissivity ϵ and absorptivity α of a real surface are equal for radiation with identical temperature and wavelength*". The equivalence of ϵ and α does suggest that a perfect absorber (the perfect black body) is also a perfect radiator.

7.8. GRAY BODY AND SELECTIVE EMITTERS

Consider two bodies, one absolutely black and the other non-black and let these be at the same temperature. The monochromatic emissive power of a non-black body varies significantly from the black body monochromatic emissive power as illustrated in Fig. 7.10.

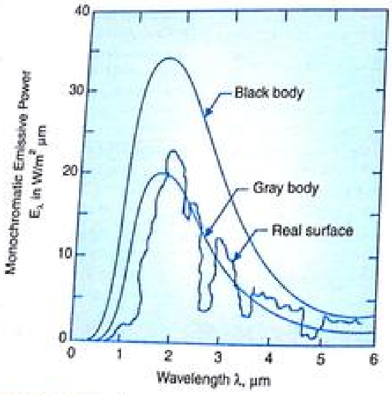


Fig. 7.10. Emission from black, gray and selective emitters

These curves indicate that a non-black body radiates less intensively than a black body. Further the radiation spectrum for a non-black body may be similar or radically different from that of a black body.

When the emissivity of non-black surface is constant at all temperatures and through out the entire range of wavelength, the surface is called a **gray body**. The radiation spectrum for a gray body, though reduced in vertical scale, is continuous and identical to the corresponding curve for a perfectly black surface; there is no shift in the peak of the curves. However for many materials the emissivity is different for the various wavelengths of the emitted energy. The radiating bodies exhibiting this behaviour are called **selective emitters**. The radiation spectrum for a selective emitter does not follow any definite pattern, and it varies entirely from that of a black body.

Stefan-Boltzman law when applied to a gray body takes the form

$$E = \sigma T^4 \qquad ...(7.16)$$

The constant σ is different for different bodies and its value depends upon the nature of the body, the state of its surface and temperature. It is always less than the radiation coefficient σ_b for a black body; it's value ranges from 0.0 to 5.67×10^{-8} W/m^2 K^4.

The emissivity of the gray surface may be expressed as :

$$\epsilon = \frac{E}{E_b} = \frac{\sigma T^4}{\sigma_b T^4} = \frac{\sigma}{\sigma_b} \qquad ...(7.17)$$

Values of emissivities range from 0.0 to 1.0.

The emitted radiant energy flux density for non-black body, as prescribed by equation 7.16 may be rewritten as :

$$E = \epsilon\, \sigma_b\, T^4 \text{ W/m}^2 \qquad ...(7.18)$$

Emissivities of real bodies : Emissivity of a surface indicates its ability to emit radiation energy in comparison with a black surface at the same temperature level; it represents the ratio of the emissive power of the surface to the emissive power of black surface at the same temperature. Based upon the direction and totality of emissive power we have :

(*i*) ***Monochromatic emissivity*** ϵ_λ : Ratio of the monochromatic emissive power of a surface to the monochromatic emissive power of a black surface at the same wavelength and temperature. For a gray body the monochromatic emissivity is independent of the wavelength of the emitted radiation, *i.e.*, the monochromatic emissive power of the surface and the monochromatic emissive power of a black surface have the same ratio for all wavelength of emitted radiation at the same temperature.

(*ii*) ***Total emissivity*** ϵ : Ratio of the total emissive power of a surface to the total emissive power of a black surface at the same temperature.

(*iii*) ***Normal total emissivity*** ϵ_n : Ratio of the normal component of the total emissive power of a surface to the normal component of the total emissive of a black body at the same temperature.

(*iv*) ***Mean and equilibrium emissivity*** : Emissivity of most of the engineering materials is influenced by temperature as well as wavelength. For a particular temperature, the average of monochromatic emissivity at various wavelengths (within a range of wavelength) is called the wavelength-mean emissivity. Like-wise for a prescribed wavelength, average value of all the monochromatic emissivity at various temperatures (within a range of temperature) is called the temperature-mean emissivity.

The emissivity of a material though varying with temperature and the nature of its surface, is not affected in any way by the nature of surface surrounding it. The total emissivity remains constant whether the material is in equilibrium with the surroundings or not. The total emissivity is, therefore, sometimes called the ***equilibrium emissivity***.

In general, the emissivity of a material is dependent upon its nature (colour, texture and roughness), temperature, wavelength of radiations, angle of emission and the nature of the surface which is influenced by the method of fabrication, thermal cycling and chemical reaction with environment. Emissivity of a metallic surface increases with temperature growth, surface roughness and with the formation of a film of impurities and a thin layer of oxide. Emissivity of a highly polished surface is quite low. For example, the pure polished copper has emissivity 0.072 but copper oxide has the emissivity value 0.8 at ordinary temperatures. Further, metals with coloured oxide (Fe, Zn, Cr) have much higher emissivity than the metals with white oxide (Cd, Al, Mg). For non-metallic surface emissivity values generally decrease with temperature rise. Non-conductors have a comparatively large (generally exceeding 0.6) emissivity. Depending upon specific material, the emissivity of a non-conductor may either increase or decrease with temperature rise.

EXAMPLE 7.18

What physical ratio determines whether a real surface is an almost specular reflector or an almost diffuse reflector ?

(b) Define emissivity. How does it vary with temperature for conductors and non-conductors?

The radiant heat transfer from a plate of 2.5 cm² area at 1250 K to a very cold enclosure is 5.0 W. Determine the emissivity of the plate at this temperature.

Solution : If the roughness dimension for a real furface is large with respect to wavelength of incident radiation the surface behaves as a diffuse reflector. If the roughness dimension is considerally smaller than the wavelength, the surface reflects specularly.

(b) A real surface has a total emissive power E less than that of a black surface E_b. The ratio of the total emissive power of a surface to that of a black surface at the same temperature is called the total emissivity ; $\epsilon = E/E_b$. Some common observatiors about the emissivity of a body are :

(i) The emissivity of the metallic surfaces is very small having the values as low as 0.02 for highly polished gold and silver.

(ii) The presence of oxide layers generally improves the emissivity of metallic surfaces.

(iii) The non-conductors have large value of emissivity, generally exceeding 0.6

(iv) The emissivity of a conducting material increases with increase in temperature, but emissivity of non-conducting materials decreases with increase in temperature

$$\text{Emissivity } \epsilon = \frac{E}{E_b} = \frac{E}{\sigma_b A T^4}$$

$$= \frac{5.0}{5.67 \times 10^{-8} \times (2.5 \times 10^{-4}) \times 1250^4}$$

$$= 0.144$$

EXAMPLE 7.19

Can a surface behave essentially like a black surface and still look practically white to the eye.

(b) A 100 watt light bulb has a tungsten filament (emissivity ϵ = 0.30) which is required to operate at 2780 K. If the bulb is completely evacuated, calculate the minimum surface area of the tungsten filament. Assume that no radiant energy strikes the bulb surface and that steady state conditions prevail.

Solution : The operating conditions stipulate that :

(i) The bulb is completely evacuated, and the vacuum precludes any heat transfer by convection.

(ii) No radiant energy strikes the bulb surface,

(iii) Steady state conditions prevail, *i.e.*, there is no storage of energy.

Accordingly the entire power supplied by the tungsten filament is dissipated by the radiation process. That is

$$E = \epsilon\, \sigma_b A T^4$$

$$100 = 0.3 \times (5.67 \times 10^{-8}) \times A \times 2780^4$$

$$= 101.6 \times 10^4 A$$

∴ Surface area of the bulb filament,

$$A = \frac{100}{101.46 \times 10^4} = 0.98 \times 10^{-4}\ m^2$$

EXAMPLE 7.20

A gray body (ϵ = 0.8) emits the same amount of heat as a black body at 1075 K. Find out the required temperature of the gray body.

(b) If a black body at 1000 K and a gray body at 1250 K emit the same amount of radiation, what should be the emissivity of the gray body.

Solution : Since the black and gray body are emitting the same amount of radiation,

$$\sigma_b A T_b^4 = \sigma_b \epsilon A T_g^4$$

Presuming the two bodies to be of same area, we have

$$T_b^4 = \epsilon T_g^4$$

(a) $$T_g = \frac{T_b}{(\epsilon)^{0.25}} = \frac{1075}{(0.8)^{0.25}}$$

$$= \frac{1075}{0.9457} = 1136.72\ K$$

(b) $$\epsilon = \left(\frac{T_b}{T_g}\right)^4 = \left(\frac{1000}{1250}\right)^4 = 0.4096$$

EXAMPLE 7.21

A polished metal pipe 5 cm outside diameter and 370 K temperature at the outer surface is exposed to ambient conditions at 295 K temperature. The emissivity of the surface is 0.2 and the convection coefficient of heat transfer is 11.35 W/m²-deg. Calcualte the heat transfer by radiation and natural convection per metre length of the pipe. Take thermal radiation constant

$$\sigma_b = 5.67 \times 10^{-8}\ W/m^2K^4$$

What would be the overall coefficient of heat transfer by the combined mode of convection and radiation ?

Solution : Surface area A of the pipe per metre run is equal to

$$\pi dl = \pi \times 0.05 \times 1 = 0.157\ m^2$$

$$Q_r\ (\text{radiation}) = \epsilon\, \sigma_b A (T_1^4 - T_2^4)$$

$$= 0.2 \times 5.67 \times 10^{-8} \times 0.157 \times (370^4 - 295^4)$$

$$= 19.88\ W$$

$$Q_c\ (\text{convection}) = h A \Delta t$$

$$= 11.35 \times 0.157\,(370 - 295)$$

$$= 133.64\ W$$

$$Q_t\ (\text{total}) = 19.88 + 133.64 = 153.52\ W$$

The total heat exchange can be expressed as :

$$Q_t = U A \Delta T$$

where U is the overall coefficient of heat transfer

$$\therefore \quad U = \frac{Q_t}{A \Delta T} = \frac{153.52}{0.157(370 - 295)}$$

$$= 13.04\ W/m^2\text{-deg}$$

EXAMPLE 7.22

Show that no surface at any temperature can have emissivity greater than unity.

For most materials, the emissivity ϵ is a function of temperature and for a certain material it has been specified as

$$\epsilon = 0.35 + \frac{T}{1535}$$

where T is in degree kelvin, over the temperature range 0 to 1500 K. Is there any reason to suspect the validity of this information ?

(b) A metal sphere of surface area 0.0225 m² is in an evaculated enclosure whose walls are held at a very low temperature. Electric current is passed through resistors imbedded in the sphere causing electrical energy to be dissipated at the rate of 75 watts. If the sphere surface temperature is measured to be 560 K, while in steady state, calculate emissivity of the sphere surface and state the assumptions made in the estimation.

Solution : The requirement that all the radiant energy striking any surface be accounted for is prescribed by the relationship :

$$\alpha + \rho + \tau = 1$$

Apparently the largest fraction of incident radiation that can be absorbed is unity (with $\rho = \tau = 0$).

From Kirchoff's law, the emissivity equals the absorptivity under the same temperature conditions. That is $\epsilon = \alpha$.

These conditions clearly indicate that emissivity of any surface cannot be greater than unity.

At 1500 K, the emissivity works out as

$$\epsilon = 0.35 + \frac{1500}{1535} = 1.327$$

However the emissivity of any material is a number between 0 and 1. Obviously, either the given expression is wrong or the quoted range of temperature is incorrect.

(b) The entire electrical energy released by the current passing through the resistors may be dissipated by the radiation process. That is

$$E = \epsilon A \sigma_b T^4$$
$$75 = \epsilon \times 0.0225 \times (5.67 \times 10^{-8}) \times 560^4$$
$$= 125.46\ \epsilon$$

∴ Emissivity of the sphere surface,

$$\epsilon = \frac{75}{125.46} = 0.598$$

The following assumptions have been made in the estimation of emissivity :

(i) The walls are held at a low temperature. This implies that the sphere emits, but does not receive radiant energy.

(ii) The enclosure is completely evacuated, and the vacuum precludes any heat transfer by convection.

(iii) Steady state conditons prevail, *i.e.*, there is no storage of energy.

EXAMPLE 7.23

A gray surface has an emissivity $\epsilon = 0.35$ at a temperature of 550 K source. If the surface is opaque, calculate its reflectivity for a black body radiation coming from a 550 K source.

(b) A small 25 mm square hole is made in the thin-walled door of a furnace whose inside walls are at 920 K. If the emissivity of the walls is 0.72, calculate the rate at which radiant energy escapes from the furnace through the hole to the room.

Solution : The requirement that all of the radiant energy striking any surface may be accounted for is :

$$\alpha + \rho + \tau = 1$$

Here :

(i) $\tau = 0$ as the surface is opaque

(ii) $\alpha = \epsilon = 0.35$

This is in accordance with Kirchoff's law which states that absorptivity equals emissivity under the same temperature conditions.

∴ Reflectivity ρ

$$= 1 - (\alpha + \tau)$$
$$= 1 - (0.35 + 0) = 0.65$$

Thus the surface reflects 65 percent of incident energy coming from a source at 550 K.

(b) The small hole acts as a black body and accordingly the rate at which radiant energy leaves the hole is

$$E = \sigma_b A T^4$$
$$= 5.67 \times 10^{-8} \times (0.025 \times 0.025) \times 920^4$$
$$= \mathbf{25.38\ watts}$$

Note: The data about the emissivity of the inside wall is not needed.

EXAMPLE 7.24

On a hot day, a car is left in sunlight with all the windows closed. After sometimes, it is found that the inside of the car is considerably warmer than the air outside. Why ?

The wall of a jet engine test cell is provided with a long 5 cm diameter cylinder of glass to examine the area near the nozzle in the combustor. The transmissivity of this glass is essentially zero except between the wavelength 0.3 to 0.8 micron where its value is 0.32. Make calculations for the fraction of incident radiant energy from the combuster that is transmitted through the glass. The radiation from the combuster approximates to radiation from a black body source at 2220 K. The pertinent data as taken from the radiation table is listed below :

λT	600	700	.....	1700	18001 μm-K
$F_{0-\lambda}$	0	0	.....	0.0285	0.0393

Solution : The windows of a car are made of glass and glass possesses the property of selective absorption of heat radiation. The amount of heat transmitted through glass depends upon the temperature of heat source. For example, it transmits about 50% of heat radiation when it is coming from a source like sun which is at high temperature. But when these radiations come from a source which is below 100°C, glass is opaque to them. The glass windows of the car allow the heat radiation from the sun to pass through them and heat the inside of car to a temperature below 100°C. The radiations emitted by the inside of car at this temperature are not allowed by the windows to pass through to the outside. Glass thus serves to trap the sun rays and the inside of car becomes hot.

(b) $\lambda_1 T = 0.3 \times 2220 = 666$ μm-K

$\lambda_2 T = 0.8 \times 2220 = 1776$ μm-K

Corresponding to these wavelength-temperature products, the fractional emissive powers can be estimated from the given data:

$$F_{0-\lambda_1} = 0.0$$

$$F_{0-\lambda_2} = 0.0285 + \frac{76}{100}(0.0393 - 0.0285)$$
$$= 0.0367$$

Therefore, the fraction of emitted radiant energy that lies in the given wavelength range is

$$= 0.0367 - 0.0 = 0.0367$$

With emissivity 0.32, the actual fraction of interest becomes

$$= 0.0367 \times 0.32 = 0.01174$$

Thus at the combustion temperature of 2220 K, approximately 1.174% of the total emission would be detected by the observer and its absolute value will be

$$= 0.01174\ \sigma_b\ T^4$$
$$= 0.01174 \times (5.67 \times 10^{-8}) \times 2220^4$$
$$= \mathbf{1.56 \times 10^4\ W/m^2}$$

EXAMPLE 7.25

The variation of monochromatic emissivity of a surface with wavelength has been prescribed as

$$\epsilon = 0 \quad for \quad \lambda < 0.3\ \mu m$$
$$= 0.85 \quad for \quad 0.3\ \mu m \le \lambda \le 1\ \mu m$$
$$= 0 \quad for \quad \lambda > 1\ \mu m$$

Make calculations for the heat flux emitted by the surface if it is maintained at 1500 K temperature.

Solution : $\lambda_1 T = 0.3 \times 1500 = 450$ μm-K

$\lambda_2 T = 1 \times 1500 = 1500$ μm-K

Corresponding to these wavelength-temperature products, the fractional emissive powers $F_{0-\lambda_1}$ and $F_{0-\lambda_2}$ can be obtained from Table 7.1 by using linear interpolations.

$$F_{0-\lambda_1} = 0.0$$

$$F_{0-\lambda_2} = \frac{0.0078 + 0.0197}{2} = 0.01375$$

Therefore, required heat flux is

$$= (F_{0-\lambda_2} - F_{0-\lambda_1})\ \epsilon\ \sigma_b\ T^4$$
$$= (0.01375 - 0.0) \times 0.85 \times (5.67 \times 10^{-8}) \times 1500^4$$
$$= \mathbf{3356\ W/m^2}$$

EXAMPLE 7.26

The monochromatic emissivity of a diffuse-surface at 1600 K varies with wavelength in the following manner :

$$\epsilon = 0.4 \quad for \quad 0 < \lambda \le 2$$
$$= 0.8 \quad for \quad 2 \le \lambda \le 5$$

Sketch this variation and determine the total emissivity and the total emissive power.

Solution : $\lambda_0 T = 0$

$\lambda_1 T = 2 \times 1600 = 3200$ μm-K

$\lambda_2 T = 5 \times 1600 = 8000$ μm-K

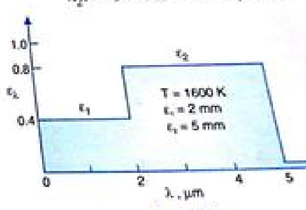


Fig. 7.11.

From Table 7.1, one obtains

$$F_{0-\lambda_1} = 0.0000$$
$$F_{0-\lambda_2} = 0.3118$$
$$F_{0-\lambda_3} = 0.8563$$

Therefore total emissivity

$$\epsilon = 0.4\ (0.3151 - 0.0000) + 0.8\ (0.8563 - 0.3118)$$
$$= 0.1272 + 0.4306 = 0.5578$$

and the total emissive power is

$$E = \epsilon\ \sigma_b\ T^4$$
$$= 0.5578 \times (5.67 \times 10^{-8}) \times 1600^4$$
$$= \mathbf{206926\ W/m^2}$$

EXAMPLE 7.27

Is the absorptivity of a surface equal to emissivity? If so, does this mean that energy absorbed is always equal to the energy emitted? Does it also mean that the net exchange of radiation energy is always equal to zero? If not, is this a violation of the Kirchoff's law?

Compute the rate of absorption and the rate of emission from a small body at 350 K which has been placed in a large furnace whose walls are maintained at 1250 K temperature. The dependence of the total emissivity of this body on temperature has been indicated below :

Temperature, K	*350*	*600*	*850*	*1250*
Absorptivity, α	*0.75*	*0.64*	*0.59*	*0.50*

Solution : The radiant energy emitted by the furnace walls is given by :

$$E = \sigma_b\ T^4$$
$$= 5.67 \times 10^{-8} \times (1250)^4$$
$$= 13.843 \times 10^4\ W/m^2$$

Of this total incident radiation, a portion will be absorbed by the small body in accordance with its absorptivity which depends upon temperature of the incident radiations. Here the body is receiving heat radiations from the furnace walls at 1250 K, and the absorptivity of the body for this radiation is 0.50.

∴ Rate of energy absorption is :

$$0.5 \times (13.843 \times 10^4) = 6.92 \times 10^4\ W/m^2$$

(*ii*) From Kirchoff's law, the emissivity equals the absorptivity under same temperature conditions. Therefore, emissivity of the body at its temperature of 350 K is 0.75.

∴ Rate of emission from the body is :

$$E = \epsilon\ \sigma_b\ T^4$$
$$= 0.75 \times (5.67 \times 10^{-8}) \times 350^4$$
$$= \mathbf{638.14\ W/m^2}$$

EXAMPLE 7.28

State Stefan-Boltzman law of total radiation from a black body. How this law can be modified to take into account radiation from a non-black body?

A steel plate is placed on a non-conducting opaque surface normal to incident solar radiation of 750 W/m². Neglecting convection effects, workout the equilibrium temperature of the plate when it is

(a) Oxidised with emissivity ∈ = 0.80, and

(b) Polished with emissivity ∈ = 0.07

Assumption may be made of the gray body behaviour.

Take σ_b = 5.67 × 10⁻⁸ W/m² K⁴. Comment on the results.

Solution : Energy emitted per unit area

$$= \sigma_b\ \epsilon\ T^4$$
$$= (5.67 \times 10^{-8}) \times 0.8 \times T^4$$

From Kirchoff's law, absorptivity α equals the emissivity ϵ of the surface.

Energy absorbed per unit area

$$= \alpha \times \text{incident radiation}$$
$$= 0.8 \times 750$$

Under equilibrium conditions

$$(5.67 \times 10^{-8}) \times 0.8 \times T^4 = 0.8 \times 750;$$
$$T = 339\ K$$

Likewise when the surface is polished

$$\alpha = \epsilon = 0.07$$
$$(5.67 \times 10^{-8}) \times 0.07 \times T^4 = 0.07 \times 750;$$
$$T = 339\ K$$

Evidently if the absorptivity and emissivity remain constant over the range of temperature and wavelength, the equilibirium temperature remains unaffected by the nature of surface. This illustration indicates that the black skin

is not necessarily of disadvantage in tropical climates. Though the black skin absorbs the sun radiations readily, the emission from it is equally strong. Actually the heat balance would favour the black skin if variation of emissivity with wavelength is taken into account.

EXAMPLE 7.29

Measurement of high temperature of a furnace are made by optical pyrometers which work on the principle of comparing the radiation energy emitted by the furnace (hot body being investigated) at a given wavelength with that of an incandescent filament. The instrument is calibrated in such a way that it measures the temperatures which a black body would have if it were emitting radiation equal to that of the body under investigation.

Make calculations for the true temperature of a body if the test results with the optical pyrometer indicate a temperature of 1675 K. It may be presumed that emissivity of the body at wavelength λ = 0.6 μm is equal to 0.58.

Solution : From Planck's law of radiation, the emissive power of a black body and that of the body being investigated (furnace) are

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T_b}\right) - 1}$$

and $$E_\lambda = \frac{\epsilon_\lambda C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right) - 1}$$

where T_b is the temperature of the black body, T is the temperature of the gray body. At $E_\lambda = (E_\lambda)_b$, T_b will be the temperature indicated by the pyrometer.

$$\frac{C_2}{\lambda T_b} = \frac{1.438 \times 10^{-2}}{0.6 \times 10^{-6} \times 1675} = 14.31 > 1$$

Neglecting the term (− 1) as compared to the value of [exp $(C_2/\lambda T)$] and equating the two emissive powers, we may write

$$\frac{\epsilon_\lambda}{\exp\left(\frac{C_2}{\lambda T}\right)} = \frac{1}{\exp\left(\frac{C_2}{\lambda T_b}\right)}$$

or $$\epsilon_\lambda \exp\left(\frac{C_2}{\lambda T_b}\right) = \exp\left(\frac{C_2}{\lambda T}\right)$$

Taking logs on both sides

$$\log_e \epsilon_\lambda + \frac{C_2}{\lambda T_b} = \frac{C_2}{\lambda T}$$

or $$\frac{1}{T_b} + \frac{\lambda}{C_2} \log_e \epsilon_\lambda = \frac{1}{T} \qquad ...(i)$$

That gives

$$T = \frac{1}{\frac{1}{T_b} - \frac{\lambda}{C_2} \log_e \frac{1}{\epsilon_\lambda}}$$

Substituting the given data :

$$T = \frac{1}{\frac{1}{1675} - \frac{0.6 \times 10^{-6}}{1.438 \times 10^{-2}} \log_e \frac{1}{0.58}}$$
$$= \frac{1}{5.97 \times 10^{-4} - 0.2273 \times 10^{-14}}$$
$$= 1741\ K$$

Therefore temperature of the body

$$= 1741 - 273 = \mathbf{1468°C}$$

EXAMPLE 7.30

Which aspect of thermal radiation forms the basis of temperature measurement by optical pyrometers?

The test results with an optical pyrometer indicated the following temperature readings of a hot source :

(i) 1725 K when the pyrometer incorporates a red light filter with λ = 0.62 micron

(ii) 1750 K when the pyrometer incorporates a green light filter with λ = 0.48 micron

Assuming the hot source to behave as a gray body, make calculations for its true temperature and emissivity.

Solution : A metallic surface is usually dark and dull coloured at room temperature. When the surface is heated, it emits radiations of different wavelengths; these radiations are however not visible at low temperatures. As the temperature is progressively increased

beyond 500°C, the surface becomes dark red, orange and finally white in colour. The high temperature is the result of concentration of radiations in a short wavelength portion of the spectrum. A colour variation with temperature growth and a shift in the wavelength of maximum emission toward shorter waves (Wien's displacement law) forms the basis of temperature measurement with optical pyrometers. Measurements of temperature are made through the act of comparison between the energy emitted by the furnace (hot body being investigated) at a given wavelength with that of a reference temperature provided by an electrically heated lamp filament. Matching of brightness of lamp filament with that of furnace is achieved by adjusting current through the lamp by changing the value of circuit resistance.

The red or green filters placed between the eye piece and filament allow only a narrow band of wavelength to pass through it (monochromatic condition)

Invoking relation (*i*) as setup in the previous example,

$$\frac{1}{T} = \frac{1}{T_b} + \frac{\lambda}{C_2} \log_e \epsilon_\lambda$$

For the optical pyrometer using red and green filters, the above identity may be written as

$$\frac{1}{T} = \frac{1}{T_{b_1}} + \frac{\lambda_1}{C_2} \log_e \epsilon_{\lambda_1}$$

$$\text{and} \quad \frac{1}{T} = \frac{1}{T_{b_2}} + \frac{\lambda_2}{C_2} \log_e \epsilon_{\lambda_2}$$

where T_{b_1} and T_{b_2} are the furnace temperatures as indicated with red and green filters.

Since the hot source (furnace) is to behave as a gray body

$$\epsilon_{\lambda_1} = \epsilon_{\lambda_2} = \epsilon$$

$$\therefore \quad \frac{1}{T} = \frac{1}{T_{b_1}} + \frac{\lambda_1}{C_2} \log_e \epsilon$$

$$\text{and} \quad \frac{1}{T} = \frac{1}{T_{b_2}} + \frac{\lambda_2}{C_2} \log_e \epsilon$$

The following expressions can be worked out when the above relations are solved for the temperature T and emissivity ϵ of the furnace.

$$T = \frac{\frac{\lambda_1}{\lambda_2} - 1}{\frac{\lambda_1}{\lambda_2} \times \frac{1}{T_{b_2}} - \frac{1}{T_{b_1}}}$$

$$\text{and} \quad \log_e \epsilon = \frac{C_2 (T_{b_1} - T_{b_2})}{T_{b_1} T_{b_2} (\lambda_1 - \lambda_2)}$$

Inserting the appropriate values we obtain :

$$T = \frac{\frac{0.62}{0.48} - 1}{\frac{0.62}{0.48} \times \frac{1}{1750} - \frac{1}{1725}} = 1841 \text{ K}$$

$$\log_e \epsilon = \frac{1.438 \times 10^{-2} (1725 - 1750)}{1725 \times 1750 (0.62 \times 10^{-6} - 0.48 \times 10^{-6})}$$

$$= -8.8506$$

$$\therefore \quad \epsilon = 0.427$$

7.9. INTENSITY OF RADIATION AND LAMBERT'S COSINE LAW

Imp.

Subtended plane and solid angles : The plane angle α is defined by a region by the rays of a circle, and is measured as the ratio of the element of arc of length l on the circle to the radius r of the circle: $\alpha = l/r$

The solid angle ω is defined by a region by the rays of a sphere, and is measured as :

$$\omega = \frac{A_n}{r^2} = \frac{A \cos\theta}{r^2} \qquad ...(7.19)$$

where A_n = projection of the incident surface normal to the line of propagation

A = area of incident surface

θ = angle between the normal to the incident surface and the line of propagation

r = length of the line of propagation between the radiating and the incident surfaces

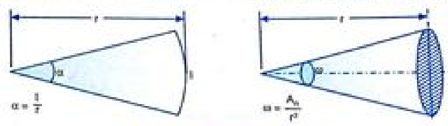


Fig. 7.12. Plane and solid angles

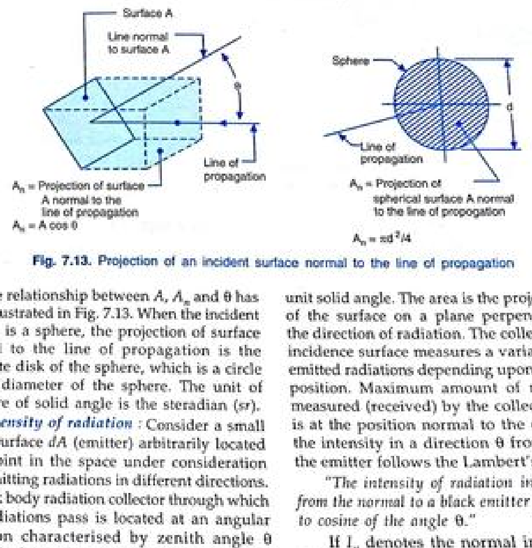


Fig. 7.13. Projection of an incident surface normal to the line of propagation

The relationship between A, A_n and θ has been illustrated in Fig. 7.13. When the incident surface is a sphere, the projection of surface normal to the line of propagation is the silhoutte disk of the sphere, which is a circle of the diameter of the sphere. The unit of measure of solid angle is the steradian (*sr*).

Intensity of radiation : Consider a small black surface dA (emitter) arbitrarily located at a point in the space under consideration and emitting radiations in different directions. A black body radiation collector through which the radiations pass is located at an angular position characterised by zenith angle θ towards the surface normal and the azimuth angle ϕ of a spherical coordinate system. Further, the collector subtends a solid angle $d\omega$ when viewed from a point on the emitter.

The intensity of radiation I is the energy emitted (of all wavelengths) in a particular direction per unit surface area and through a unit solid angle. The area is the projected area of the surface on a plane perpendicular to the direction of radiation. The collector or the incidence surface measures a variation of the emitted radiations depending upon its angular position. Maximum amount of radiation is measured (received) by the collector when it is at the position normal to the emitter, and the intensity in a direction θ from normal to the emitter follows the Lambert's cosine law :

"*The intensity of radiation in a direction θ from the normal to a black emitter is proportional to cosine of the angle θ.*"

If I_n denotes the normal intensity and I_θ represents the intensity at angle θ from the normal, then

$$I_\theta = I_n \cos\theta \qquad ...(7.20)$$

Apparently the energy radiated out decreases with increase in θ and becmes zero at $\theta = 90°$.

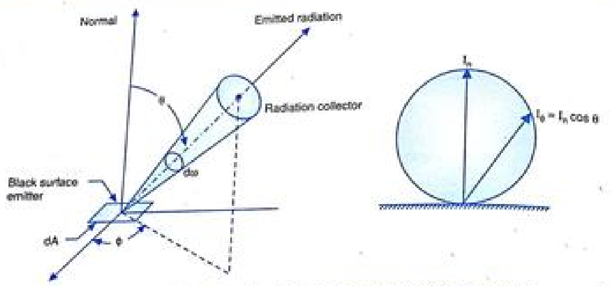


Fig. 7.14. Spatial distribution of radiations emitted from a surface element

When the collector is oriented at an angle θ_1 from the normal to the emitter, then the radiations striking and being absorbed by the collector can be expressed as :

$$(dE_b)_{\theta_1} = I_{\theta_1} d\omega_1 \, dA$$
$$= I_n \cos\theta_1 \, d\omega_1 \, dA \quad ...(7.21)$$

where $d\omega_1$ is the solid angle subtended by the collector at the surface of the emitter.

The collector could be located at different angular positions and still maintain the same radial distance from the emitter. Let it subtend a solid angle $d\omega_2$ at the emitter surface when located in a direction θ_2 from the normal. Then the rate of flow of energy through it will be:

$$(dE_b)_{\theta_2} = I_{\theta_2} d\omega_2 \, dA$$
$$= I_n \cos\theta_2 \, d\omega_2 \, dA \quad ...(7.22)$$

It follows from equations 7.21 and 7.22 that for any surface located at an angle θ from the normal and subtending a solid angle $d\omega$ at the emitter dA,

$$(dE_b)_\theta = I_n \cos\theta \, d\omega \, dA \quad ...(7.23)$$

Relation between the normal intensity and emissive power : To establish a relation between the normal intensity and the emissive power, we relate the differential solid angle $d\omega$ to the zenith and azimuth angles by noting that for a spherical surface :

area of collector

$$= (r\,d\theta) \times (r \sin\theta \, d\phi)$$
$$= r^2 \sin\theta \, d\theta \, d\phi$$

$\therefore$ solid angle $d\omega$

$$= \frac{r^2 \sin\theta \, d\theta \, d\phi}{r^2} = \sin\theta \, d\theta \, d\phi$$

Then the radiations leaving the emitter and striking the collector is :

$$dE_b = I_n \cos\theta \sin\theta \, d\theta \, d\phi \, dA \quad ...(7.24)$$

The total energy E_b radiated by the emitter and passing through a spherical region can be worked out be integrating equation 7.24 over the limits.

$$\theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

and $\phi = 0$ to $\phi = 2\pi$

Thus,

$$\int_{sphere} dE_b = I_n \, dA \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{2\pi} d\phi$$

$$E_b = I_n \, dA \times \frac{1}{2}(2\pi) = I_n \, \pi \, dA \quad ...(7.25)$$

But the total emissive power of the emitter with area dA and temperature T is also given by:

$$E_b = \sigma_b \, T^4 \, dA$$

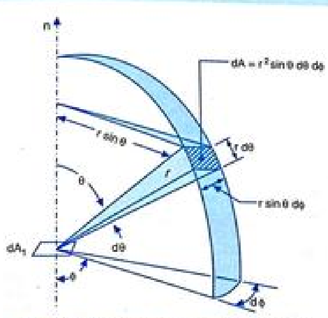


Fig. 7.15. Differential solid angle in terms of zenith and azimuthal angles

Combining this with equation 7.25, we get :

$$I_n = \frac{\sigma_b T^4}{\pi dA} \times dA = \frac{\sigma_b T^4}{\pi} \quad ...(7.26)$$

Thus for a unit surface, the ***intensity of normal radiation I_n is the 1/π times the emissive power E_b***.

EXAMPLE 7.31.

Define intensity of radiation and prove that the intensity of normal radiation is 1/π times the total emissive power.

A black body of 0.2 m^2 area has an effective temperature of 800 K. Calcualte (a) the total rate of energy emission (b) the intensity of normal radiation (c) the intensity of radiation along a direction 60° to the normal, and (d) the wavelength of maximum monochromatic emissive power.

Solution : From Stefan-Boltzman law, the rate of energy emission E_b is :

$$E_b = \sigma_b \, AT^4$$
$$= (5.67 \times 10^{-8}) \times 0.2 \times 800^4$$
$$= \mathbf{4644.86 \ W}$$

(*ii*) The intensity of normal radiation is

$$I_n = \frac{\sigma_b T^4}{\pi}$$
$$= \frac{5.67 \times 10^{-8} \times 800^4}{\pi}$$
$$= 7396.28 \ W/m^2 \text{ steradian}$$

(*iii*) From Lambert's cosine law :

$$I_\phi = I_n \cos\phi$$
$$= 7396.28 \times \cos 60°$$
$$= 3698.14 \ W/m^2 \text{ steradian}$$

(*iv*) From Wien's displacement law, the wavelength λ_{max} for maximum monochromatic emissive power is :

$$\lambda_{max} = \frac{2.898 \times 10^{-3}}{T}$$
$$= \frac{2.898 \times 10^{-3}}{800} = \mathbf{36.25 \times 10^{-6} \ m}$$

EXAMPLE 7.32

A small surface of area A_1 = 0.0015 m^2 emits diffusely, and measurements indicate that the total intensity associated with emission in the normal direction I_n = 6500 W/m^2-sr. The radiation thus emitted is intercepted by three surfaces of areas $A_2 = A_4$ = 0.001 m^2 and A_3 = 0.00125 m^2 which are at a distance of 0.6 m from A_1 and their orientation is as indicated in the figure given below. Make calculations for : (a) intensity associated with emission in each of the three directions, (b) solid angles subtended by the intercepting surfaces when viewed from the emitting surface, and (c) rates at which radiation emitted by A_1 is intercepted by the three surfaces A_2, A_3 and A_4.

Solution : (*a*) For a diffused emitter, the intensity of the emitted radiation is independent of direction. Therefore

$$I = 6500 \ W/m^2\text{-sr}$$

for each of the three directions.

(*b*) In terms of differential surface and the distance radiation travels, the solid angle is given by:

$$d\omega = \frac{dA_n}{r^2} = \frac{dA \cos\theta}{r^2}$$

where θ is the angle between the surface normal and the direction of radiation. Therefore solid angle subtended by surface A_2 with respect to surface A_1 is

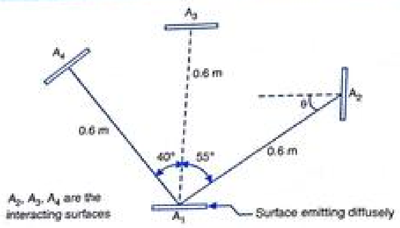


Fig. 7.16.

$$\omega_{2-1} = \frac{0.001 \times \cos(90-55)}{0.6^2}$$
$$= 2.275 \times 10^{-3} \text{ sr}$$

Similarly,

$$\omega_{3-1} = \frac{0.00125 \cos 0^\circ}{0.6^2} = 3.47 \times 10^{-3} \text{ sr}$$

$$\omega_{4-1} = \frac{0.001 \cos 0^\circ}{0.6^2} = 2.78 \times 10^{-3} \text{ sr}$$

(c) The rate at which radiation is intercepted by each of three surfaces A_2, A_3 and A_4 can be obtained from the expression

$$Q_{1-j} = I A_1 \cos \theta_1 \, \omega_{j-1}$$

where θ_1 is the angle between normal to the emitting surface A_1 and the direction of propagation of radiation. Therefore

$$Q_{1-2} = 6500 \times 0.0015 \cos 55 \times 2.275 \times 10^{-3}$$
$$= 12.72 \times 10^{-3} \text{ W}$$
$$Q_{1-3} = 6500 \times 0.0015 \cos 0 \times 3.47 \times 10^{-3}$$
$$= 33.83 \times 10^{-3} \text{ W}$$

and
$$Q_{1-4} = 6500 \times 0.0015 \cos 40 \times 2.78 \times 10^{-3}$$
$$= 20.76 \times 10^{-3} \text{ W}$$

7.10. SOLAR RADIATIONS

The sun is a source of heat radiations and it emits radiations in all directions. The atmosphere absorbs a part of the heat radiations and air, clouds, dust particles etc in the atmosphere scatter the heat and light radiations falling on them. Obviously, the earth receives only a fraction of the energy emitted by the sun. The solar radiation that is felt at the earth's surface includes direct radiation that has passed through the atmosphere; diffuse radiation from the sky; reflected radiation from water, snow and other such materials on the surface. The approximate distribution of the flow of sun's energy to the earth's surface is :

- 9% is scattered
- 15% is absorbed in the atmosphere and out of it 4% reaches the earth's surface by convection
- 43% is transmitted to the earth directly and by diffuse radiation
- 33% is reflected back to space

When the sun lies at a mean distance from the earth, the heat flux from the sun to the outer edge of the atmosphere has been found to be about 1350 W/m^2 and 47% of this (or 635 W/m^2) would reach the earth's surface. A large fraction of the sun's energy reaches the earth's surface in ultra-voilet and visible wavelength; and re-radiation from the cool surface of the earth would be in wavelength that are generally far larger.

Green House Effect

Much of the solar radiation is transmitted through the glass or plastic covering and

absorbed by the objects within the enclosure. As their temperature rises, they too radiate energy which, however, is mostly in the higher wavelength band to which the glass or plastic is opaque. Most of the thermal radiation emitted at low temperature is reflected back and remains inside. Because of this one-way action of heat exchange of the glass or plastic, the temperature within the enclosure becomes considerably higher than the ambinet temperature outside. The phenomenon is commonly referred to as the "green house effect", and obviously it is a manifestation of transmission of low wavelength energy (from the sun) and absorption or reflection of higher wavelength emission at low temperature.

Selective absorbing surfaces have been developed for use in solar collectors; these surfaces absorb much of the incident radiation without reradiating it.

The sky creates a partial green house effect if it is heavily loaded with CO_2, H_2O and to a lesser extent, ozone.

Solar constant : From the quantity of heat radiations received by the earth, it is possible to estimate the temperature of the sun. For that certain ideal conditions are taken into consideration and a parameter called solar constant is introduced.

Solar constant is the amount of heat energy (radiation) absorbed per unit time by unit area of a perfectly black body surface placed at a mean distance of the earth from the sun, in the absence of the atmosphere; the surface being held perpendicular to the sun's rays.

The heat energy absorbed by a known area in a fixed time is determined with the help of an instrument called ***pyrheliometer***. The effects of absorption by the atmosphere are eliminated by finding the value of the solar constant at various altitudes of the sun on the same day under similar sky conditions.

The observed solar constant S_0, the true solar constant S and the angular elevation (altitude) Z of the sun are related by the expression,

$$S_0 = S\, a^{\sec Z} \qquad (a \text{ is a constant})$$

or
$$\log S_0 = \log S + \sec Z \log a$$

From the straight line graph between $\log S_0$ along the y-axis and $\sec Z$ along the x-axis, the intercept $\log S$ on the y-axis is found and the true solar constant is evaluated there from. The solar constant varies between 1335 and 1815 W/m^2.

Let R = mean distance of earth from the sun

r = radius of the sun

Then the total amount of heat energy received by the sphere of radius R is $4\pi R^2 S$, and the amount of heat energy radiated by unit surface area of the sun in the same time works out as

$$E = \frac{4\pi R^2 S}{4\pi r^2} = \left(\frac{R}{r}\right)^2 S$$

Taking $R = 15 \times 10^{10}$ m ;

$r = 7 \times 10^8$ m

and $S = 1650$ W/m^2

$$E = \left(\frac{15 \times 10^{10}}{7 \times 10^8}\right)^2 \times 1650$$
$$= 75.76 \times 10^6 \text{ W/m}^2$$

Also,
$$E = \sigma_b T^4 = 5.67 \times 10^{-8} T^4$$

$$\therefore \quad T = \left(\frac{75.76 \times 10^6}{5.67 \times 10^{-8}}\right)^{1/4} = 6046 \text{ K}$$

This value represents the effective temperature of the sun acting as a black body.

Temperature of the sun can also be worked out from Wien's displacement law :

$$\lambda_{max} T = 2.89 \times 10^{-3} \text{ mK}$$

The wavelength of radiation for which the energy is maximum is 0.49 micron. The temperature of the sun then works out as

$$T = \frac{2.898 \times 10^{-3}}{0.49 \times 10^{-6}} = 5914 \text{ K}$$

The sun consists of a central hot portic surrounded by the photosphere. T temperature of the photosphere, referred as effective temperature of the sun, is usu taken as 6000 K.

EXAMPLE 7.33

Explain how the temperature inside a glass or plastic shield enclosure can be considerably higher than the ambinet temperature outside.

Consider a deep-space probe constructed as 1 m diameter polished aluminium sphere. Estimate the equilibrium temperature that the probe reaches if the solar energy received is 300 W/m². For solar radiation, absorptivity of aluminium is 0.3 and the average emissivity appropriate for aluminium at low temperature is 0.04.

Solution : The heat input from the sun is determined by the projected area of the sun.

$$Q_{in} = \alpha\, q\, A_p$$

$$= 0.3 \times 300 \times \frac{\pi}{4}\,(1)^2 = 70.7 \text{ W}$$

The outgoing radiation goes to space where the temperature is low enough to be neglected.

$$Q_{out} = \epsilon\, \sigma_b\, AT^4$$

where A is the total area of the sphere.

$$= 0.04 \times (5.67 \times 10^{-8}) \times 4\pi\,(0.5)^2\, T^4$$

$$= 0.712 \times 10^{-8}\, T^4$$

Energy balance gives :

$$0.712 \times 10^{-8}\, T^4 = 70.7$$

$$T = \left[\frac{70.7}{0.712 \times 10^{-8}}\right]^{1/4} = \mathbf{315.67\ K}$$

The probe will attain an equilibrium temperature of **315.67 K**.

EXAMPLE 7.34

Give a brief account of the effects of temperature and surface condition on emissivity.

An artificial satellite flying round the earth has absorptivity α with respect to incident solar radiation and surface emissivity ϵ. The satellite has no inner heat source and has uniform absolute temperature T all over its surface. Set up a relation between T, α and ϵ. You may ignore the effects due to solar radiation reflected from the earth and the radiations emitted by the earth.

Presuming that $\alpha = 0.18$, $\epsilon = 0.12$ and incident soalr radiation $E_{in} = 1560$ W/m², calculate the surface temperature of satellite. How this temperature will change if the satellite surface is considered as gray ?

Solution : Under steady state conditions of heat transfer from the satellite surface,

radiant energy absorbed by the satellite
= energy emitted by the satellite into the space

$$\alpha\, E_{in}\, A_p = \epsilon\, \sigma_b\, A_s\, T^4 \quad \ldots(i)$$

where A_p is projected area of the satellite on the plane normal to incident radiation and A_s is the surface area of satellite. If r is the radius of satellite, then

$$A_p = \pi r^2 \quad \text{and} \quad A_s = 4\pi r^2$$

From expression (*i*)

$$T = \left[\frac{A_p}{A_s} \times \frac{\alpha}{\epsilon} \times \frac{E_{in}}{\sigma_b}\right]^{1/4}$$

$$= \left[\frac{\pi r^2}{4\pi r^2} \times \frac{\alpha}{\epsilon} \times \frac{1560}{5.67 \times 10^{-8}}\right]^{1/4}$$

$$= 288\left(\frac{\alpha}{\epsilon}\right)^{1/4}$$

which is the required relation between T, α and ϵ.

For given values of $\alpha = 0.18$ and $\epsilon = 0.12$

$$T = 288\left(\frac{0.18}{0.12}\right)^{1/4} = \mathbf{318.72\ K}$$

For gray surface $\alpha = \epsilon$, In that case :

$$T = \mathbf{288\ K}$$

EXAMPLE 7.35

An artificial spherical satellite orbiting the earth is shifted towards mars. What shall be the temperature as it approaches the mars if its temperature near the earth was 315 K ? The pertinent data is :

Distance of earth from the sun
$= 149.6 \times 10^6$ *km*

Distance of Mars from the sun
$= 227.9 \times 10^6$ *km*

The emissivity of the satellite does not vary with temperature.

Solution : Under steady state conditions of heat transfer from the satellite surface,

radiant energy absorbed by the satellite
= energy emitted by the satellite into the space.

∴ Near the earth :

$$\alpha\, E_1\, A_p = \epsilon\, \sigma_b\, A_s T_1^4 \quad \ldots(i)$$

Near the Mars :

$$\alpha\, E_2\, A_p = \epsilon\, \sigma_b\, A_s T_2^4 \quad \ldots(ii)$$

where E_1 and E_2 are the intensities of solar radiation near the earth and mars, A_p is the projected area of the satellite on the plane near to incident radiation and A_s is the surface area of satellite.

Upon division, these identities give

$$\left(\frac{T_2}{T_1}\right)^4 = \frac{E_2}{E_1} = \left(\frac{S_1}{S_2}\right)^2$$

where S_1 and S_2 are distances of earth and mars from the sun

$$T_2 = T_1\left(\frac{S_1}{S_2}\right)^{\frac{1}{2}} = 315\left[\frac{149.6 \times 10^6}{227.9 \times 10^6}\right]^{1/2}$$

$$= 255.21 \text{ K} = \mathbf{-\ 17.79°C}$$

EXAMPLE 7.36

It has been observed that when the sun is overhead the earth's surface on a clear day, the radiation received by the earth's surface is 1 kW/m² and an additional 0.3 kW/m² is absorbed by the earth's atmosphere. Assuming the sun to be a black body, determine the temperature of the sun.

Given :

dia of sun $= 1.39 \times 19^9$ *m*

dia of earth $= 12.6 \times 10^6$ *m*

distance between the sun and earth
$= 1.5 \times 10^{11}$ *m*

Solution : Let T_{sun} be the temperature of the sun considered as black body. Then energy radiated by the sun,

$$Q_{sun} = \epsilon\, A\, \sigma_b\, T^4 = 1 \times 4\pi\,(1.39 \times 10^9)^2 \times (5.67 \times 10^{-8}) \times T_{sun}^4$$

$$= 34.4 \times 10^{10}\, T_{sun}^4$$

Considering sun to be point source, the mean area over which the radiation is distributed is

$$= 4\pi\, s^2$$

where s is the distance of sun from the earth

$$= 4\pi\,(1.5 \times 10^{11})^2$$

$$= 28.26 \times 10^{22} \text{ m}^2$$

Then solar flux

$$\frac{Q_{sun}}{A} = \frac{34.4 \times 10^{10}\, T_{sun}^4}{28.26 \times 10^{22}}$$

$$= 1.217 \times 10^{-12}\, T_{sun}^4$$

Total radiation from the sun

$$= 1 + 0.3 \text{ kW/m}^2$$

$$= 1.3 \times 10^3 \text{ W/m}^2$$

$$\therefore 1.217 \times 10^{-12}\, T_{sun}^4 = 1.3 \times 10^3$$

That gives : approximate temperature of sun,

$$= \left[\frac{1.3 \times 10^3}{1.217 \times 10^{-12}}\right]^{1/4} = \mathbf{5716.67\ K}$$

EXAMPLE 7.37

The sun may be regarded as a black body with a surface temperature of 5600 K at a mean distance of 15×10^{10} m from the earth. The diameter of the sun is 1.4×10^9 m and that of the earth is 12.8×10^6 m. Make calculations for (a) the total energy radiated by the sun, (b) the energy received per m² just outside the earth's atmosphere, (c) the total energy the earth would receive if no energy were blocked by the earth's atmosphere, and (d) the energy received by a 1.25×1.25 m solar collector whose perpendicular is inclined at 35° to the sun. The energy loss through the atmosphere is 35% and the diffuse radiation is 15% of direct radiation.

Solution : For the sun :

$$\epsilon = 1 \text{ (black body)}$$

and surface area

$$= 4\pi r^2 = 4\pi\,(0.7 \times 10^9)^2$$

∴ Energy radiated by the sun, Q

$$= \epsilon\, \sigma_b\, A\, T^4$$

$= 1 \times (5.67 \times 10^{-8})$
$\times 4\pi\,(0.7 \times 10^{9})^2 \times (5600)^4$

$= \mathbf{3.43 \times 10^{26}\ W}$

(*b*) The sun may be regarded as a point source at a distance of 15×10^{10} from the earth. The mean area over which the radiation is distributed becomes $4\pi\,(15 \times 10^{10})^2$

$\therefore$ Radiation received at this distance

$$= \frac{3.43 \times 10^{26}}{4\pi(15 \times 10^{10})^2}$$

$= \mathbf{1.213 \times 10^3\ W/m^2}$

(*c*) The earth is nearly spherical and as such the energy received by it will be proportional to the perpendicular projected area, *i.e.*, that of a circle.

$\therefore$ Energy received by the earth

$= 1.213 \times 10^3 \times \pi\,(6.4 \times 10^6)^2$

$= \mathbf{1.56 \times 10^{17}\ W}$

(*d*) Direct energy reaching the earth,

$$= \left(1 - \frac{35}{100}\right) \times 1.213 \times 10^3$$

$= 0.788 \times 10^3\ W/m^2$

Diffused radiation,

$$= \frac{15}{100} \times 0.788 \times 10^3$$

$= 0.1182 \times 10^3\ W/m^2$

Total radiation reaching the plate,

$(0.788 + 0.1182)\,10^3 = 0.9062 \times 10^3\ W/m^2$

Since the plate surface is not oriented perpendicular to the incoming radiations, the relevant area is equivalent to the projected perpendicular surface area.

Projected plate area,

$= A \cos\theta$

$= 1.25 \times 1.25 \times \cos 35 = 1.28\ m^2$

$\therefore$ Energy received by the plate

$= 0.9062 \times 10^3 \times 1.28$

$= \mathbf{1.16 \times 10^3\ W}$

A reduction in energy received due to inclination explains the variation in solar intensity with season and much reduced solar intensity at the poles of earth.

SALIENT POINTS

1. Thermal radiation is the electromagnetic radiation emitted by a body due to its temperature.

   The wavelength λ, speed C and frequency f of radiation waves are correlated by the expression:

   $$C = f\lambda$$

   The radiation waves propagate with the speed of light, and their wavelength ranges from 0.1 to 100 µm.

2. A radiant energy incident on a body is partially absorbed, partially reflected and partially transmitted

   $\alpha + \rho + \tau = 0$ for all bodies

   The factors absorptivity α, reflectivity ρ and transmissivity are dimesionless and vary from 0 to 1.

   (*i*) A black body absorbs all the radiations incident upon it ($\alpha = 1$ and $\rho = \tau = 0$)

   (*ii*) A body that does not allow any radiation to pass through it ($\tau = 0$ and $\alpha + \rho = 1$) is called an opaque body.

   (*iii*) The body is called transparent or diathermaneous if it allows all the radiations to pass through it ($\tau = 1$ and $\alpha = \rho = 0$).

   (*iv*) A body that reflects all the incident thermal radiations is called a specular body (if the reflection is regular) or an absolutely white body (if the reflection is diffused). For such bodies $\rho = 1$ and $\alpha = \tau = 0$.

   - **Regular reflection:** angle between the reflected beam and normal to the surface equals the angle made by the incident radiation with the same normal.
   - **Diffused reflection:** the incident beam is reflected in all directions, *i.e.*, there is directional independence of the reflected beam.

   (*v*) When a surface absorbs certain fixed percentage of impinging radiations, the surface is called the gray body. The absorpitivity of a gray body remains constant over the entire range of temperature and wavelength of incident radiation.

3. The rate of energy radiated per unit area of the surfaces per unit wavelength is called the monochromatic emissive power (E_λ).

   The total emissive power is then given by

   $$E = \int_0^{\infty} E_\lambda\, d\lambda$$

   Based upon extensive experimental evidence, Max Planck suggested the following expression for the spectural distribution of emissive power for a black body

   $$(E_\lambda)_b = 2\pi c^2 h \frac{\lambda^{-5}}{\exp\left[\frac{ch}{k\lambda T}\right] - 1}$$

   By integration of monochromatic emissive power over the entire band width of wavelength from $\lambda = 0$ to $\lambda = \infty$, the following equation can be set up for the total emissive power for a black body

   $$E_b = \sigma_b T^4 = 5.64 \times 10^{-8} T^4$$

   This expression is referred to as Stefan Boltzman law.

   (*i*) The wavelength at which emissive power is maximum is prescribed by Wien's displacement law prescribed as

   $$\lambda_{max} T = 0.0029\ mK$$

   A combination of Planck's law and Wien's displacement law yields the following correlation for maximum monochromatic emissive power for a black body.

   $$(E_\lambda)_{max} = 1.285 \times 10^{-5}\, T^5\ W/m^2$$

   per meter wavelength

4. Emissivity $\in$ of a surface is defiend as the ratio of total emissive power of a surface to that of a black surface at the same temperature.

   $$\text{Emissivity } \in = \frac{E}{E_b} = \frac{E}{\sigma_b A T^4}$$

   For a gray body, the emissivity is constant at all temperatures and throughout the entire range of wavelength.

5. Kirchoff's law states that the ratio of emissive power E to the absorptivity α is same for all the bodies, and is equal to emissive power of a black body at the same temperature.

   $$\frac{E}{\alpha} = E_b; \quad \alpha = \frac{E}{E_b} = \in$$

   That is the emissivity $\in$ and absorptivity α of a real surface are equal for radiation with identical temperature and wavelength.

6. A real surface always radiats less than a black body at the same temperature. For a gray body (a body with constant emissivity $\in$), the emissive power would be

   $$E_g = \in \sigma_b T^4$$

7. The intensity of radiation is defined as the rate of energy leaving a surface in a given direction per unit solid angle per unit area of the emitting surface normal to the mean direction.

   A solid angle is a portion of the space inside a sphere enclosed by a conical surface with the vertex of the cone at the centre of the sphere.

8. For a unit surface the intensity of normal radiation I_n is $\frac{1}{\pi}$ times the emissive power E_b.

   Further, according to Lambert's cosine law,

   $$I_\theta = I_n \cos\theta$$

   where I_θ represents the intensity at angle θ from the normal.

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. Enumerate some salient features of thermal radiation. What position does thermal radiation occupy in the electromagnetic spectrum ? What limits this band width on the short and long wavelength sides ?
2. State the relationship between wavelength and frequency of radiation propagating in a medium.
3. Explain the terms absorptivity, reflectivity and transmissivity of radiant energy. How are the related to each other for a black body and a opaque body ?

   (*b*) By qualitative reasoning establish t expression :

   $$\alpha + \rho + \tau = 1$$

where α = absorptivity, ρ = reflectivity and τ is transmissivity of the body. Comment upon the validity of the statement that this relation is valid only for a gray-diffuse surface.

4. Based upon the reradiating properties of absorptivity, reflectivity and transmissi-vity, how would you distinguish between the following :

   black body, white body, transparent body and opaque body

5. Define a black body. Give examples of some surfaces which donot appear black but have high values of absorptivities.
6. Two pieces of wood are placed in sun light; one piece is painted white and the other black. Which piece will absorb more heat ? If the same two pieces, but at room temperature, are laid on the ground in the mid winter and at night, which piece will cool faster ? Give argument in support of your answer.
7. What do you understand by a 'black body' and a 'gray body' as applied to radiation problems. Explain how radiation streaming out of a small hole in a large hollow body can be considered as black body radiation.
8. A black body may be approximated by means of a large hollow container with a small opening. Exactly which surface of the arrangement can be considered as black body ?
9. How does regular or specular reflection differ from diffuse reflection?
10. Define monochromatic and total emissive power. How is the later related to the absolute temperature ? Describe how the monochromatic emissive power varies with the wavelength for emissions from a black body ? At what wavelength is the black body monochromatic emissive power the maximum ?
11. State and explain the following laws relating to thermal radiation and temperature of a radiating body :

    Planck's law : Stefan Boltzman law and Wien's displacement law

12. State and prove Kirchoff's law of radiation. What restrictive conditions are inherent in the derivation of Kirchoff's law ?
13. Define Lambert's cosine law of radiation and prove that the intensity of radiation is always constant at any angle of emission for a diffused surface.
14. Define the following terms as applied to radiation heat transfer :
    (i) Black, gray and real surface
    (ii) Spectral and spacial distribution of energy
    (iii) Specular and diffused reflection
    (iv) Emissive power and intensity of radiation.
15. Asphalt pavements on hot summer days exhibit temperatures of approximately 50°C. Consider such a surface to emit as a black body, and calculate the emitted radiant energy per unit surface area.

    (Ans. 6.17 kW)

16. After sunset, radiant energy can be sensed by a person standing near a brick wall. Such walls have surface temperatures around 320 K, and typical brick emissivity values are of the order of 0.92. Make calculations for the radiant thermal flux per square metre from a brick wall at this temperature.

    (Ans. 5.47 kW)

17. A black body, in the form of a cube 1 m long on the side has a temperature 1000 K. Find the radiant energy flux density and the total energy emitted by the black body cube.

    (Ans. 5.67 kW/m², 340.2 kW)

18. The filament of a 75 W light bulb may be considered as a black body radiating into a black enclosure at 70°C. The filament diameter is 0.10 mm and the length is 5 cm. Considering only radiation, make calculations for the filament temperature.
19. What is the radiant energy flux from a steel products at 1000 K ? The product may be treated as a gray body with constant emissivity ϵ = 0.7. Also workout the wavelength corresponding to the maximum spectral intensity of radiation.

    (Ans. 39.7 kW/m², 2.90 km)

20. For a black body maintained at 115°C, make calculations for the (a) total emissive power, (b) wavelength at which the maximum monochromatic power occurs, and (c) maximum monochromatic emissive power.
21. The maximum spectral intensity of radiation for a gray surface at 1100 K is given to be 1.37×10^{10} W/m²-m of wavelength. Determine the emissivity of the body surface and the wavelength at which the maximum spectral intensity of radiation occurs.

    (Ans. 0.662, 2.636×10^{-6} m)

B. Fill in the blanks with appropriate word/words :

1. No .... contact is required for radiation heat transfer but the surefaces should be in .... constant for direct radiation transfer.
2. Thermal radiation is limited to a range of wavelength between .... of the spectrum of electromagnetic radiation.
3. All bodies above absolute zero temperature emit radiations. This statement is based on .... of heat exchange.
4. A body that allows all the incident radiation to pass through it is called ....
5. The absorptivity of a .... does not vary with temperature and wavelength of incident radiation.
6. The ratio of total emissive power to the absorptivity is constant for all real surfaces with identical temperature and wavelength. This statement is referred to as ....
7. Emissivity and absorptivity of a surface are equal for radiation with equal .... and ....
8. The .... is the energy emitted (of all wavelengths) in a particular direction per unit surface area and through a unit solid angle.
9. The intensity of radiation in a direction θ from the normal to a black emitter is proportional to .... of the angle θ.
10. For a unit surface, the intensity of normal radiation is $\frac{1}{\pi}$ times the ....
11. The law governing the distribution of radiant energy over wavelength for a black body at fixed temperature is referred to as ....
12. The surface temperature of the sun is nearly .... degree Kelvin.

Answers : 1. physical, visual; 2. 0.1 to 100 μm; 3. Prevsot theory; 4. transparent or diathermaneous; 5. gray; 6. Kirchoff's law; 7. temperature and wavelength; 8. intensity of radiation; 9. cosine; 10. emissive power; 11. Planck's law; 12. 6000°K.

C. Multiple choice questions :

1. Radiation heat transfer is characterised by :
   (a) energy transport as a result of bulk fluid motion
   (b) thermal energy transfer as vibrational energy in the lattice structure of the material
   (c) movement of discrete packets of energy as electromagnetic waves
   (d) circulation of fluid motion by buoyancy effects
2. Thermal radiations occur in the portion of electromagnetic spectrum between the wavelengths
   (a) 10^{-2} to 10^{-4} micron
   (b) 10^{-1} to 10^{-2} micron
   (c) 0.1 to 10^{2} micron
   (d) 10^{2} micron onwards
3. A perfectly black body
   (a) absorbs all the incident radiation
   (b) allows all the incident radiation to pass through it
   (c) reflects all the incident radiation
   (d) has its surface coated with lamp black or graphite
4. For a perfectly black body
   (a) absorptivity $\alpha = 1$, reflectivity $\rho = 0$ and transmissivity $\tau = 0$
   (b) $\rho = 1$ and $\alpha = \tau = 0$
   (c) $\tau = 1$ and $\alpha = \rho = 0$
   (d) $\alpha + \tau = 1$ and $\rho = 0$
5. For an absolutely white or specular body
   (a) absorptivity $\alpha = 1$, reflectivity $\rho = 0$ and transmissivity $\tau = 0$
   (b) $\rho = 1$ and $\alpha = \tau = 0$
   (c) $\tau = 1$ and $\alpha = \rho = 0$
   (d) $\alpha + \tau = 1$ and $\rho = 0$
6. For a transparent or diathermanous body
   (a) absorptivity $\alpha = 1$, reflectivity $\rho = 0$ and transmissivity $\tau = 0$
   (b) $\rho = 1$ and $\alpha = \tau = 0$
   (c) $\tau = 1$ and $\alpha = \rho = 0$
   (d) $\alpha + \tau = 1$ and $\rho = 0$
7. A diathermanous body
   (a) shines as a result of incident radiation
   (b) gets heated up as a result of absorption of incident radiation
   (c) allows all the incident radiation to pass through it
   (d) partly absorbs and partly reflects the incident radiation

8. A body which partly absorbs and partly reflects but does not allow any radiation to pass through it ($\alpha + \rho = 1$ and $\tau = 0$) is called
(a) diathermanous (b) opaque
(c) grey (d) specular

9. Choose the false statement :
(a) snow is nearly black to thermal radiation
(b) absorption of radiation occurs in a very thin layer of material near the surface
(c) transmissivity varies with wavelength of incident radiation, *i.e.*, a material may be non-transparent for a certain wavelength band and be transparent for another
(d) most of the engineering materials have rough surfaces and these rough surfaces give regular (specular) reflections

10. Gases have poor
(a) absorptivity
(b) reflectivity
(c) transmissivity
(d) absorptivity as well as transmissivity

11. With an increase in wavelength, the monochromatic emissive power of a black body
(a) increases
(b) decreases
(c) increases, reaches a maximum and then decreases
(d) decreases, reaches a minimum and then increases

12. With an increase in the temperature of source, the wavelength at which the monochromatic emissive power is maximum
(a) increases continuously
(b) decreases continuously
(c) increases, reaches a maximum and then decreases
(d) decreases, reaches a minimum and then increases

13. Absorptivity of a body is equal to its emissivity
(a) for a polished body
(b) under thermal equilibrium condition
(c) at one particular temperature
(d) at shorter wavelengths

14. The ratio of total emissive power of body to the total emissive power of a black body at the same temperature is called
(a) absorptivity (b) transmissivity
(c) reflectivity (d) emissivity

15. A surface for which emissivity is constant at all temperatures and throughout the entire range of wavelength is called
(a) opaque (b) grey
(c) specular (d) diathermanous

16. Four identical pieces of copper painted with different colour of paints were heated to the same temperature and then left in the environment to cool. Which of the following paints will give fast cooling ?
(a) white (b) rough
(c) black (d) shining

17. For a grey surface
(a) emissivity is constant
(b) absorptivity equals reflectivity
(c) emissivity equals transmissivity
(d) reflectivity equals emissivity

18. What is the basic equation of radiation from which all other equations of radiation equations can be derived?
(a) Stefan-Boltzman equation
(b) Planck's equation
(c) Wien's equation
(d) Rayleigh-Jeans formula

19. The law governing the distribution of radiant energy over wavelength for a black body at fixed temperature is referred to as
(a) Planck's law (b) Wien's formula
(c) Kirchoff's law (d) Lambert's law

20. The thermal radiation propagates in the form of discrete quanta; each quanta having an energy of $E = h\nu$ where ν is the frequency of quantum. The Planck's constant h has the dimensions
(a) MLT (b) MLT^{-1}
(c) MLT^{-2} (d) ML^2T^{-1}

21. The emissivity and the absorptivity of a real surface are equal for radiation with identical temperature and wavelength. This law is referred to as
(a) Lambert's law
(b) Kirchoff's law
(c) Planck's law
(d) Wien's displacement law

22. A thermally transparent surface of transmissivity 0.15 receives 2000 kJ/min of radiation and reflects back 800 kJ/min out of it. The emissivity of the surface is then
(a) 0.15 (b) 0.4
(c) 0.45 (d) 0.55

23. The intensity of solar radiation on earth is
(a) 1 kW/m^2 (b) 2 kW/m^2
(c) 5 kW/m^2 (d) 10 kW/m^2

24. The relationship, $\lambda_{max}\, T$ = constant, between the temperature of a black body and the wavelength at which maximum value of monochromatic emissive power occurs is known as
(a) Planck's law (b) Wien's law
(c) Kirchoff's law (d) Lambert's law

25. The Stefan-Boltzman constant has units of
(a) kcal/m^2-hr-K^4 (b) kcal/m-hr-K^4
(c) kcal/hr-K^4 (d) kcal/m^2-K^4

26. The temperature of a solid surface changes from 27°C to 627°C. The emissive power changes would then conform to the ratio
(a) 6 : 1 (b) 9 : 1
(c) 27 : 1 (d) 81 : 1

27. If the temperature of a hot body is increased by 50%, the amount of radiation emitted by it would increase by nearly
(a) 50% (b) 100%
(c) 200% (d) 500%

28. The following figure 7.17 was generated from experimental data relating spectral black body emissive power to wavelength at the three temperatures T_1, T_2 and T_3 ($T_1 > T_2 > T_3$).

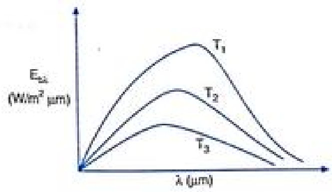


Fig. 7.17.

What conclusion can be drawn with respect to experimental data?
(a) correct because the maximum in $E_{b\lambda}$ shows the correct trend
(b) correct because Planck's law is satisfied
(c) wrong because the Stefan Boltzman law is not satisfied
(d) wrong because Wien's displacement law is not satisfied

29. A body at 500 K cools by radiating heat to ambient atmosphere maintained at 300 K. When the body has cooled to 400 K, the cooling rate as a percentage of original rate is about
(a) 31.1 (b) 41.5
(c) 50.3 (d) 80.4

30. For a hemisphere, the solid angle is measured in
(a) radian and its maximum value is π
(b) degree and its maximum value is 180°
(c) steradian and its maximum value is 2π
(d) steradian and its maximum value is π

31. The energy emitted (of all wavelengths) in a particular direction per unit surface area and through a unit solid angle is called
(a) total emissive power
(b) monochromatic emmissive power
(c) radiant flux
(d) intensity of radiation

32. The emissive power is multiplied with the factor .... to obtain the intensity of normal radiation for a unit surface
(a) $1/\sqrt{\pi}$ (b) $1/\pi$
(c) $1/2\pi$ (d) $\sqrt{\pi}$

33. Two spheres A and B of same material have radii 1 m and 4 m, and temperatures 4000 K and 2000 K respectively. Then the eneregy radiated by sphere A is
(a) greater than that of sphere B
(b) less than that of sphere B
(c) equal to that of sphere B
(d) two times that of sphere B

Answers :

1. (c)	2. (c)	3. (a)	4. (a)	5. (b)
6. (c)	7. (c)	8. (b)	9. (d)	10. (b)
11. (c)	12. (b)	13. (b)	14. (d)	15. (b)
16. (c)	17. (a)	18. (b)	19. (a)	20. (d)
21. (b)	22. (c)	23. (a)	24. (b)	25. (a)
26. (d)	27. (d)	28. (d)	29. (a)	30. (d)
31. (d)	32. (b)	33. (c)		

HINTS AND COMMENTS

9(*d*):

Rough surfaces give diffused reflections. Reflections from highly polished and smooth surfaces have regular (specular) characteristics.

16(*c*):

The emissivity of a black paint is highest (close to unity). Consequently, the emitted radiant energy will be maximum when painted black. Higher the emitted radiation, fast will be the cooling.

26(*d*):

$$\frac{E_2}{E_1} = \frac{\sigma_b A T_2^4}{\sigma_b A T_1^4}$$

$$= \left(\frac{T_2}{T_1}\right)^4 = \left(\frac{273+627}{273+27}\right)^4$$

$$= (3)^4 = 81$$

27(*d*):

$$\frac{E_2}{E_1} = \frac{\sigma_b A T_2^2}{\sigma_b A T_1^4}$$

$$= \left(\frac{T_2}{T_1}\right)^4 = (1.5)^4 = 5.06$$

The amount of radiation emitted would increase nearly by 500%.

28(*d*):

According to Wien's displacement law $\lambda_m T$ = constant. As λ_m increases, T decreases and accordingly $E_{b\lambda}$ decreases. As such, the correct diagram relating spectral black body emissive power to wavelength would be as shown below:

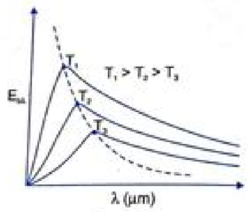


Fig. 7.18.

29(*a*):

$$\frac{Q_2}{Q_1} = \frac{T_2^4 - T_\infty^4}{T_1^4 - T_\infty^4}$$

$$= \frac{400^4 - 300^4}{500^4 - 300^4}$$

$$= 0.32 \text{ or } 32\%$$

33(*c*):

$$\frac{E_A}{E_B} = \frac{\sigma\, 4\pi R_A^2 T_A^4}{\sigma\, 4\pi R_B^2 T_B^4}$$

$$= \frac{R_A^2 T_A^4}{R_B^2 T_B^4}$$

$$= \frac{1^2 \times 4000^4}{4^2 \times 2000^4} = 1$$

As such $E_A = E_B$

Chapter

8

Radiation : Exchange Between Surfaces

Learning objectives : Attention has been focussed in this chapter to make the reader familiar with

- *heat exchange between black bodies – configuration factor*
- *shape factor algebra and salient facts about shape factor*
- *heat exchange between non-black bodies; interchange factor and geometric factor*
- *electrical network approach for radiation heat exchange, radiation shields, adiabatic and reradiating surfaces*
- *coefficient of radiant heat transfer and radiation combined with convection*

Engineering problems of practical interest are involved with heat exchange between two or more surfaces, and this exchange is strongly dependent upon their radiative properties, temperature levels, surface geometrics (size and shape) and orientation relative to each other. For black surfaces, it is necessary to determine what portion of radiation emitted by one will be intercepted by the other. If surfaces are non-black, considerations have to be given to the emissivity, reflectivity and transmissivity of the radiating surfaces. The problem becomes more complex if there is an absorbing or radiating medium between the surfaces. Attention would be directed in this chapter on the geometric features of radiation exchange, and the relations would be setup by presuming that the surfaces are separated by non-participating medium. A non-participating medium neither emits, absorbs nor scatters radiations and obviously it has no effect on the radiation heat exchange. Most of the gases meet this requirement to an excellent approximation; exceptions are carbon dioxide and water vapours which have high absorptivites at certain wave lengths of infrared radiation. Some reflections have also been made on the electrical network approach for radiation heat exchange and the utility of radiation shields.

8.1. HEAT EXCHANCE BETWEEN BLACK BODIES : CONFIGURATION FACTOR

Consider heat exchange between elementary areas dA_1 and dA_2 of two black radiating bodies having areas A_1 and A_2 respectively. The elementary areas are at a distance r apart and the normals to these areas make angles θ_1 and θ_2 with the line joining them. The surface dA_1 is at temperature T_1 and the surface dA_2 is at temperature T_2.

If the surface dA_2 subtends a solid angle $d\omega_1$ at the centre of the surface dA_1, then radiant energy emitted by dA_1 and impinging on (and absorbed by) the surface dA_2 is :

$$dQ_{12} = I_{\theta_1}\, d\omega_1\, dA_1$$

$$= I_{n1} \cos\theta_1\, d\omega_1\, dA_1$$

Projected area of dA_2 normal to the line joining dA_1 and $dA_2 = dA_2 \cos\theta_2$

solid angle $d\omega_1 = \dfrac{dA_2 \cos\theta_2}{r^2}$

$$\therefore \quad dQ_{12} = I_{n1} \cos\theta_1 \frac{dA_2 \cos\theta_2}{r^2} dA_1$$

$$= I_{n1} \frac{\cos\theta_1 \cos\theta_2 \, dA_1 \, dA_2}{r^2}$$

But $I_{n1} = \dfrac{\sigma_1 T_1^4}{\pi}$ and therefore

$$dQ_{12} = \frac{\sigma_1 T_1^4}{\pi} \frac{\cos\theta_1 \cos\theta_2 \, dA_1 \, dA_2}{r^2} \quad ...(8.1)$$

Integration of equation 8.1 over finite areas A_1 and A_2 gives :

$$Q_{12} = \frac{\sigma_1 T_1^4}{\pi} \int_{A_1} \int_{A_2} \cos\theta_1 \cos\theta_2 \frac{dA_1 \, dA_2}{r^2} \quad ...(8.2)$$

The solution to this equation is simplified by introducting a term called ***radiation shape factor, geometrical factor, configuration factor or view factor.*** The configuration factor depends only on the specific geometry of the emitter and collection surfaces, and is defined as :

"The fraction of the radiative energy that is diffused from one surface element and strikes the other surface directly with no intervening reflections."

The radiation shape factor is represented by the symbol F_{ij} which means the shape factor form a surfacxe, A_i to another surface, A_j. Thus the radiation shape factor F_{12} of surface A_1 to surface A_2 is

$$F_{12} = \frac{\text{direct radiation from surface 1 incident upon surface 2}}{\text{total radiation from emitting surface 1}}$$

$$= \frac{Q_{12}}{\sigma_1 A_1 T_1^4}$$

$$= \frac{1}{\sigma_1 A_1 T_1^4} \times \frac{\sigma_1 T_1^4}{\pi} \times \int_{A_1} \int_{A_2} \cos\theta_1 \cos\theta_2 \frac{dA_1 \, dA_2}{r^2}$$

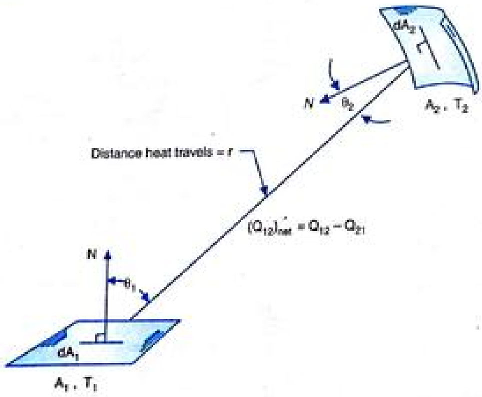


Fig. 8.1. Radiant heat exchange between two black surfaces

$$= \frac{1}{A_1 \pi} \int_{A_1} \int_{A_2} \cos\theta_1 \cos\theta_2 \frac{dA_1 \, dA_2}{r^2} \quad ...(8.3)$$

Thus the amount of radiation leaving A_1 and striking A_2 may be written as :

$$Q_{12} = A_1 F_{12} \sigma_1 T_1^4 \quad ...(8.4)$$

Similarly the energy leaving A_2 and arriving A_1 is :

$$Q_{21} = A_2 F_{21} \sigma_2 T_2^4 \quad ...(8.5)$$

and the net energy exchange from A_1 to A_2 is :

$$(Q_{12})_{net} = A_1 F_{12} \sigma_1 T_1^4 - A_2 F_{21} \sigma_2 T_2^4$$

When the surfaces are maintained at the same temperatures, $T_1 = T_2$, there can be no heat exchange.

$$\therefore \quad 0 = (A_1 F_{12} - A_2 F_{21}) \sigma_b T_1^4$$

[because $\sigma_1 = \sigma_2 = \sigma_b$]

Since σ_b and T_1 are both non-żero quantities,

$$A_1 F_{12} - A_2 F_{21} = 0$$

or

$$A_1 F_{12} = A_2 F_{21} \quad ...(8.6)$$

The above result is known as a *reciprocity theorem*. It indicates that the net radiant interchange may be evaluated by computing one way configuration factor from either surface to the other. Thus the net heat exchange between surfaces A_1 and A_2 is

$$(Q_{12})_{net} = A_1 F_{12} \sigma_b (T_1^4 - T_2^4)$$

$$= A_2 F_{21} \sigma_b (T_1^4 - T_2^4) \quad ...(8.7)$$

Equation 8.7 applies only to black surfaces and must not be used for surfaces having emissivities very different from unity.

The evaluation of the integral of equation 8.3 for determining the geometrical factor is rather complex and cumbersome. Accordingly results have been obtained and presented in graphical form for the geometries normally encountered in engineering practice. Geometrical factors for parallel planes (disks and rectangles) directly opposed and those for radiation between perpendicular rectangles with a common edge are depicted in Figs. 8.2 to 8.4.

8.2. SHAPE FACTOR ALGEBRA AND SALIENT FEATURES OF THE SHAPE FACTOR

The shape factors for complex geometries (for which shape factor charts or equations are

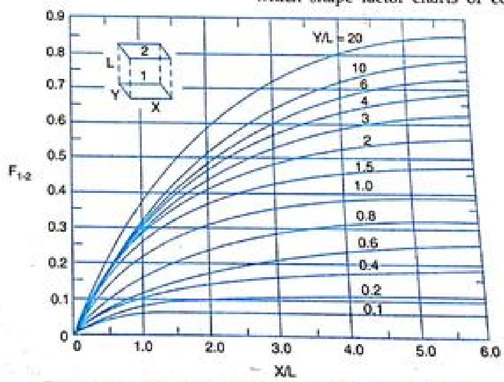


Fig. 8.2. Radiation shape factor for aligned parallel plates

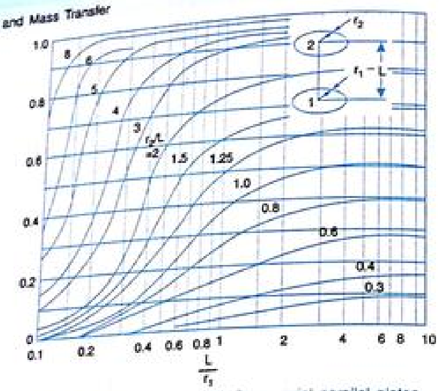


Fig. 8.3. Radiation shape factor for co-axial parallel plates

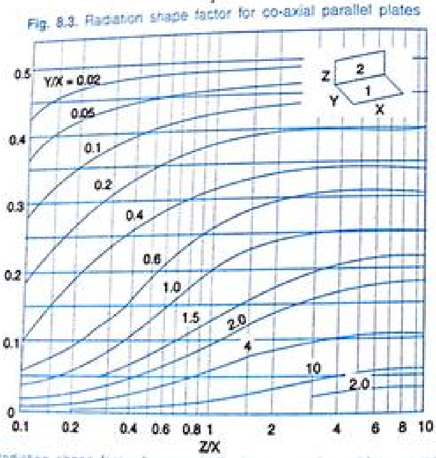


Fig. 8.4. Radiation shape factor for perpendicular rectangles with a common edge

not available) can be derived in terms of known shape factors for other geometries. For that the complex shape is divided into sections for which the shape factor is either known or can be readily evaluated.

The unknown configuration factor is worked out by adding and subtracting known factors of related geometries. The method is based on the definition of shape factor, the

reciprocity principle and the energy conservation law.

The inter-relation between various shape factors is called shape factor algebra

The following facts and properties will be useful for the calculation of shape factors of specific geometries and for the analysis of radiant heat exchange between surfaces :

(*i*) The value of shape factor depends only on the geometry and orientation of surfaces with respect to each other. Once the shape factor between two surfaces is known, it can be used for calculating the radiant heat exchange between the surfaces at any temperature.

(*ii*) The net heat exchange between surfaces A_1 A_2 is

$$(Q_{12})_{net} = A_1 F_{12} \sigma_1 T_1^4 - A_2 F_{21} \sigma_2 T_2^4$$

When the surfaces are thought to be black ($\sigma_1 = \sigma_2 = \sigma_b$) and are maintained at the same temperature ($T_1 = T_2 = T$), there is no heat exchange and as such

$$0 = (A_1 F_{12} - A_2 F_{21}) \sigma_b T_1^4$$

Since σ_b and T are both non-zero entities,

$$A_{F12} = A_2 F_{21}$$

This reciprocal relation is particularly useful when one of the shape factors is unity.

(*iii*) All the radiation streaming out from a convex surface 1 is intercepted by the enclosing surface 2. As such the shape factor of convex surface with respect to the enclosure F_{12} is unity. Then in conformity with reciprocity theorem, the other shape factor F_{21} is merely the ratio of areas.

(*iv*) The radiant energy emitted by one part of concave surface is intercepted by another part of the same surface. Accordingly a concave surface has a shape factor with respect to itself. The shape factor of a surface with respect to itself is denoted by F_{11}.

For a flat or convex surface, the shape factor with respect to itself is zero. This aspect stems from the fact that for any part of flat or convex surface, one cannot see any other part of the same surface.

(*v*) If one of the two surfaces (say A_i) is divided into sub areas $A_{i1}, A_{i2}, \ldots A_{in}$, then

$$A_i F_{ij} = \Sigma A_{in} F_{in\,j} \quad \ldots(8.8)$$

Thus with respect to Fig. 8.5 (*a*), wherein the radiating surface A_1 has been split up into areas A_3 and A_4,

$$A_1 F_{12} = A_3 F_{32} + A_4 F_{42}$$

Obviously $F_{12} \neq F_{32} + F_{42}$

Thus if the transmitting surface is sub divided, the shape factor for that surface with respect to the receiving surface is not equal to the sum of the individual shape factors.

If the receiving surface A_2 [Fig. 8.5*b*] is divided into subareas A_3 and A_4, we have

$$A_1 F_{12} = A_1 F_{13} + A_1 F_{14}$$

or $\quad F_{12} = F_{13} + F_{14}$

Apparently the shape factor from a radiating surface to a subdivided receiving

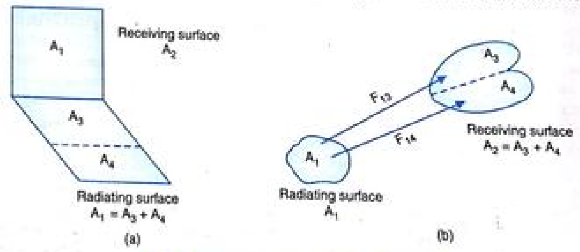


Fig. 8.5. Relation between shape factors

surface is simply the sum of the individual shape factors.

(vi) Any radiating surface will have finite area and therefore will be enclosed by many surfaces. The total unit radiation being emitted by the radiating surface will be received and absorbed by each of the confining surfaces. Since a shape factor is the fraction of total radiation leaving the radiating surface and falling upon a particular receiving surface, the energy balance would give :

$$\sum_{j=1}^{n} F_{ij} = 1 \; ; \; i = 1, 2, \ldots n \qquad \ldots(8.9)$$

The interior surface of a complete enclosed space has been subdivided in n-parts-each part having a finite area A_1, A_2 .... A_n. Thus

$$\begin{array}{l} F_{11} + F_{12} + F_{13} \cdots + F_{1n} = 1 \\ F_{21} + F_{22} + F_{23} \cdots + F_{2n} = 1 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad\quad \vdots \\ F_{n1} + F_{n2} + F_{n3} \cdots + F_{nn} = 1 \end{array}$$

Accordingly when a radiating surface exchanges heat with a number of black surfaces comprising the enclosure, the net heat transfer from the radiating surface will be

$$(Q_{12})_{net} = A_1 F_{12} \sigma_b \left(T_1^4 - T_2^4\right) + A_1 F_{13} \sigma_b \left(T_1^4 - T_3^4\right) + A_1 F_{14} \sigma_b \left(T_1^4 - T_4^4\right)$$

EXAMPLE 8.1

The sun is nearly spherical radiation source that is approximately 1.385 × 10⁹ m diameter and located at 1.50 × 10¹¹ m from the earth. On a clear day, the energy flux associated with solar radiation incident on the outer surface of earth has been accurately measured and known to be 1135 W/m² with an additional 285 W/m² absorbed by the earth atmosphere. Assuming that the sun emits as black body, estimate its surface temperature.

Solution : The radiant heat exchange between the sun and the earth may be written as :

$$Q_{12} = \sigma_b \left(T_1^4 - T_2^4\right) \frac{\cos\theta_1 \cos\theta_2 \, dA_1 \, dA_2}{\pi r^2}$$

where the suffex 1 and 2 pertain to sun and earth respectively. From the given data :

(i) $\theta_1 = \theta_2 = 90°$; hence $\cos\theta_1 = \cos\theta_2 = 1$

(ii) Temperature T_2 of the earth can be neglected in comparison to temperature T_1 of the sun.

(iii) The sun, emitting diffusely, appears as a disk of area $dA_1 = \frac{\pi}{4} d_1^2$ and

(iv) The heat interchange corresponds to unit area of the earth surface and so $dA_2 = 1$.

$$\therefore \quad Q_{12} = \frac{\sigma_b T_1^4}{\pi r^2} \frac{\pi}{4} d_1^2 = \frac{\sigma_b T_1^4}{4 \pi r^2} d_1^2$$

$$(1135 + 285) = \frac{5.67 \times 10^{-8} \times T_1^4}{4\pi \times \left(1.50 \times 10^{11}\right)^2} \times \left(1.385 \times 10^9\right)^2$$

Solution gives :

$$T_1 = 5854.8 \text{ K}$$

The sun's temperature is generally accepted to be of this approximate magnitude.

EXAMPLE 8.2

Define solar constant and obtain its value from the data given below :

Emissive power of sun's surface $= 22.7 \times 10^7$ *kJ/hr-m²*

Distance between sun and earth $= 1.5 \times 10^{11}$ *m*

Diameter of sun $= 1.4 \times 10^9$ *m*

Solution : The solar constant is defined as the amount of heat energy (radiation) absorbed per unit time by unit area of the earth surface; the surface being held normal to the sun's rays :

Consider small areas dA_1 and dA_2 on the surfaces of sun and earth respectively. The heat flow from dA_1 and dA_2 is prescribed by the relation

$$dQ_{12} = E_1 \frac{\cos\theta_1 \cos\theta_2 \, dA_1 \, dA_2}{\pi r^2}$$

where θ_1 and θ_2 are the angles measured from the normal to the surfaces dA_1 and dA_2 by

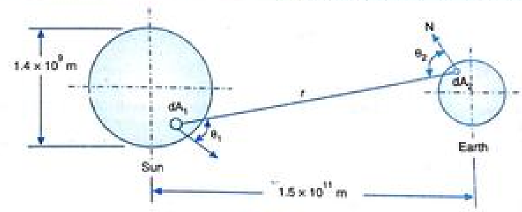


Fig. 8.6.

the beam of radiant energy, r is the radial distance between the sun and earth, and E_1 is the emissive power of the radiating surface *i.e.*, the sun's surface. Upon re-arrangement and integration

$$\int \frac{dQ_{12}}{dA_2 \cos\theta_2} = \frac{E_1}{\pi r^2} \int \cos\theta_1 \, dA_1$$

The factor $dA_2 \cos\theta_2$ is the projected area of A_2 normal to the line joining dA_1 and dA_2 and as such left side of the above identity represents the rate at which energy radiated by dA_1 falls on unit area of earth's surface and hence

$$\text{solar constant} = \frac{E_1}{\pi r^2} \int \cos\theta_1 \, dA_1$$

The integral $\int \cos\theta_1 \, dA_1$ is the projected area of the sun as seen from the earth and it equals $\frac{\pi}{4} d^2$ where d is the diameter of the sun.

∴ Solar constant is equal to,

$$\frac{E_1}{\pi r^2} \times \frac{\pi}{4} d^2 = \frac{E_1}{4} \left(\frac{d}{r}\right)^2 = \frac{22.7 \times 10^7}{4} \times \left(\frac{1.4 \times 10^9}{1.5 \times 10^{11}}\right)^2 = 4943 \text{ kJ/hr-m}^2$$

EXAMPLE 8.3

Calculate the shape factors for the configurations shown in the figure given below :

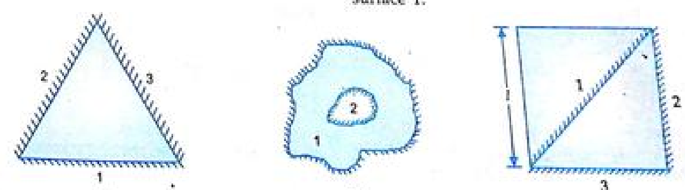
(a) long tube with cross-section of an equilateral triangle

(b) black body inside a black enclosure

(c) diagonal partition within a long square duct.

Solution : The desired shape factors can be worked out by invoking the summation rule, the reciprocity theorem and from inspection of the geometry.

(a) By summation rule for radiation from surface 1.



(a) (b) (c)

Fig. 8.7.

$$F_{11} + F_{12} + F_{13} = 1$$

The flat surface 1 cannot see itself and so $F_{11} = 0$. That gives

$$F_{12} + F_{13} = 1$$

Due to symmetry, the radiation from surface 1 is equally divided between surfaces 2 and 3, and therefore

$$F_{12} = F_{13} = 0.5$$

Likewise, considering radiation from surface 2 :

$$F_{21} + F_{23} = 1 \qquad \text{(because } F_{22} = 0)$$

$$F_{23} = 1 - F_{21}$$

By reciprocity :

$$F_{21} = \frac{A_1}{A_2} F_{12} = F_{12} \qquad \text{(because } A_1 = A_2)$$

$$\therefore \quad F_{23} = 1 - F_{12} = 1 - 0.5 = 0.5$$

(*b*) The concave surface 1 can see itself and the rest of radiation falls on the enclosed surface 2. Invoking the conservation principle (summation rule) for surface 1

$$F_{11} + F_{12} = 1$$

By reciprocity :

$$A_1 F_{12} = A_2 F_{21} \quad \text{or} \quad F_{12} = \frac{A_2}{A_1} F_{21}$$

$$\therefore \quad F_{11} = 1 - F_{12} = 1 - \frac{A_2}{A_1} F_{21}$$

Further, all the radiations coming out from the black surface 2 are intercepted by the enclosing surface, Therefore $F_{21} = 1$, and so

$$F_{11} = 1 - \frac{A_2}{A_1}$$

(*c*) From summation rule:

$$F_{11} + F_{12} + F_{13} = 1$$

By inspection $F_{11} = 0$ as the flat surface cannot see itself

$$\therefore F_{12} + F_{13} = 1$$

Due to symmetry, the radiation form surface 1 is equally divided between the surfaces 2 and 3 and therefore

$$F_{12} = F_{13} = 0.5$$

By reciprocity :

$$F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\sqrt{2}\,l}{l} \times 0.5 = 0.71$$

EXAMPLE 8.4

Establish a relation for the shape factor of a cavity with respect to itself. The cavity is closed on its outer surface with a flat surface.

Solution : Invoking the conservation principle (summation rule) for surface 1 of the cavity,

$$F_{11} + F_{12} = 1 \qquad \text{...(i)}$$

For the closing surface 2 :

$$F_{21} + F_{22} = 1$$

But $F_{22} = 0$ as the surface is flat and cannot see itself.

$$\therefore \quad F_{21} = 1 - F_{22} = 1 - 0 = 1$$

By reciprocity :

$$F_{12} = \frac{A_2}{A_1} F_{21} = \frac{A_2}{A_1}$$

$\therefore$ From expression (*i*)

$$F_{11} = 1 - F_{12} = 1 - \frac{A_2}{A_1}$$

The above relation is valid fol all types of cavities as shown in Fig. 8.8.

(*i*) For a cylindrical cavity of depth of h and diameter d,

$$F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{\frac{\pi}{4} d^2}{\frac{\pi}{4} d^2 + \pi d h}$$

$$= 1 - \frac{d}{d + 4h} = \frac{4h}{4h + d}$$

(*ii*) For a conical cavity of diameter d and depth h,

$$F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{\frac{\pi}{4} d^2}{\frac{\pi d l}{2}} = 1 - \frac{d}{2l}$$

$$= 1 - 2 \sin \alpha$$

where l is the slant height of the cone and α is half vertex angle.

In terms of depth h

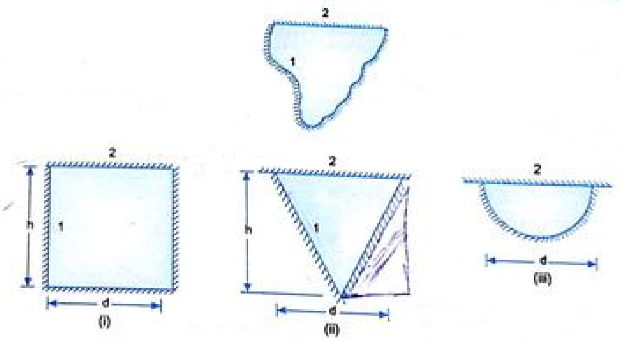


Fig. 8.8.

$$F_{11} = 1 - \frac{d}{2\sqrt{h^2 + \frac{d^2}{4}}}$$

$$= 1 - \frac{d}{\sqrt{4h^2 + d^2}}$$

(*iii*) For a hemispherical bowl of diameter d

$$F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{\frac{\pi}{4} d^2}{\frac{\pi d^2}{2}} = 1 - \frac{1}{2} = 0.5$$

Thus, half of the radiation from hemispherical bowl falls on itself and the remaining half is intercepted by the plane closing surface.

EXAMPLE 8.5

Calculate the shape factors for the configurations shown in the figure given below :

(*a*) *sphere of diameter d inside a cubical box of length $l = d$*

(*b*) *hemispherical surface closed by a plane surface*

(*c*) *end and side of circular tube of equal length and diameter.*

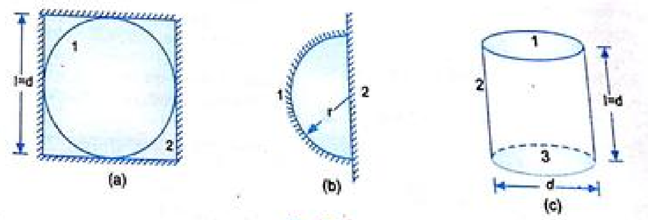


Fig. 8.9.

Solution : The desired shape factors can be worked out by invoking the summation rule, the reciprocity theorem and from inspection of the geometry.

(a) The convex spherical surface cannot see itself and the entire radiation emitted by this surface falls on the enclosing box. Invoking the conservation principle (summation rule) for surface 1.

$$F_{11} + F_{12} = 1 \quad \text{or} \quad F_{12} = 1 \qquad \text{(because } F_{11} = 0\text{)}$$

By reciprocity :

$$F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\pi d^2}{6l^2} \times 1 = \frac{\pi}{6} \qquad \text{(because } l = d\text{)}$$

(b) The hemispherical surface 1 can see itself and the rest of radiation falls on the flat plane surface.

$$\therefore F_{11} + F_{12} = 1$$

Further all the radiations emitted from the plane surface 2 (which cannot see itself) are intercepted by the hemispherical surface and so $F_{21} = 1$

By reciprocity :

$$F_{12} = \frac{A_2}{A_1} F_{21} = \frac{\pi r^2}{\frac{1}{2} \times 4\pi r^2} \times 1 = 0.5$$

$$\therefore \quad F_{11} = 1 - F_{12} = 1 - 0.5 = 0.5$$

(c) From the given geometry,

$$\frac{l}{r_1} = 2 \quad \text{and} \quad \frac{r_2}{l} = 0.5$$

With reference to Fig. 8.3, the shape factor F_{13} is then read as = 0.17

Applying summation rule:

$$F_{11} + F_{12} + F_{13} = 1$$

The flat surface 1 cannot see itself and so $F_{11} = 0$. That gives

$$F_{12} = 1 - F_{13} = 1 - 0.17 = 0.83$$

By reciprocity :

$$F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\frac{\pi}{4} d^2}{\pi d l} \times 0.83 = \frac{0.83}{4} = 0.21 \qquad \text{(because } l = d\text{)}$$

EXAMPLE 8.6

Consider a system of concentric spheres of radius r_1 and r_2 ($r_2 > r_1$). If $r_1 = 5$ cm, determine the radius r_2 if it is desired to have the value of shape factor F_{21} equal to 0.6.

Solution : For the configuration of concentric cylinders as depicted in, Fig. 8.10,

$F_{12} = 1$

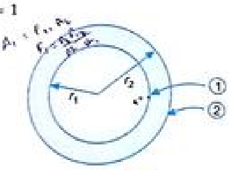


Fig. 8.10.

From reciprocity theorem :

$$A_1 F_{12} = A_2 F_{21}$$

Substituting the relevent data,

$$\frac{\pi}{4}(0.05)^2 \times 1 = \frac{\pi}{4} r^2 \times 0.6$$

$$\therefore \quad r_2 = \left[\frac{0.05^2}{0.6}\right]^{\frac{1}{2}} = 0.0645 \text{ m}$$

$$= 6.45 \text{ cm}$$

EXAMPLE 8.7 (a)

Consider a very long isocelles triangular duct shown in the accompanying Fig. 8.11. For the given dimensions :

$$ab = ac = x \quad \text{and} \quad bc = \frac{x}{2}$$

Show that $F_{12} = 0.75$

Solution : The duct is very long and as such the radiation lost out of the end of the duct can be ignored. This implies that the shape factors of the large surfaces with respect to the triangular end surfaces are vanishingly small

From the summation rule for flat surfaces (1) and (3)

$$F_{12} + F_{13} = 1 \qquad \ldots(i)$$

$$F_{31} + F_{32} = 1 \qquad \ldots(ii)$$

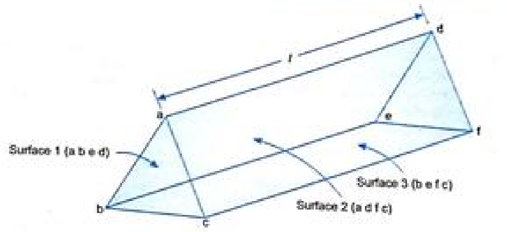


Fig. 8.11.

By symmetry

$$F_{31} = F_{32}$$

Then from identity (*ii*), we obtain

$$F_{31} = 0.5$$

By reciprocity theorem ;

$$A_1 F_{13} = A_3 F_{31}$$

or

$$F_{13} = \frac{A_3}{A_1} F_{31} = \frac{l \times \frac{x}{2}}{l \times x} \times 0.5 = 0.25$$

Therefore from identity (*i*)

$$F_{12} = 1 - F_{13} = 1 - 0.25 = 0.75$$

EXAMPLE 8.7(b)

Show that the radiation shape factor between a disc of radius r located at a distance f from the centre of a small sphere is given by

Solution : With reference to Fig. 8.12, consider an elemental area dA_2 of included angle $d\theta$ at an angle θ.

$$dA_2 = (2\pi R \sin\theta)\, R\, d\theta$$

Then area of the surface of the sector ab of the sphere

$$A_2 = \int_0^{\alpha} (2\pi R \sin\theta) R\, d\theta$$

$$= 2\pi R^2 \int_0^{\alpha} \sin\theta\, d\theta$$

$$= 2\pi R^2 (1 - \cos\alpha)$$

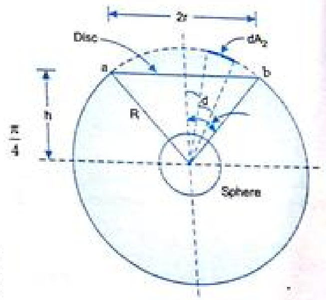


Fig. 8.12.

Area of the sphere of radius R = 4 πR^2

$$\therefore F_{\text{sphere - disc}} = \frac{2\pi R^2 (1 - \cos\alpha)}{4\pi R^2}$$

$$= \frac{1}{2}(1 - \cos\alpha)$$

$$= \frac{1}{2}\left[1 - \frac{h}{R}\right]$$

$$= \frac{1}{2}\left[1 - \frac{h}{\sqrt{r^2 + h^2}}\right]$$

EXAMPLE 8.8

Consider a thin hollow cylinder of 8 cm diameter and 10 cm length. If the radiant shape factor of the circular surface of this cylinder is 0.172, make calculations for the shape factor of the curved surface of the cylinder with respect to itself.

Solution : Refer Fig. 8.13 for the nomenclature of the surfaces of the thin hollow cylinder

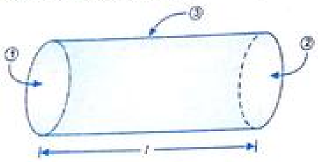


Fig. 8.13.

For the surfaces 1 and 2,

$A_1 F_{12} = A_2 F_{21}$ (Reciprocity theorem)

Since $A_1 = A_2$, we get

$F_{12} = F_{21} = 0.172$

Also $F_{11} = F_{22} = 0$

and $F_{31} = F_{32}$ because of symmetry.

By summation rule, the shape factor relations among the three surfaces are given by

$F_{11} + F_{12} + F_{13} = 1$

or $F_{12} + F_{13} = 1$ $(\because F_{11} = 0)$

$\therefore$ $F_{13} = 1 - F_{12} = 1 - 0.172 = 0.828$

Also, $F_{31} + F_{32} + F_{33} = 1$

or $F_{31} + F_{31} + F_{33} = 1$ $(\because F_{31} = F_{32})$

$\therefore$ $F_{33} = (1 - 2F_{31})$

Invoking reciprocity theorm :

$A_1 F_{13} = A_3 F_{31}$

$$F_{31} = \frac{A_1}{A_3} F_{13} = \frac{\pi r^2}{2\pi r l} F_{13}$$

$$= \frac{r}{2l} F_{13}$$

$$= \frac{4}{2 \times 10} \times 0.828 = 0.1656$$

is $F_{33} = 1 - 2 \times 0.1656 = 0.6688$

EXAMPLE 8.9

A truncated cone of height 10 cm has top and bottom diameters of 8 cm and 16 cm respectively. The bottom surface is stated to intercept 15 percent of radiation leaving the top surface. Determine the shape factor between the (i) top surface and the conical side surface, and (ii) the side surface and itself

Solution : Area of the curved surfaces

$$A = \pi (r_1 + r_2) \sqrt{(r_2 - r_1)^2 + h^2}$$

$$= \pi (4 + 8) \sqrt{(8-4)^2 + 10^2}$$

$$= 12\pi \sqrt{116} = 405.83 \text{ cm}^2$$

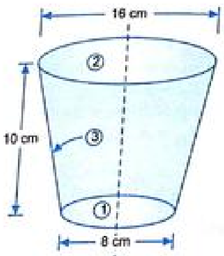


Fig. 8.14.

(*i*) As the bottom surface 1 intercepts 15 percent of heat radiations leaving the top surface 2, we have

$F_{21} = 0.15$

By reciprocity theorem :

$A_1 F_{12} = A_2 F_{21}$

$$\therefore \quad F_{12} = \frac{A_2}{A_1} F_{21} = \frac{\pi \times 8^2}{\pi \times 4^2} \times 0.15 = 0.6$$

Further, $F_{11} + F_{12} + F_{13} = 1$

$F_{12} + F_{13} = 1$ $(\because F_{11} = 0)$

or $F_{13} = 1 - F_{12} = 1 - 0.6 = 0.4$

Again, $F_{21} + F_{22} + F_{23} = 1$

$F_{21} + F_{23} = 1$ $(\because F_{22} = 0)$

$F_{23} = 1 - F_{21} = 1 - 0.15 = 0.85$

(*ii*) By reciprocity theorem :

$$F_{32} = \frac{A_2}{A_3} F_{23} = \frac{\pi \times 8^2}{405.83} \times 0.85 = 0.421$$

$$F_{31} = \frac{A_1}{A_3} F_{13} = \frac{\pi \times 4^2}{405.83} \times 0.4 = 0.0495$$

From the identity,

$F_{31} + F_{32} + F_{33} = 1$

$F_{33} = 1 - F_{31} - F_{32}$

$= 1 - 0.0495 - 0.421 = 0.5295$

EXAMPLE 8.10

Two areas 1 and 2 are in the form of circular rings, coaxial and are in two parallel planes at a distance of 10 cm. For area 1 the inner radius is 5 cm, the outer radius is 10 cm and the corresponding values for area 2 are 8 cm and 20 cm respectively. Make calculations for the shape factor between these two areas.

The following relation may be used for calculating the shape factor between two circular areas located coaxially in two parallel planes :

$$F_{1\text{-}2} = \frac{1}{2B^2}\left[X - \sqrt{X^2 - 4B^2C^2}\right]$$

where $B = \frac{R_1}{H}$; $C = \frac{R_2}{H}$; $X = 1 + B^2 + C^2$ *and*

H is the distance between the two areas

Solution : Refer Fig. 8.15 for the configuration and nomenclature

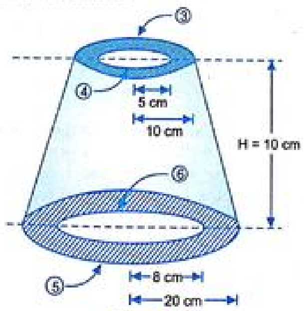


Fig. 8.15.

Surface 1 = $A_3 - A_4$

Surface 2 = $A_5 - A_6$

$F_{12} = F_{35} - F_{46}$ ...(*i*)

For surfaces 3 and 5 :

$R_3 = 10$ cm ;

$R_5 = 20$ cm and $H = 10$ cm

$B = \frac{10}{10} = 1$; $C = \frac{20}{10} = 2$ and

$X = 1 + 1^2 + 2^2 = 6$

$$\therefore \quad F_{35} = \frac{1}{2 \times 1^2}\left[6 - \sqrt{6^2 - 4 \times 1^2 \times 2^2}\right]$$

$$= \frac{1}{2}(6 - 4.47) = 0.765$$

For surfaces 4 and 6

$R_4 = 5$ cm ;

$R_6 = 8$ cm and $H = 10$ cm

$B = \frac{5}{10} = 0.5$;

$C = \frac{8}{10} = 0.8$

and $X = 1 + 0.5^2 + 0.8^2 = 1.89$

$$\therefore \quad F_{46} = \frac{1}{2 \times 0.5^2}\left[1.89 - \sqrt{1.89^2 - 4 \times 0.5^2 \times 0.8^2}\right]$$

$$= \frac{1}{0.5}(1.89 - 1.71) = 0.36$$

Then from expression (*i*),

$F_{12} = F_{35} - F_{46} = 0.765 - 0.36 = 0.405$

EXAMPLE 8.11

A large black enclosure consists of a box as shown in the adjoining figure. The bottom surface 1 is at 530 K, the top surface 2 is at 450 K and all vertical surfaces 3 (including the back wall) are at 475 K. Find the net heat transfer rates Q_{12} and Q_{13}.

Solution : The net interchange of heat between two surfaces is given by,

$$(Q_{ij})_{net} = F_{ij} A_i \sigma_b \left(T_i^4 - T_j^4\right)$$

(a) Net heat transfer rate from the bottom surface 1 to top surface 2 :

From Fig. 8.2 with X/L = 20/25 = 0.8 and Y/L = 25/25 = 1,

Shape factor F_{12} = 0.168

$\therefore (Q_{12})_{net} = 0.168 \times (0.25 \times 0.20) \times 5.67 \times 10^{-8}\,(530^4 - 450^4)$

$= 18.03$ W

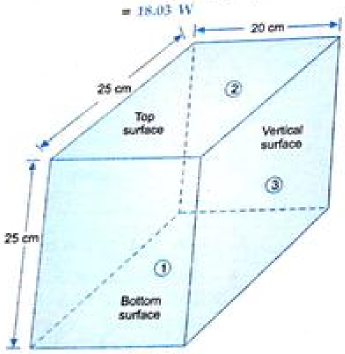


Fig. 8.16.

(b) Net heat transfer rate from the bottom surface 1 to vertical surface 3

From Fig. 8.4 with $Y/X = 25/20 = 1.25$ and $Z/X = 25/20 = 1.25$.

Shape factor F_{13} = 0.185

$\therefore (Q_{13})_{net} = 0.185 \times (0.25 \times 0.20) \times 5.67 \times 10^{-8}\,(530^4 - 475^4)$

$= 14.68$ W

EXAMPLE 8.12

A small sphere with a surface temperature of 550 K is located at the geometric centre of a large sphere with an inner surface temperature of 280 K. The outside diameter of the small sphere is 5 cm, and the inside diameter of the large sphere is 25 cm. Assuming that both sides approach black body behaviour, determine how much of the emission from the inner surface of the large sphere is incident upon the outer surface of the small sphere.

Solution : Let suffix 1 designate the small sphere and suffix 2 denote the large sphere.

We are asked to determine the configfuration factor F_{21}.

All the radiation being emitted by the small sphere is incident upon and absorbed by the inner surface of the large sphere. Therefore configuration factor between 1 and 2 is $F_{12} = 1$

By reciprocity theorem,

$$A_1 F_{12} = A_2 F_{21}$$

$$4\pi r_1^2 F_{12} = 4\pi r_2^2 F_{21}$$

$$\therefore \quad F_{21} = F_{12} \times \frac{r_1^2}{r_2^2} = \frac{1 \times (2.5)^2}{(12.5)^2} = 0.04$$

Thus 4% of the emission from the inner surface of the large sphere is incident upon the small sphere and absorbed by it.

From energy balance for the large sphere,

$F_{21} + F_{22} = 1$

or $F_{22} = 1 - F_{21} = 1 - 0.04 = 0.96$

Thus 96% of emission from the large sphere is absorbed by the inner surface of the sphere itself. The net interchange of heat between the two sphere is :

$$(Q_{12})_{net} = F_{12} A_1 \sigma_b \left(T_1^4 - T_2^4\right)$$
$$= 1 \times (4\pi \times 0.025^2) \times 5.67 \times 10^{-8}\,(550^4 - 280^4)$$
$$= 37.99 \text{ W}$$

EXAMPLE 8.13

A 2.5 cm hole has been drilled completely through a 5 cm thick metal plate that is maintained at a uniform temperature of 500 K. Workout the loss of energy to the surroundings at 300 K temperature. Both the metallic surfaces and the surroundings have black body characteristics. Two ends of the hole may be treated as disks.

Solution : The arrangement of the system is shown in Fig. 8.17. Let suffix 1 designate the cavity and the suffices 2 and 3 denote the two ends of the 2.5 cm diameter hole.

With both ends of the hole treated as disks,

$$\frac{L}{r_2} = \frac{5}{1.25} = 4$$

$$\frac{r_3}{L} = \frac{1.25}{5} = 0.25$$

Fig. 8.17.

With reference to Fig. 8.3, the configuration factor F_{23} is then read as 0.065.

Considering radiations from end 2 of the hole.

$F_{21} + F_{22} + F_{23} = 1$

Now $F_{22} = 0$ as the end conforms to a flat surface and cannot see itself

$\therefore \quad F_{21} = 1 - F_{23} = 1 - 0.065 = 0.935$

From reciprocity theorem ,

$$F_{12} = \frac{A_2}{A_1} F_{21}$$

$$= \frac{\frac{\pi}{4}(0.025)^2}{\pi \times 0.025 \times 0.05} d_1^2 \times 0.935$$

$$= 0.1166$$

Adopting the same procedure or by symmetry : $F_{13} = 0.1166$. The total energy loss equals the sum of the heats escaping through each end of the hole. That is

Total energy loss,

$$= F_{12} A_1 \sigma_b \left(T_1^4 - T_{surr}^4\right) + F_{13} A_1 \sigma_b \left(T_1^4 - T_{surr}^4\right)$$
$$= 2F_{12} A_1 \sigma_b \left(T_1^4 - T_{surr}^4\right) \quad (\text{because } F_{12} = F_{13})$$
$$= 2 \times 0.1166 \times (\pi \times 0.025 \times 0.05) \times (5.67 \times 10^{-8})\,(500^4 - 300^4)$$
$$= 2.82 \text{ W}$$

EXAMPLE 8.14

Find the shape factor F_{12} for the arrangement shown in the adjoining figure 8.18. The areas A_1 and A_2 are perpendicular but do not share the common edge.

Solution : The evaluation of shape factor for such cases is made by introducting

hypothetical areas A_3 and A_4 so that the arrangement of perpendicular surfaces has a common edge. Further,

$$A_5 = A_1 + A_3$$
and $$A_6 = {}_2 + A_4$$

The sequence of solution is

$$A_5 F_{56} = A_1 F_{16} + A_3 F_{36}$$
$$= A_1 F_{14} + A_1 F_{12} + A_3 F_{36}$$
$$= A_5 F_{54} - A_3 F_{34} + A_1 F_{12} + A_3 F_{36}$$
$$\therefore A_1 F_{12} = (A_5 F_{56} + A_3 F_{34}) - (A_5 F_{54} + A_3 F_{36}) \quad \text{...(i)}$$

Each of the configuration facter on the right hand side of this expression can be read from Fig. 8.4. as they correspond to perpendicular surfaces having a common intersection line. The values are tabulated below :

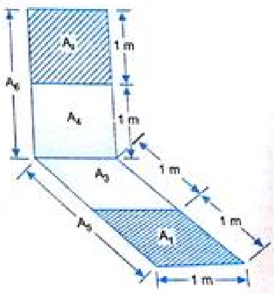


Fig. 8.18.

Surfaces (A_{ij})	$\frac{Z}{X}$	$\frac{Y}{X}$	F_{ij}
A_{56}	$\frac{2}{1} = 2$	$\frac{2}{1} = 2$	0.15
A_{34}	$\frac{1}{1} = 1$	$\frac{1}{1} = 1$	0.20
A_{54}	$\frac{1}{1} = 1$	$\frac{2}{1} = 2$	0.11
A_{36}	$\frac{2}{1} = 2$	$\frac{1}{1} = 1$	0.24

Substituting these values in expression (i), we get

$$A_1 F_{12} = (2 \times 0.15 + 1 \times 0.20) - (2 \times 0.11 + 1 \times 0.24)$$

$$= 0.04$$

$$\therefore \quad F_{12} = \frac{0.04}{1} = 0.04$$

EXAMPLE 8.15

Find the shape factor between a small area A_1 and a circulare segment of area A_2 of a spherical surface of radius R. The area A_1 is located symmetrically at the centre of sphere and the segment subtends an angle 2α at A_1.

(b) A small disc-shaped earth satellite; 1 metre in diameter, circles the earth (radius 6.25×10^6 m) at a distance of 3×10^5 m from the surface. The flat surface of the disc is oriented tangential to the earth's surface; the satellite surface has an emissivity of 0.32 and is at 250 K temperature. Make calculations for the net rate at which energy is leaving the satellite. It may be presumed that the average earth surface temperature is 300 K and the earth behaves as a black body; the satellite is in the shadow of the earth; and that part of the satellite surroundings not occupied by the earth is black and at 0 K.

Solution : The arrangement of the system is shown in Fig. 8.19. The radiation shape factor of the small area A_1 relative to the circular segment of area A_2 is worked out from the relation

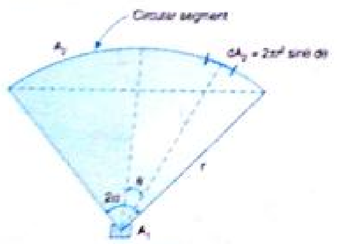


Fig. 8.19.

$$F_{12} = \frac{1}{A_1}\int_{A_1}\int_{A_2}\frac{\cos\theta_1\cos\theta_2}{\pi r^2}\,dA_1\,dA_2$$

$$= \int_{A_2}\frac{\cos\theta_1\cos\theta_2}{\pi r^2}\,dA_2$$

(because $dA_1 = A_1$)

Considering an elementary circular ring on the sphere's surface at any arbitrary angle θ, we have

$$\theta_1 = 0 \quad \text{and} \quad \theta_2 = \infty$$

$$dA_2 = 2\pi r^2 \sin\theta\, d\theta$$

$$\therefore \quad F_{12} = \int_0^{\alpha}\frac{\cos\theta\cos\theta}{\pi r^2}\,2\pi r^2\sin\theta\,d\theta$$

$$= \int_0^{\alpha} 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\alpha}\sin 2\theta\,d\theta$$

$$= \left[-\frac{\cos 2\theta}{2}\right]_0^{\alpha} = \left[-\frac{\cos 2\alpha}{2} + \frac{1}{2}\right]$$

$$= \left[\frac{1}{2} - \frac{1-2\sin^2\alpha}{2}\right] = \sin^2\alpha$$

(b) The arrangement of the system of earth, satellite and the surroundings is shown in Fig. 8.20.

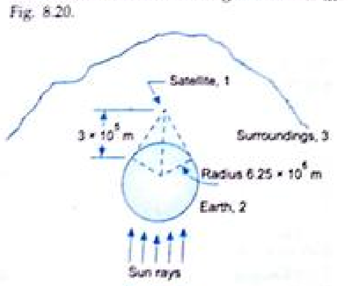


Fig. 8.20.

The rate of emission of radiant energy from the satellite surfaces is

$$= 2\,\epsilon_1\,A_1\,\sigma_b\,T_1^4$$

The factor 2 accounts for two surfaces of the satellite

$$= 2 \times 0.32 \times \frac{\pi}{4}\,(1)^2 \times 5.67 \times 10^{-8} \times (250)^4$$

$$= 111.27 \text{ W}$$

The earth and the surroundings are stated to be black and as such the above calculated radiant energy would be completely absorbed by these surfaces. The satellite would not receive back any of the radiation emitted by it.

The earth at temperature T_2 emits radiations equal to $(A_2\,F_{12}\,\sigma_b\,T_2^4)$. Upon reaching the satellite a portion $(\alpha_1\,A_2\,F_{21}\,\sigma_b\,T_2^4)$ would be absorbed by the satellite. From Kirchoff's law, absorptivity α_1 of the satellite equals its emissivity ϵ_1. Therefore the rate at which the satellite receives and absorbs energy coming from the earth is

$$= \epsilon_1\,A_2\,F_{12}\,\sigma_b T_2^4$$

By reciprocity theorem :

$$A_1\,F_{12} = A_2\,F_{21}$$

Since the satellite is small in relation to the earth's surface, its radiation shape factor (as derived above in part a) would be

$$F_{12} = \sin^2\alpha = \left(\frac{6.25\times 10^6}{6.25\times 10^6 + 3\times 10^5}\right)$$

$$= 0.910$$

$\therefore$ Radiant energy received by the satellite from the earth is

$$= \epsilon_1\,A_1\,F_{12}\,\sigma_b\,T_2^4$$

$$= 0.32 \times \frac{\pi}{4}\,(1)^2 \times 0.910 \times 5.67 \times 10^{-8} \times 300^4$$

$$= 104.98 \text{ W}$$

Since the surroundings are at 0 K, they do not emit any radiations.

Therefore the net rate at which energy leaves the satellite surface is

$$= 111.27 - 104.98 = 6.29 \text{ W}$$

EXAMPLE 8.16

Two diffuse surfaces, a small disk of area A_1 and a large disk of area A_2, are parallel to each other and directly opposed, i.e., a line joining their centres is normal to both the surfaces. The large disk has a radius R and is located at a height L form the smaller disk. Obtain an expression for the configuration factor of small disk with respect to the large disk.

Solution : The arrangement is shown in Fig. 8.21 and the radiation factor of the small surface relative to the large circular disk may be obtained from the equation,

$$F_{12} = \frac{1}{A_1}\int_{A_1}\int_{A_2}\frac{\cos\theta_1\cos\theta_2}{\pi s^2}\,dA_1\,dA_2$$

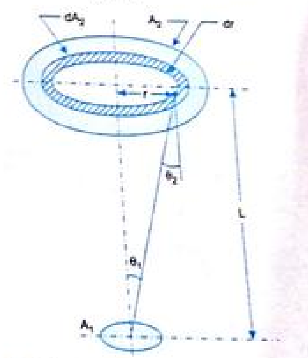


Fig. 8.21. Configuration factor between two diffuse surfaces : A small disk and a large disk

Recognising that θ_1, θ_2 and s are approximately independent of position on A_1 (as the disk is small), this expression reduces to

$$F_{12} = \int_{A_2}\frac{\cos\theta_1\cos\theta_2}{\pi s^2}\,dA_2$$

and with $\theta_1 = \theta_2 = \theta$

$$F_{12} = \int_{A_2}\frac{\cos^2\theta}{\pi s^2}\,dA_2$$

From the geometry of the arrangement,

$$s^2 = r^2 + L^2$$

$$\cos\theta = \frac{L}{s} = \frac{1}{\sqrt{r^2 + L^2}}$$

and $\quad dA_2 = 2\pi r\, dr$

$$\therefore \quad F_{12} = \int_0^R \frac{L^2}{(r^2 + L^2)^2} 2r\,dr$$

To evaluate the integral, we substitute

$$r^2 + L^2 = t\,; \quad 2r\,dr = dt$$

$$\therefore \int \frac{L^2}{t^2} dt = \left|\frac{-L^2}{t}\right| = \left|\frac{-L^2}{r^2 + L^2}\right|_0^R = \frac{R^2}{R^2 + L^2}$$

Therefore the configuration factor F_{12} of a small surface relative to a large circular disk

$$F_{12} = \frac{R^2}{R^2 + L^2} = \left(\frac{R}{\sqrt{R^2 + L^2}}\right)^2$$

$$= \sin^2\theta$$

Note: The configuration factor F_{12} of a small surface relative to a circular ring of inner radius R_1 and outer radius R_2 (Fig 8.22) would be

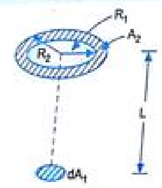


Fig. 8.22.

$$F_{12} = \int_{R_1}^{R_2} \frac{L^2}{(r^2 + L^2)^2} = 2r\,dr$$

$$= -L^2 \left|\frac{1}{r^2 + L^2}\right|_{R_1}^{R_2}$$

$$= -L^2 \left[\frac{1}{R_2^2 + L^2} - \frac{1}{R_1^2 + L^2}\right]$$

$$= L^2 \left[\frac{R_2^2 - R_1^2}{(R_1^2 + L^2)(R_2^2 + L^2)}\right]$$

EXAMPLE 8.17

Determine the geometrical factor of a bead-shaped thermocouple to the inside wall of a circular duct.

A small thermocouple having a spherical shape 2 mm diameter is placed at the centre of a circular duct 0.25 m long and 10 cm in diameter. The thermocouple reads 185°C when the duct wall is at 140°C and gas at 200 °C flows along the duct. Determine the convective coefficient of heat transfer between the gas and the thermocouple bead. The duct walls and the thermocouple bead may be assumed to have characteristics of black radiating surfaces.

Solution : The arrangement is shown in Fig. 8.23 and the configuration factor of the bead shaped thermocouple relative to the duct may be obtained from the equation :

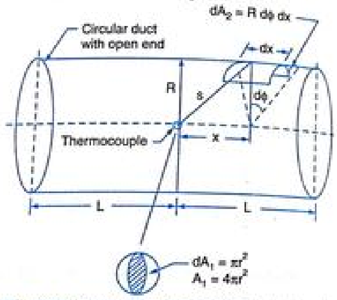


Fig. 8.23. Geometrical factor of a bed shaped thermocouple

$$F_{12} = \frac{1}{A_1} \int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\pi s^2} dA_1\, dA_2$$

Recognising that area of the thermocouple bead is very small and will be constant for any area of the duct; $\cos\theta_1 = 1$ and this expression reduces to

$$F_{12} = \frac{dA_1}{A_1} \int_{A_1} \frac{\cos\theta_2}{\pi s^2} dA_2$$

From the geometry of the arrangement,

$$\cos\theta_2 = \frac{R}{s}\,; \quad dA_2 = R\,d\phi\, dx$$

$$s^2 = R^2 + x^2$$

$$\therefore \quad F_{12} = \frac{dA_1}{A_1} \int_L^{-L} \int_0^{2\pi} \frac{R^2\, d\phi\, dx}{\pi (R^2 + x)^{3/2}}$$

$$= 2R^2 \frac{dA_1}{A_1} \int_L^{-L} \frac{dx}{(R^2 + x^2)^{3/2}}$$

To evaluate the integration, we substitute

$$x = R\tan\theta\,; \quad dx = R\sec^2\theta\, d\theta$$

$$\therefore \int \frac{dx}{(R^2 + x^2)^{3/2}} = \int \frac{R\sec^2\theta\, d\theta}{R^3 \sec^3\theta}$$

$$= \frac{1}{R^2} \int \cos\theta\, d\theta$$

$$= \frac{1}{R^2} \sin\theta$$

$$= \frac{1}{R^2} \frac{x}{\sqrt{R^2 + x^2}}$$

$$\therefore 2R^2 \int_L^{-L} \frac{dx}{(R^2 + x^2)^{3/2}}$$

$$= 2R^2 \frac{1}{R^2} \left[\frac{x}{\sqrt{R^2 + x^2}}\right]_L^{-L}$$

$$= 2\left[\frac{L}{\sqrt{R^2 + L^2}} + \frac{L}{\sqrt{R^2 + L^2}}\right]$$

$$= \frac{4L}{\sqrt{R^2 + L^2}}$$

Therefore, configuration factor of the thermocouple bead relative to the duct walls will be:

$$F_{12} = \frac{4L}{\sqrt{R^2 + L^2}} \frac{dA_1}{A_1}$$

$$= \frac{4L}{\sqrt{R^2 + L^2}} \times \frac{\pi r^2}{4\pi r^2}$$

$$= \frac{L}{\sqrt{R^2 + L^2}}$$

(*b*) Convective heat flow from gas to the thermocouple

$$= hA\,\Delta t$$

$$= hA\,(200 - 185) = 15\,hA$$

Radiation heat exchange between the thermocouple and the duct walls

$$= F_{12}\, A\, \sigma_b \left(T_1^4 - T_2^4\right)$$

$$= \frac{L}{\sqrt{R^2 + L^2}} A\, \sigma_b \left(T_1^4 - T_2^4\right)$$

$$= \frac{0.125}{\sqrt{0.05^2 + 0.125^2}} A \times 5.67 \times 10^{-8} \left(458^4 - 413^4\right)$$

$$= 786.06\, A$$

Under steady state conditions, there exists an equilibrium between the convective heat flow from gas to thermocouple, and the heat radiated by the thermocouple to the pipe wall. Thus

$$15\,hA = 786.06\,A\,;$$

Solution gives: $h = 52.40$ W/m²-deg

Therefore convective coefficient of heat transfer is 52.40 W/m²-deg

Note : The heat radiated by the gas to the thermocouple is very small as compared with convective heat flow and hence neglected.

8.3. HEAT EXCHANGE BETWEEN NON-BLACK BODIES

The black bodies absorb the entire incident radiation and this aspect makes the calculation procedure of heat exchange between black bodies rather simple. One has only to determine shape factor, *i.e.*, how much of radiation leaving one surface is actually incident on another. However the real (non-black) surfaces do not absorb the whole of incident radiation; a part is reflected back to the radiating surface and a part may be reflected out of the system. The back and forth reflection between surfaces may go on several times. Further, the absorptivities and emissivities are not uniform in all directions

and for all wavelength. The problem is simplified to some extent if the bodies are considered gray for which the absorptivities and emissivities are constant over the entire wavelength spectrum, and accordingly their average values would be equal irrespective of temperature, $\alpha = \epsilon \neq f(\lambda)$. Heat exchange for some common situations involving opaque gray bodies has been presented in this section.

8.3.1. Infinite Parallel Planes

The analysis of radiant heat exchange between two non-black parallel surfaces shall be based on the following assumptions :

(i) The surfaces are arranged at small distance from each other and are of equal area so that practically all radiation emitted by one surface falls on the other. The configuration factor of either surface is therefore unity

(ii) The surfaces are diffuse and uniform in temperature, and that the reflective and emissive properties are constant over all the surface

(iii) The surfaces are separated by a non-absorbing medium such as air.

The surface 1 emits radiant energy E_1 which strikes the surface 2. From it a part $\alpha_2 E_1$ is absorbed by the surface 2 and the remainder $(1 - \alpha_2) E_1$ is reflected back to surface 1. On reaching surface 1, a part $\alpha_1 (1 - \alpha_2) E_1$ is absorbed and the remainder $(1 - \alpha_1)(1 - \alpha_2) E_1$ is reflected and so on. The amount of radiant energy which left surface 1 per unit time is :

$$Q_1 = E_1 - \left[\alpha_1 (1-\alpha_2) E_1 + \alpha_1 (1-\alpha_1)(1-\alpha_2)^2 E_1 + \alpha_1 (1-\alpha_1)^2 (1-\alpha_2)^2 E_1 + ...\right]$$

$$= E_1 - \alpha_1 (1 - \alpha_2) E_1 \left[1 + (1-\alpha_1)(1-\alpha_2) + (1-\alpha_1)^2 (1-\alpha_2)^2 + ...\right]$$

$$= E_1 - \alpha_1 (1 - \alpha_2) E_1 \times [1 + P + P^2 + ...]$$

where $P = (1 - \alpha_1)(1 - \alpha_2)$

Since P is less than unity, the series $1 + P + P^2 + ...$, when extended to infinity gives $1/(1 - P)$

$$\therefore \quad Q_1 = E_1 - \frac{\alpha_1 (1-\alpha_2) E_1}{1-P}$$

$$= E_1 \left[1 - \frac{\alpha_1 (1-\alpha_2)}{1-(1-\alpha_1)(1-\alpha_2)}\right]$$

From Kirchoff's law, emissivity and absorptivity of a surface are equal and so

$$\alpha_1 = \epsilon_1$$

$$\therefore \quad Q_1 = E_1 \left[1 - \frac{\epsilon_1 (1-\epsilon_2)}{1-(1-\epsilon_1)(1-\epsilon_2)}\right]$$

$$= E_1 \left[\frac{\epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}\right] \qquad ...(8.10)$$

Similarly the surface 2 emits radiation of emissive power E_2. From it a part $\alpha_1 E_2$ is absorbed by surface 1 and the remainder $(1 - \alpha_1) E_2$ is reflected back to it. On reaching surface 2, a part $\alpha_2 (1 - \alpha_1) E_2$ is absorbed and the rest $(1 - \alpha_1)(1 - \alpha_2) E_2$ is reflected and so on. Proceeding exactly in the same way, we can determine the amount of heat which leaves surface 2 per unit time.

$$Q_2 = E_2 \left[\frac{\epsilon_1}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}\right] \qquad ...(8.11)$$

The net heat flow from surface 1 to surface 2 per unit time is then given by

$$Q_{12} = Q_1 - Q_2$$

$$= \frac{E_1 \epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} - \frac{E_2 \epsilon_1}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}$$

$$= \frac{E_1 \epsilon_2 - E_2 \epsilon_1}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} \qquad ...(8.12)$$

From Stefan-Blotzman law for non-black surfaces,

$$E_1 = \epsilon_1 \sigma_b T_1^4 \text{ and } E_2 = \epsilon_2 \sigma_b T_2^4$$

$$\therefore \quad Q_{12} = \frac{\epsilon_1 \sigma_b T_1^4 \epsilon_2 - \epsilon_2 \sigma_b T_2^4 \epsilon_1}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}$$

$$= \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} \sigma_b \left(T_1^4 - T_2^4\right)$$

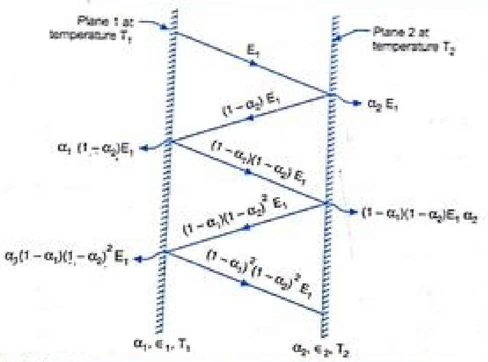


Fig. 8.24. Radiant heat exchange between two non-black parallel surfaces

$$= f_{12} \sigma_b \left(T_1^4 - T_2^4\right) \qquad ...(8.13)$$

where, $$f_{12} = \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

and is called the interchange factor for the radiation from surface 1 to surface 2

8.3.2. Infinite Long Concentric Cylinders

Consider two large concentric cylinders of areas A_1 and A_2, emissivities ϵ_1 and ϵ_2 and their surfaces maintained at temperatures T_1 and T_2 respectively

From reciprocity throrem :

$$A_1 F_{12} = A_2 F_{21}$$

The inner cylinder is completely enclosed by the outer cylinder and as such the entire heat radiations emitted by the inner cylinder are intercepted by the outer cylinder *i.e.*, $F_{12} = 1$ and therefore,

$$F_{12} = \frac{A_1}{A_2}$$

Consider the energy emitted per unit area by the inner cylinder. All this emitted energy at any instant will eventually come to rest either back to inner cylinder or in the outer cylinder. The process involves the following sequence of absorption and reflection

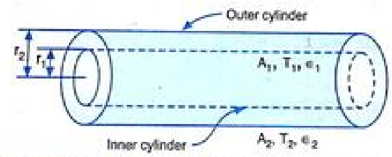


Fig. 8.25. Radiant heat exchange between two large concentric cylinders

Surface 1 emits : E_1

Surface 2 absorbs : $\alpha_2 E_1 = \epsilon_2 E_1$

Surface 2 reflects :

$$= E_1 - \epsilon_2 E_1$$
$$= E_1 (1 - \epsilon_2)$$

Surface 1 absorbs :

$$= E_1 (1 - \epsilon_2) F_{21} \alpha_1$$
$$= E_1 (1 - \epsilon_2) \frac{A_1}{A_2} \epsilon_1$$

Surface 1 reflects :

$$= E_1 (1 - \epsilon_2) - E_1 (1 - \epsilon_2) \epsilon_1 \frac{A_1}{A_2}$$
$$= E_1 (1 - \epsilon_2) \left[1 - \epsilon_1 \frac{A_1}{A_2}\right]$$

It can be shown that the energy absorbed by surface 1 (inner cylinder) on the second reflection would be

$$= E_1 (1 - \epsilon_2)^2 \epsilon_1 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\epsilon_1\right)$$

Continuation of this process would show that total energy lost by inner cylinder per unit area is

$$= E_1 - E_1 (1 - \epsilon_2) \epsilon_1 \frac{A_1}{A_2}$$
$$- E_1 (1 - \epsilon_2)^2 \epsilon_1 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\epsilon_1\right) + \ldots\ldots$$
$$= E_1\left[1 - \frac{A_1}{A_2}\epsilon_1 (1 - \epsilon_2) - (1 - \epsilon_2)^2 \times \epsilon_1 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\epsilon_1\right)\ldots\ldots\right]$$
$$= E_1\left[1 - \frac{A_1}{A_2}\epsilon_1 (1 - \epsilon_2) \times \left\{1 + (1 - \epsilon_2)\left(1 - \frac{A_1}{A_2}\epsilon_1\right)\right\}\ldots\ldots\right]$$
$$= E_1\left[1 - \frac{A_1}{A_2}\epsilon_1 (1 - \epsilon_2) \times \left\{1 - (1 - \epsilon_2)\left(1 - \frac{A_1}{A_2}\epsilon_1\right)\right\}^{-1}\right]$$
$$= E_1\left[1 - \frac{\left(\frac{A_1}{A_2}\right)\epsilon_1 (1 - \epsilon_2)}{\left\{1 - (1 - \epsilon_2)\left[1 - \left(\frac{A_1}{A_2}\right)\epsilon_1\right]\right\}}\right]$$
$$= \frac{E_1 \epsilon_2}{\left(\frac{A_1}{A_2}\right)\epsilon_1 + \epsilon_2 - \left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2} \qquad \ldots(8.14a)$$

Similarly the heat energy lost by the outer cylinder per unit area would workout

$$= \frac{\epsilon_1 E_2 \left(\frac{A_1}{A_2}\right)}{\left(\frac{A_1}{A_2}\right)\epsilon_1 + \epsilon_2 - \left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2} \qquad \ldots(8.14b)$$

The net heat flow from inner cylinder to outer cylinder is then given by

$$Q_{12} = A_1\left[\frac{E_1 \epsilon_2}{\left(\frac{A_1}{A_2}\right)\epsilon_1 + \epsilon_2 - \left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2}\right] - A_2\left[\frac{\epsilon_1 E_2 \left(\frac{A_1}{A_2}\right)}{\left(\frac{A_1}{A_2}\right)\epsilon_1 + \epsilon_2 - \left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2}\right]$$
$$= \frac{A_1 E_1 \epsilon_2 - A_1 E_2 \epsilon_1}{\left(\frac{A_1}{A_2}\right)\epsilon_1 + \epsilon_2 - \left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2}$$

From Stefan-Boltazman law for non-black bodies,

$$E_1 = \epsilon_1 \sigma_b T_1^4 \quad \text{and} \quad E_2 = \epsilon_2 \sigma_b T_2^4$$

$$\therefore Q_{12} = \frac{A_1 \epsilon_1 \epsilon_2 \sigma_b T_1^4 - A_1 \epsilon_1 \epsilon_2 \sigma_b T_2^4}{\left(\frac{A_1}{A_2}\right)\epsilon_1 + \epsilon_2 - \left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2}$$
$$= \frac{\epsilon_1 \epsilon_2 A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\epsilon_2 + \left[\left(\frac{A_1}{A_2}\right)\epsilon_1 \epsilon_2 \left(\frac{1}{\epsilon_2} - 1\right)\right]}$$

$$= \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \left[\left(\frac{A_1}{A_2}\right)\left(\frac{1}{\epsilon_2} - 1\right)\right]}$$
$$= f_{12} A_1 \sigma_b \left(T_1^4 - T_2^4\right) \qquad \ldots(8.15)$$

where, $f_{12} = \dfrac{1}{\frac{1}{\epsilon_1} + \left(\frac{A_1}{A_2}\right)\left(\frac{1}{\epsilon_2} - 1\right)}$

is the interchange factor or equivalent emissivity for radiant heat exchange between infinite long concentric cylinders

Equation 8.15 is equally applicable to concentric spheres except that for concentric cylinders of equal length

$$\frac{A_1}{A_2} = \frac{\pi d_1 l}{\pi d_2 l} = \frac{d_1}{d_2} = \frac{r_1}{r_2}$$

and for concentric spheres

$$\frac{A_1}{A_2} = \frac{4\pi r_1^2}{4\pi r_2^2} = \left(\frac{r_1}{r_2}\right)^2$$

8.3.3. Small Gray Bodies

Consider two small gray bodies having emissivities ϵ_1 and ϵ_2, and absorptivities α_1 and α_2 respectively. The small size of the bodies does signify that their size is very small compared to the distance between them. The radiant energy emitted by surface 1 would be partly absorbed by surface 2, and the unabsorbed reflected portion would be lost in space. It will not be reflected back to surface 2 because of its small size and large distance between the two surfaces.

Energy emitted by body 1

$$= A_1 \epsilon_1 \sigma_b T_1^4$$

Energy incident upon body 2

$$= F_{12} A_1 \epsilon_1 \sigma_b T_1^4$$

Energy absorbed by body 2

$$= \alpha_2 F_{12} A_1 \epsilon_1 \sigma_b T_1^4$$

Since $\alpha_2 = \epsilon_2$, the radiant energy transfer from body 1 to body 2 is

$$Q_1 = \epsilon_1 \epsilon_2 A_1 F_{12} \sigma_b T_1^4$$

Likewise the energy transfer from body 2 to body 1 would be

$$Q_2 = \epsilon_1 \epsilon_2 A_2 F_{21} \sigma_b T_2^4$$

Therefore, the net radiant heat exchange between the two bodes is

$$Q_{12} = \epsilon_1 \epsilon_2 A_1 F_{12} \sigma_b T_1^4 - \epsilon_1 \epsilon_2 A_2 F_{21} \sigma_b T_2^4$$

From reciprocity theorem :

$$A_1 F_{12} = A_2 F_{21}$$

$$\therefore \quad Q_{12} = \epsilon_1 \epsilon_2 A_1 F_{12} \left(T_1^4 - T_2^4\right)$$
$$= f_{12} A_1 F_{12} \left(T_1^4 - T_2^4\right) \qquad \ldots(8.16)$$

where $f_{12} = \epsilon_1 \epsilon_2$ respresents the equivalent emissivity or interchange factor for radiant heat exchange between two small gray bodies.

8.3.4. Small Body in a Large Enclosure

The large gray enclosure acts like a black body; it absorbs practically all the radiation incident upon it and reflects negligibly small energy back to the small gray body. Further, the entire radiations emitted by the small body would be intercepted by the outer large enclosure and as such $F_{12} = 1$.

Therefore,

Energy emitted by small body 1 and absorbed by large enclosure 2

$$= A_1 \epsilon_1 \sigma_b T_1^4$$

Energy emitted by enclsoure

$$= A_2 \epsilon_2 \sigma_b T_2^4$$

Energy incident upon small body

$$= F_{21} A_2 \epsilon_2 \sigma_b T_2^4$$

Energy absorbed by small body

$$= \alpha_1 F_{21} A_2 \epsilon_2 \sigma_b T_2^4$$
$$= \epsilon_1 \epsilon_2 A_2 F_{21} \sigma_b T_2^4$$

(because $\alpha_1 = \epsilon_1$

$\therefore$ Net exchange of energy

$$Q_{12} = \epsilon_1 A_1 \sigma_b T_1^4 - \epsilon_1 \epsilon_2 A_2 F_{21} \sigma_b T$$

Considering the trivial case in which the small body and its surrounding large enclosure are at the same temperature, *i.e.*, $T_1 = T_2$ and $Q_{12} = 0$, we get :

$$A_1 = A_2\, \epsilon_2\, F_{21}$$

$$\text{and so } Q_{12} = \epsilon_1\, A_1\, \sigma_b \left(T_1^4 - T_2^4\right)$$

$$= f_{12}\, A_1\, \sigma_b \left(T_1^4 - T_2^4\right) \quad ...(8.17)$$

where $f_{12} = \epsilon_1$ represents the equivalent emissivity or interchange factor for radiation heat exchange between a small body and a large enclosure.

While calculating the radiant interchange between two gray surfaces, both the interchange factor f_{12} and the geometric factor F_{12} are considered and net heat interchange is computed from the relation:

$$Q_{net} = f_{12}\, F_{12}\, \sigma_b\, A_1 \left(T_1^4 - T_2^4\right) \quad ...(8.18)$$

Values of interchange factor and geometric factor for some situations have been listed in Table 8.1. It is to be noted that the value of geometric factor for some of the most important engineering problems is unity.

Table 8.1. Radiation between solid surfaces

$$Q_{net} = f_{12}\, F_{12}\, \sigma_b\, A_1 \left(T_1^4 - T_2^4\right)$$

Configuration	Geometric Factor F_{12}	Interchange Factor f_{12}
(1) Infinite parallel planes	1	$\dfrac{1}{\dfrac{1}{\epsilon_1}+\dfrac{1}{\epsilon_2}-1}$
(2) Body 1 completely enclosed by body 2; body 1 small	1	ϵ_1
(3) Body 1 completely enclosed by body 2; body 1 large	1	$\dfrac{1}{\dfrac{1}{\epsilon_1}+\dfrac{A_1}{A_2}\left(\dfrac{1}{\epsilon_2}-1\right)}$
(4) Concentric spheres or infinitely long concentric cylinders	1	$\dfrac{1}{\dfrac{1}{\epsilon_1}+\dfrac{A_1}{A_2}\left(\dfrac{1}{\epsilon_2}-1\right)}$
(5) Two rectangles with common side at right angles to each other	1	$\epsilon_1\, \epsilon_2$

8.4. ELECTRICAL NETWORK APPROACH FOR RADIATION HEAT EXCHANGE

The solutions to several problems in radiation heat exchange are obtained easily by first reducing the actual system to an equivalent electrical network and then solving that network. To illustrate the concept of this technique, the following terms need to be introduced and defined:

- **Radiosity** (J) indicates the total radiant energy leaving a surface per unit time per unit surface area. It comprises the original emittance from the surface plus the reflected portion of any radiation incident upon it.
- **Irradiaton** (G) denotes the total radiant energy incident upon a surface per unit time per unit area; some of it may be reflected to become a part of the radiosity of the surface.

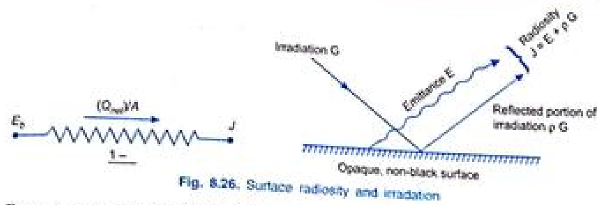


Fig. 8.26. Surface radiosity and irradation

For an opaque non-black surface of constant radiation characteristics, the total radiant energy (J) leaving the surface is the sum of its original emittance (E) and the energy reflected (ρG) by it out of the irradiation (G) impinging on it. Hence

$$J = E + \rho G = \epsilon\, E_b + \rho G \quad ...(8.19)$$

where E_b is the emissive power of a perfect black body at the same temperature. Since no energy is transmitted through an opaque surface, $\alpha + \rho = 1$ and therefore

$$J = \epsilon\, E_b + (1 - \alpha)\, G$$

Invoking Kirchoff's law, the absorptivity α of the surface equals its emissivity ϵ. Therefore

$$J = \epsilon\, E_b + (1 - \epsilon)\, G$$

$$\text{or} \quad G = \frac{J - \epsilon E_b}{1 - \epsilon} \quad ...(8.20)$$

The net rate at which the radiation leaves the surface is given by the difference between its radiosity and the incoming irradiation.

$$\frac{Q_{net}}{A} = J - \frac{J - \epsilon E_b}{1-\epsilon} = \frac{J - J\epsilon - J + \epsilon E_b}{1-\epsilon}$$

$$= \frac{\epsilon (E_b - J)}{1-\epsilon}$$

$$Q_{net} = \frac{A\epsilon}{1-\epsilon}(E_b - J) = \frac{E_b - J}{(1-\epsilon)/A\epsilon} \quad ...(8.21)$$

This equation can be represented in the form of an electrical network (Fig. 8.26). The factor $(1 - \epsilon)/A\,\epsilon$ is related to the surface properties of the radiating body and is called the *surface resistance* to radiation heat transfer.

Now consider the radiant heat exchange between two non-black surfaces. Of the total radiation J_1 leaving surface 1, only a fraction $J_1\, A_1 F_{12}$ is received by the other surface 2. Likewise the heat radiated by surface 2 and received by surface 1 is $J_2\, A_2\, F_{21}$. Net interchange of heat between the two surfaces will be

$$Q_{12} = J_1\, A_1\, F_{12} - J_2\, A_2\, F_{21} \quad ...(8.22)$$

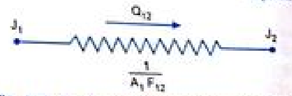


Fig. 8.27. Electrical network representing space resistance to radiation

From the reciprocity theorem :

$$A_1\, F_{12} = A_2\, F_{21}$$

$$\therefore \quad Q_{12} = (J_1 - J_2)\, A_1\, F_{12}$$

$$= \frac{(J_1 - J_2)}{1/A_1 F_{12}} \quad ...(8.23)$$

Equation 8.23 has been represented by an electrical circuit in Fig. 8.27. The quantity $\dfrac{1}{A_1 F_{12}}$ is the space resistance due to the distance between and geometry of the radiating bodies.

The electrical network corresponding to surface resistances of two radiating bodies and

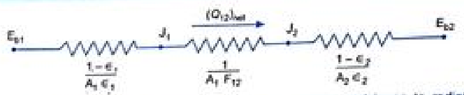


Fig. 8.28. Electrical network representing space and surface resistance to radiation

the space resistance between them has been illustrated in Fig. 8.28. The net heat exchange between the two gray surfaces may be written as :

$$(Q_{12})_{net} = \frac{E_{b_1} - E_{b_2}}{\frac{1-\epsilon_1}{A_1\epsilon_1} + \frac{1}{A_1F_{12}} + \frac{1-\epsilon_2}{A_2\epsilon_2}}$$

$$= \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$= (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4) \quad ...(8.24)$$

where factor $(F_g)_{12}$ is equal to

$$\boxed{\frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}}$$

and is referred to as **gray body factor**. A reflection on equation ~~8.24 does show~~ that the radiant heat exchange between two gray surfaces depends upon the temperature and emissivity of the surfaces and the geometry of the enclosure.

When the heat exchange is between two black surfaces, the surface resistance becomes zero as $\epsilon_1 = \epsilon_2 = 1$. The factor F_g then takes the value F_{12} which is the configuration factor, and the expression for net heat exchange gets transformed to:

$$(Q_{12})_{net} = A_1 F_{12} \sigma_b (T_1^4 - T_2^4)$$

for black surfaces

Regarding heat exchange between gray bodies, the cases of practical interest are :

(*i*) The radiating surfaces are infinite parallel planes. All the radiations emitted by one plane reach and are absorbed by the other plane. Hence $F_{12} = F_{21} = 1$, and also $A_1 = A_2$.

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$= \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + \frac{1-\epsilon_2}{\epsilon_2}}$$

$$= \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} \quad ...(8.25)$$

(*ii*) The radiating bodies are concentric cylinders or spheres. The inner cylinder or sphere of area A_1 sees only the outer surface and not itself. Therefore $F_{12} = 1$ and hence

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}} \quad ...(8.26)$$

For concentric cylinders of equal length,

$$\frac{A_1}{A_2} = \frac{\pi d_1 l}{\pi d_2 l} = \frac{d_1}{d_2} = \frac{r_1}{r_2}$$

For concentric spheres,

$$\frac{A_1}{A_2} = \frac{4\pi r_1^2}{4\pi r_2^2} = \left(\frac{r_1}{r_2}\right)^2$$

(*iii*) The heat exchange is between a small body in a large enclosure.

$$F_{12} = 1 \; ; A_1 <<< A_2 \text{ and } \therefore \frac{A_1}{A_2} \to 0$$

$$\therefore (F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1} = \epsilon_1 \quad ...(8.27)$$

Practical example of this situation corresponds to a pipe carrying steam in a large room or a thermocouple bead located inside a duct for measurement of fluid temperature.

In all the cases listed above, the configuration factor had the unit value, and as such the value of $(F_g)_{12}$ are essentially the values of the interchange factor for heat exchange between gray bodies.

EXAMPLE 8.18

A ring (ϵ = 0.85) of 8 cm inner and 16 cm outer diameter is placed in a horizontal plane. A small element (ϵ = 0.7) of 1 cm² is placed concentrically 8 cm vertically below the centre of the ring. The temperature of the ring is 800 K and that of small area is 400 K. Make calculations for the radiant heat gain by the small ring.

Solution : The radiant heat flow between two surfaces is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

$$= \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}} \quad ...(i)$$

The shape factor F_{12} for the given geometry is

$$F_{12} = L^2 \left[\frac{R_2^2 - R_1^2}{(R_2^2 + L^2) + (R_1^2 + L^2)}\right]$$

$$= 0.08^2 \times \left[\frac{0.08^2 - 0.04^2}{(0.08^2 + 0.08^2) + (0.04^2 + 0.08^2)}\right]$$

$$= 0.0064 \times \left[\frac{0.0064 - 0.0016}{(0.0064 + 0.0064) + (0.0016 + 0.0064)}\right]$$

$$= 0.0064 \times \frac{0.0048}{0.0128 + 0.008}$$

$$= 0.00148$$

Further, $A_1 = 1 \text{ cm}^2 = 1 \times 10^{-4} \text{ m}^2$

$$A_2 = \pi(R_2^2 - R_1^2)$$

$$= \pi(0.08^2 - 0.04^2)$$

$$= 0.01507 \text{ m}^2$$

Substituting the relevant data in expression (i), we get

$$Q_{12} = \frac{(1 \times 10^{-4}) \times 5.67 \times 10^{-8} (400^4 - 800^4)}{\left(\frac{1}{0.7} - 1\right) + \frac{1}{0.00148} + \left(\frac{1}{0.85} - 1\right)\frac{1 \times 10^{-4}}{0.01507}}$$

$$= \frac{-2.177}{0.428 + 675.67 + 0.00117}$$

$$= -0.00322 \text{ W (J/s)}$$

$$= -0.00322 \times 3600 = -11.59 \text{ J/hr}$$

The negative sign indicates heat gain by the ring.

EXAMPLE 8.19

Two opposed, parallel, infinite planes are maintained at 420 K and 480 K respectively. Calculate the net heat flux between these planes if one has an emissivity of 0.8 and other an emissivity of 0.7. Does it matter which plate has which emissivity ? How this heat flux will be affected if

(i) the temperature difference is doubled by raising the temperature 480 K to 540 K.

(ii) the planes are assumed to be black ?

Solution : The rate of heat interchange between the two plates is given by :

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The gray body factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

For infinite long parallel planes which see each other and nothing else,

$$F_{12} = 1 \text{ and } A_1 = A_2$$

$$\therefore (F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.87} + \frac{1}{0.7} - 1} = 0.59$$

$Q_{12} = 0.59 \times 1 \times 5.67 \times 10^{-8} (480^4 - 420$

$= 734.86 \text{ W/m}^2$

It is immaterial which plane has which emissivity since the emissivities are independent of temperature.

If T_2 = 540 K and the surfaces have the given emissivities, then

$$Q_{12} = 0.59 \times 1 \times (5.67 \times 10^{-8}) \times (540^4 - 420^4)$$

$$= 1803.55 \text{ W/m}^2$$

Obviously the heat flux increases by a factor of 1803.55/734.86 = 2.45 when the temperature is doubled.

If the surfaces are black,

$$Q_{12} = A_1 \sigma_b (T_1^4 - T_2^4)$$

$$= 1 \times (5.67 \times 10^{-8}) \times (480^4 - 420^4)$$

$$= 1245.52 \text{ W/m}^2$$

EXAMPLE 8.20

Distinguish between the configuration factor and the interchange factor.

Determine the radiation heat flux between two closely spaced, black parallel plates radiating only to each other if their temperatures are 850 K and 425 K respectively. Recalculate the heat flux presuming that each of the parallel plates has an emissivity of 0.5. In each case, the plates have an area of 4 m².

Solution : The configuration factor F_{12} considers the orientation and geometry of the black radiating surfaces; how the two surfaces view each other and to what extent the two surfaces radiate solely to each other. The interchange factor f_{12} allows for the departure of the two surfaces from complete blackness; a function of the emissivities of the two surfaces.

For the black parallel plates radiating only to each other, F_{12} = 1 and then the radiant heat exchange is :

$$Q_{12} = F_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

$$= 1 \times 4 \times (5.67 \times 10^{-8}) \times (850^4 - 425^4)$$

$$= 10 \times 10^{-3} \text{ W}$$

For the gray surfaces, the heat exchange is :

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

For the given configuration of parallel plates which see each other and nothing else,

$$F_{12} = 1 \quad \text{and} \quad A_1 = A_2$$

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_1} \times \frac{A_1}{A_2}}$$

$$= \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + \frac{1-\epsilon_2}{\epsilon_2}}$$

$$= \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.5} + \frac{1}{0.5} - 1} = 0.333$$

$$\therefore \quad Q_{12} = 0.333 \times 4 \times (5.67 \times 10^{-8}) \times (850^4 - 425^4)$$

$$= 37 \times 10^{-3} \text{ W}$$

It may be noted that if the emissivity of each plate is one-half of a black body, heat flux is reduced by a factor of 3.

EXAMPLE 8.21

A large plane, perfectly insulated on one face and maintained at a fixed temperature T_1 on the bare face, has an emissivity of 0.84 and loses 250 W/m² when exposed to surroundings at nearly 0 K. The radiant heat loss from another plane of the same size is 125 W/m² when bare face having emissivity 0.42 and maintained at temperature T_2 is exposed to the same surroundings. Subsequently these two planes are brought together so that the parallel bare faces lie only 1 cm apart and the heat supply to each is so regulated that their respective temperatures T_1 and T_2 remain unchanged. Determine the net heat flux between the planes.

Solution : From Stefan-Boltzman law

$$E = \epsilon \sigma_b T^4$$

That gives

$$E_1 = \epsilon_1 \sigma_b T_1^4 \text{ and } E_2 = \epsilon_2 \sigma_b T_2^4$$

$$\therefore \left(\frac{T_1}{T_2}\right)^4 = \left(\frac{E_1}{E_2}\right)\left(\frac{\epsilon_2}{\epsilon_1}\right) = \frac{250}{125} \times \frac{0.42}{0.84} = 1$$

That gives :

$$T_1 = T_2$$

The net radiant heat transfer between two infinite parallel planes is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

Since $T_1 = T_2$, we get:

$$\frac{Q_{12}}{A_1} = 0$$

That is, the net heat flux between the planes is zero.

EXAMPLE 8.22

A thermos flask has a double walled bottle and the space between the walls is evacuated so as to reduce the heat flow. The bottle surfaces are silver plated and the emissivity of each surface is 0.025. If the contents of the bottle are at 375 K, find the rate of heat loss from the thermos bottle to the ambient air at 300 K. What thickness of cork (k = 0.03 W/m-deg) would be required if the same insulating effect is to be achieved by the use of cork ?

Solution : The rate of heat interchange between the two bottle surfaces is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The gray body factor $(F_g)_{12}$ is equal to

$$\frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

For infinite long parallel planes which see each other and nothing else $F_{12} = 1$ and $A_1 = A_2$

$$\therefore (F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + \frac{1-\epsilon_2}{\epsilon_2}}$$

$$= \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.025} + \frac{1}{0.025} - 1} = 0.01266$$

$$\therefore \quad Q_{12} = 0.01266 \times 1 \times (5.67 \times 10^{-8}) \times (375^4 - 300^4)$$

$$= 8.38 \text{ W}$$

(b) Let δ be the required thickness of cork.

Then

$$Q = \frac{kA(t_1 - t_2)}{\delta}$$

$$\text{or} \quad 8.38 = \frac{0.03 \times 1 \times (375 - 300)}{\delta}$$

$$\therefore \quad \delta = \frac{0.03 \times 1 \times 75}{8.38}$$

$$= 0.268 \text{ m} = 26.8 \text{ cm}$$

EXAMPLE 8.23

An electric heating system is installed in the ceiling of a room that measures 5 m × 5 m with a height of 2.5 m. The temperature of the ceiling is maintained at 320 K whereas under equilibrium conditions, the walls are at 300 K. Workout the radiant heat loss from the ceiling to the walls. The floor is non-sensitive to radiations and the emissivities of the ceiling and wall are 0.7 and 0.6 respectively.

Solution : The rate of radiant interchange between the ceiling (suffix 1) and a single wall (suffix 2) is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The gray body factor

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

In the given problem :

$$A_1 = \text{area of ceiling} = 5 \times 5 = 25 \text{ m}^2$$

$$A_2 = \text{area of single wall} = 5 \times 2.5 = 12.5 \text{ m}^2$$

The ceiling and the wall are perpendicular surfaces with common edge for which

$$\frac{Z}{X} = \frac{2.5}{5} = 0.5 \text{ and } \frac{Y}{X} = \frac{5}{5} = 1$$

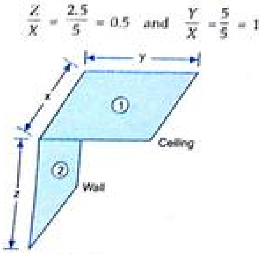


Fig. 8.29.

Corresponding to these parameters, the shape factor F_{12} as read from Fig 8.4 equals 0.15

$$\therefore (F_g)_{12} = \frac{1}{\frac{1-0.7}{0.7} + \frac{1}{0.15} + \frac{1-0.6}{0.6} \times \frac{25}{12.5}}$$

$$= \frac{1}{0.428 + 6.667 + 1.333} = 0.1186$$

$$\therefore \quad Q_{12} = 0.186 \times 25 \times (5.67 \times 10^{-8}) \times (320^4 - 300^4)$$

$$= 401 \text{ W}$$

For all the four walls,

$$Q_{12} = 4 \times 401 = 1604 \text{ W} = \mathbf{1.604 \text{ kW}}$$

EXAMPLE 8.24

Determine the net heat exchange between areas A_1 and A_2 which are perpendicular but do not share the common edge (Fig 8.30). The relevant data is :

Surface A_1 :

$$T_1 = 650 \text{ K} \quad \text{and} \quad \epsilon_1 = 0.8$$

Surface A_2 :

$$T_2 = 450 \text{ K} \quad \text{and} \quad \epsilon_1 = 0.85$$

Solution : The evaluation of such cases is made by introducting hypothetical areas A_3 and A_4 so that the arrangement of perpendicular surfaces has a common edge.

The squence of solution is :

$$A_5 F_{56} = A_1 F_{16} + A_3 F_{36}$$
$$= A_1 F_{14} + A_1 F_{12} + A_3 F_{36}$$
$$= A_5 F_{54} - A_3 F_{34} + A_1 F_{12} + A_3 F_{36}$$

$$\therefore A_1 F_{12} = (A_5 F_{56} + A_3 F_{34}) - (A_5 F_{54} - A_3 F_{36}) \quad \ldots(a)$$

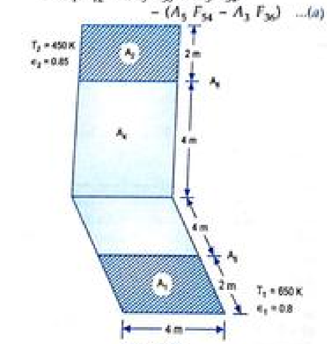


Fig. 8.30.

Each of the configuration factor on the right hand side of this expression can be read from Fig. 8.4 as they correspond to perpendicular surfaces having a common intersection line. The values are tabulated below :

Surfaces (A_{ij})	$\frac{Z}{X}$	$\frac{Y}{X}$	F_{ij}
A_{56}	$\frac{6}{4} = 1.5$	$\frac{4}{4} = 1.0$	0.22
A_{34}	$\frac{4}{4} = 1.0$	$\frac{2}{4} = 0.5$	0.35
A_{54}	$\frac{4}{4} = 1.0$	$\frac{4}{4} = 1.0$	0.20
A_{36}	$\frac{6}{4} = 1.5$	$\frac{2}{4} = 0.5$	0.37

Substituting these values in expression (*a*) :

$$A_1 F_{12} = (4 \times 4 \times 0.22) + (4 \times 2 \times 0.35) - (4 \times 4 \times 0.20) - (4 \times 2 \times 0.37)$$
$$= 3.52 + 2.80 - 3.20 - 2.96 = 0.16$$

$$\therefore \quad F_{12} = \frac{0.16}{4 \times 2} = \mathbf{0.02}$$

The rate of radiant interchange between the surfaces A_1 and A_2 is given by :

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The gray body factor $(F_g)_{12}$ is equal to

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$= \frac{1}{\frac{1-0.8}{0.8} + \frac{1}{0.2} + \frac{1-0.85}{0.85} \times \frac{8}{8}}$$

$$= \frac{1}{0.25 + 0.5 + 0.176} = 1.08$$

$$\therefore \quad Q_{12} = 1.08 \times 8 \times (5.67 \times 10^{-8}) \times (650^4 - 450^4)$$

$$= 67360 \text{ W} = \mathbf{67.36 \text{ kW}}$$

EXAMPLE 8.25

A 250 × 250 mm ingot casting, 1.5 m high and at 1225 K temperature, is stripped from its mold. The casting is made to stand on end on the floor of a large foundry whose wall, floor and roof can be assumed to be at 300 K temperature. Make calculation for the rate of radiant heat interchange between the casting and the room. The casting material has an emissivity of 0.85. Take Stefan Boltzman constant $\sigma_b = 5.67 \times 10^{-8}$ W/m² K⁴.

Solution : The rate interchange between the ingot and the room is given by :

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The configuration corresponds to a completely enclosed body, and small compared with the enclosing body, *i.e.*, $A_1 <<< A_2$ and $F_{12} = 1$. Hence

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$= \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + 0} = \epsilon_1 = 0.85$$

Area A_1 of the ingot radiating energy,

$$A_1 = (0.25 \times 0.25) + (4 \times 0.25 \times 1.5)$$
$$= 0.0625 + 1.5$$
$$= 1.5625 \text{ m}^2$$

$$\therefore \quad Q_{12} = 0.85 \times 1.5625 \times 5.67 \times 10^{-8} \times (1225^4 - 300^4)$$

$$= \mathbf{171.12 \times 10^3 \text{ W}}$$

EXAMPLE 8.26

A domestic hot water tank, 0.5 m diameter and 1 m high, is located in a large space effectively forming black surrounding. The surface emissivity and temperature are 0.8 and 350 K, and the temperature of surroundings is 295 K. Estimate the heat loss by radiation from the tank, and suggest a possibility to reduce this heat loss.

Solution : The rate of radiant interchange between the tank and enclosure is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

where the suffix 1 and 2 denote the conditions at the tank and the enclosure respectively

The configuration corresponds to a completely enclosed body, and small compared with the enclosing body, *i.e.*, $A_1 <<< A_2$ and $F_{12} = 1$. Hence

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$= \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + 0} = \epsilon_1 = 0.8$$

Area A_1 of the tank radiating energy,

$$A_1 = \pi dl + 2\left(\frac{\pi}{4} d^2\right)$$

$$= (\pi \times 0.5 \times 1) + 2\left(\frac{\pi}{4} \times 0.5^2\right)$$

$$= 1.96 \text{ m}^2$$

$$\therefore \quad Q_{12} = 0.8 \times 1.96 \times (5.67 \times 10^{-8}) \times (350^4 - 295^4)$$

$$= \mathbf{660.86 \text{ W}}$$

Some aluminium paints have an emissivity of about 0.3 and a coating of this paint on the tank would reduce the heat loss to

$$= 660 \times \frac{0.3}{0.8} = 247.82$$

(all other parameters remain the same)

$\therefore$ Percentage reduction in heat loss

$$= \frac{660.86 - 247.82}{660.82}$$
$$= 0.625 \quad \text{or} \quad \mathbf{62.5\%}$$

EXAMPLE 8.27

An enclosure measures 1.5 m × 1.75 m with a height of 2 m. Under steady state equilibrium conditions, the walls and ceiling are maintained at 525 K and floor at 400 K. Determine the net radiation to floor.

ϵ_1 (emissivity of ceiling and wall) = 0.85

ϵ_2 (emissivity of floor) = 0.75

Solution : The rate of radiant heat exchange between the ceiling and walls (suffix 1) and the floor (suffix 2) is given by :

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The gray body factor $(F_g)_{12}$ is equal to,

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

As per given data :

A_1 = total area of four walls and ceiling
$= 2\,(1.75 \times 2) + 2\,(1.5 \times 2) + (1.5 \times 1.75)$
$= 15.625\ m^2$

A_2 = area of floor $= 1.5 \times 1.75 = 2.625\ m^2$

The floor is completely enclosed by the walls and ceiling.

$\therefore \quad F_{21} = 1$

From reciprocity theorem ;

$$A_1 F_{12} = A_2 F_{21}$$

$$F_{12} = \frac{A_2}{A_1} F_{21} = \frac{2.625}{15.625} \times 1 = 0.168$$

$$(F_g)_{12} = \frac{1}{\frac{1-0.85}{0.85} + \frac{1}{0.168} + \frac{1-0.75}{0.75} \times \frac{15.625}{2.625}}$$

$$= \frac{1}{0.176 + 5.952 + 1.984} = 0.123$$

$$\therefore \quad Q_{12} = 0.123 \times 15.625 \times (5.67 \times 10^{-8}) \times (525^4 - 400^4)$$
$$= 5488\ W = \mathbf{5.488\ kW}$$

EXAMPLE 8.28

The flat floor of a hemispherical furnace is at 800 K and has an emissivity of 0.5. The corresponding values for the hemispherical roof are 1200 K and 0.25. Determine the net radiation heat transfer from the roof to floor.

Solution : Refer Fig. 8.31 for the configuration of hemispherical furnace and its nomenclature. The net radiative heat transfer froam floor to roof is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

where the suffix 1 and 2 denote the conditions at the floor and roof respectively, and

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

All the radiations from the floor (suffix 1) reach the roof (suffix 2) and hence $F_{12} = 1$

Fig. 8.31.

That gives

$$(F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

For the given configuration

$$A_1 = \pi r^2$$

and $$A_2 = \frac{1}{2}(4\pi r^2) = 2\pi r^2$$

$$\frac{A_1}{A_2} = \frac{\pi r^2}{2\pi r^2} = 0.5$$

$$(F_g)_{12} = \frac{1}{\frac{1}{0.5} + \left(\frac{1}{0.25} - 1\right) \times 0.5}$$

$$= \frac{1}{2 + 1.5} = 0.286$$

$$\therefore \quad Q_{12} = 0.286 \times 1 \times 5.67 \times 10^{-8} (800^4 - 1200^4)$$
$$= \mathbf{-\ 26984\ W/m^2}$$

The negative sign indicates that heat flow is from roof to floor *i.e.* the floor gains the heat

EXAMPLE 8.29

A sphere of 50 mm outside diameter and with a surface temperature of 600 K is located at the geometric centre of another sphere of 300 mm inside diameter and an inner surface temperature of 300 K. How much of emission from the inner surface of the large sphere is incident upon the outer surface of the small sphere ? Also calculate the net interchange of heat between the two spheres.

Assume black body behaviour for both sides of the two spheres.

Solution : All the radiations emitted by the small sphere are incident upon and absorbed by the inner surface of the large sphere. Accordingly $F_{12} = 1$

From reciprocity theorem :

$$A_1 F_{12} = A_2 F_{21}$$

$$\therefore \quad F_{21} = \frac{A_1}{A_2} F_{12} = \frac{4\pi r_1^2}{4\pi r_2^2} F_{12}$$

$$= \frac{r_1^2}{r_2^2} F_{12} = \frac{25^2}{150^2} \times 1$$

$$= 0.0278$$

Obviously 2.78 percent of emission from the inner surface of the large sphere is incident upon the small sphere and absorbed by it

Further, from energy balance of the large sphere

$$F_{21} + F_{22} = 1$$

or $$F_{22} = 1 - F_{21} = 1 - 0.0278 = 0.9722$$

Obviously 97.22 perecnt of emission from the large sphere is absorbed by inner surface of the sphere itself.

Net interchange of heat between the two spheres is,

$$Q_{net} = F_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

$$= 1 \times 4\pi\,(0.025)^2 \times 5.67 \times 10^{-8} (600^4 - 300^4)$$
$$= \mathbf{54.08\ W}$$

EXAMPLE 8.30

A steel tube, 5 cm outside diameter and 2 m long, is at 500 K temperature. This tube is located centrally in (i) a large brick room having wall temperature 300 K and (ii) a square brick conduit of 20 cm side and at 300 K. If the emissivities of steel and brick are 0.8 and 0.95 respectively, make calculations for the rate of heat loss by radiation from the tube in each case and comment on the results.

Solution : The rate of radiant interchange between the steel tube and brick enclosure is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

where the suffix 1 and 2 denote the conditions at the tube and enclosure respectively, and

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

The steel tube sees only the brick enclosure and not itself and therefore $F_{12} = 1$.

That gives

$$(F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

(i) The configuration corresponds to a completely enclosed body and much smaller compared to the enclosing body, *i.e.*, $A_1 <<< A_2$. Hence

$$(F_g)_{12} = \epsilon_1$$

$$\therefore \quad Q_{12} = \epsilon_1 A_1 \sigma_b (T_1^4 - T_2^4)$$
$$= 0.8 \times (\pi \times 0.05 \times 2) \times 5.67 \times 10^{-8} (500^4 - 300^4)$$
$$= \mathbf{774.82\ W}$$

(ii) The situation corresponds to a relatively large body completely enclosed.

$$\frac{A_1}{A_2} = \frac{\pi d l}{P l} = \frac{\pi d}{P} = \frac{\pi \times 0.05}{4 \times 0.2} = 0.196$$

$$(F_g)_{12} = \frac{1}{\frac{1}{0.8} + \left(\frac{1}{0.95} - 1\right) \times 0.196}$$

$$= \frac{1}{1.25 + 0.0103} = 0.793$$

$$\therefore \quad Q_{12} = 0.793 \times (\pi \times 0.05 \times 2) \times 5.67 \times 10^{-8} \times (500^4 - 300^4)$$
$$= 768.04 \text{ W}$$

Comment : *Radiation heat transfer depends both on the tube area and the conduit area.*

EXAMPLE 8.31

A 0.25 mm diameter electric wire having surface emissivity 0.38 is placed within a tube of 2.5 mm diameter, negligible thickness and 0.65 surface emissivity. This tube in turn is placed concentrically within a 5 mm diameter tube of emissivity 0.72. The surface of the outer tube is maintained at 280 K and the annular faces between the tubes are evacuated completely. If the temperature at the inner tube is required to be 400 K, make calculations for the temperature of the electric wire.

Solution : Under steady state conditions: radiant heat flow from wire to inner tube = radiant heat flow from inner tube to outer tube

$$(F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4) = (F_g)_{23} A_2 \sigma_b (T_2^4 - T_3^4)$$

or
$$\frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}} = \frac{A_2 \sigma_b (T_2^4 - T_3^4)}{\left(\frac{1}{\epsilon_2} - 1\right) + \frac{1}{F_{23}} + \left(\frac{1}{\epsilon_3} - 1\right)\frac{A_2}{A_3}}$$

For the given arrangement

$$F_{12} = F_{23} = 1$$

$$\frac{A_1}{A_2} = \frac{\pi d_1 l}{\pi d_2 l} = \frac{d_1}{d_2} = \frac{0.25}{2.5} = 0.1$$

$$\frac{A_2}{A_3} = \frac{d_2}{d_3} = \frac{2.5}{5} = 0.5$$

Substituting the relevant data in the above identity, we get

$$\frac{(\pi \times 0.00025 \times 1) \times 5.67 \times 10^{-8} (T_1^4 - 400^4)}{\left(\frac{1}{0.38} - 1\right) + 1 + \left(\frac{1}{0.65} - 1\right) \times 0.1} = \frac{(\pi \times 0.0025 \times 1) \times 5.67 \times 10^{-8} (400^4 - 280^4)}{\left(\frac{1}{0.65} - 1\right) + 1 + \left(\frac{1}{0.72} - 1\right) \times 0.5}$$

or
$$\frac{0.00025 (T_1^4 - 400^4)}{1.63 + 1 + 0.0538} = \frac{0.0025 (400^4 - 280^4)}{0.538 + 1 + 0.194}$$

or
$$\frac{T_1^4 - 400^4}{2.6838} = \frac{10 (400^4 - 280^4)}{1.732}$$

or
$$T_1^4 = \frac{10 \times 2.6838}{1.732} \times (400^4 - 280^4) + 400^4$$

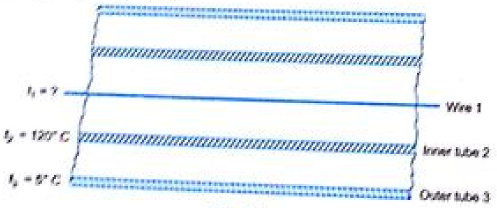


Fig. 8.32.

$$= 3014.38 \times 10^8 + 256 \times 10^8$$
$$= 3270.38 \times 10^8$$

$\therefore$ Temperature of electric wire, T_1

$$= (3270.38 \times 10^8)^{1/4} = 756.2 \text{ K}$$

EXAMPLE 8.32

A 30 mm diameter spherical container used for storing liquid nitrogen under atmospheric conditions (boiling point = 90 K) is insulated by enclosing it concentrically within another sphere of 45 cm diameter. The intervening annular space between the spheres is completely evacuated and the material for both spheres has surface emissivity of 0.3. Make calculations for the radiant heat flow if the temperature of the outer container is 300 K. Proceed to calculate the surface emissivity of the container material if heat flow rate is to be reduced by 80 percent.

Solution : The radiant heat flow between two concentric spheres is given by

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

where the suffix 1 and 2 refer to inner and outer containers respectively and the gray factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

For the given configuration,

$$F_{12} = 1 ;$$

$$A_1 = 4\pi r_1^2 = 4\pi (0.15)^2 = 0.2826 \text{ m}^2$$

$$\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{0.15}{0.225}\right)^2 = 0.445$$

Accordingly

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{0.3} - 1\right) + 1 + \left(\frac{1}{0.3} - 1\right) \times 0.445}$$

$$= \frac{1}{3.33 - 1 + 1 + 1.038} = 0.229$$

$$\therefore \quad Q_{12} = 0.229 \times 0.2826 \times 5.67 \times 10^{-8} (90^4 - 300^4)$$
$$= -29.48 \text{ W}$$

The –ve sign implies that the flow of heat is from outside to inside

(ii) Since the heat flow is to be reduced by 80%,

$$Q_{12} = (1 - 0.8) \times (-29.48)$$
$$= -5.896 \text{ W}$$

$$\therefore -5.896 = \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

$$= \frac{0.2826 \times 5.67 \times 10^{-8} (90^4 - 300^4)}{\left(\frac{1}{\epsilon} - 1\right) + 1 + \left(\frac{1}{\epsilon} - 1\right) \times 0.445}$$

or
$$\frac{1}{\epsilon} + \left(\frac{1}{\epsilon} - 1\right) \times 0.445 = \frac{0.2826 \times (5.67 \times 10^{-8}) \times (90^4 - 300^4)}{(-5.896)}$$

$$= 21.83$$

or
$$\frac{1}{\epsilon} + \frac{0.445}{\epsilon} = 21.83 + 0.445 = 22.275$$

$$\therefore \quad \epsilon = \frac{1.445}{22.275} = 0.0649$$

EXAMPLE 8.33

Three hollow thin walled cylinders having radii 5 cm, 10 cm and 15 cm are arranged concentrically. The temperatures of the innermost and outermost cylinder surfaces are 1000 K and 300 K respectively. Assuming vacuum between the annular spaces, determine the steady state temperature attained by the cylinder surface having radius of 10 cm. Proceed to calculate the heat flow per m² area of cylinder with radius 5 cm.

Take $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.05$.

Solution : With reference to Fig. 8.33 and for steady state heat flow condition, one may write

$$Q_{12} = Q_{23}$$

$$\text{or}\quad \frac{A_1\sigma_b(T_1^4-T_2^4)}{\left(\frac{1}{\epsilon_1}-1\right)+\frac{1}{F_{12}}+\left(\frac{1}{\epsilon_2}-1\right)\frac{A_1}{A_2}}$$

$$=\frac{A_2\sigma_b(T_2^4-T_3^4)}{\left(\frac{1}{\epsilon_2}-1\right)+\frac{1}{F_{23}}+\left(\frac{1}{\epsilon_3}-1\right)\frac{A_2}{A_3}}$$

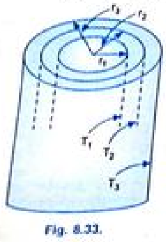


Fig. 8.33.

For the given figuration $F_{12} = F_{23} = 1$

That gives

$$\frac{A_1\sigma_b(T_1^4-T_2^4)}{\frac{1}{\epsilon_1}+\left(\frac{1}{\epsilon_2}-1\right)\frac{A_1}{A_2}}=\frac{A_2\sigma_b(T_2^4-T_3^4)}{\frac{1}{\epsilon_2}+\left(\frac{1}{\epsilon_3}-1\right)\frac{A_2}{A_3}} \quad ...(i)$$

All the areas are surface areas of the cylinders and therefore

$$\frac{A_1}{A_2}=\frac{2\pi r_1 l}{2\pi r_2 l}=\frac{r_1}{r_2}=\frac{5}{10}=0.5$$

$$\text{and}\quad \frac{A_2}{A_3}=\frac{r_2}{r_3}=\frac{10}{15}=0.67$$

Substituting the given data in expression (i).

$$\frac{(2\pi\times 0.05\times l)\times 5.67\times 10^{-8}(1000^4-T_2^4)}{\frac{1}{0.05}+\left(\frac{1}{0.05}-1\right)\times 0.5}$$

$$=\frac{(2\pi\times 0.1\times l)\times 5.67\times 10^{-8}(T_2^4-300^4)}{\frac{1}{0.05}+\left(\frac{1}{0.05}-1\right)\times 0.67}$$

$$\text{or}\quad \frac{0.05(1000^4-T_2^4)}{20+9.5}=\frac{0.1(T_2^4-300^4)}{20+12.73}$$

$$\text{or}\quad 1000^4-T_2^4=\frac{29.5}{32.73}\times\frac{0.1}{0.05}(T_2^4-300^4)$$

$$=1.8(T_2^4-300^4)$$

$$\text{or}\quad 2.8\,T^4 = 1000^4 + 1.8\times 300^4$$

$$= 1\times 10^{12} + 0.01458\times 10^{12}$$

$$= 1.01458\times 10^{12}$$

$$\text{or}\quad T^4=\frac{1.01458\times 10^{12}}{2.8}$$

$$= 0.36235\times 10^{12}\ ;\ T_2 = 775.8\ \text{K}$$

(i) Heat flow per m^2 area of cylinder 1 is

$$Q_{12}=\frac{A_1\sigma_b(T_1^4-T_2^4)}{\frac{1}{\epsilon_1}+\left(\frac{1}{\epsilon_2}-1\right)\frac{A_1}{A_2}}$$

$$=\frac{1\times(5.67\times 10^{-8})\times(1000^4-775.8^4)}{\frac{1}{0.05}+\left(\frac{1}{0.05}-1\right)0.5}$$

$$=\frac{5.67(10^4-7.758^4)}{20+9.5}=1225.8\ \text{W}$$

EXAMPLE 8.34

Consider an enclosure formed by three surfaces having the following values of shape factors, emissivities and temperatures :

Surface	*Shape*	*Emissivity*	*Temperature*
1	Curved cylindrical	0.75	800 K
2	Closing disc end	0.8	700 K
3	Closing disc end	0.8	700 K

The closing flat discs are 25 mm in diameter and they have interspacing distance equal to 100 mm. If the shape factor between these two identical discs is 0.05, calculate the net rate of radiant heat flow from the curve surface to each of the closing end surfaces.

Solution : From the shape factor relation : $F_{21} + F_{23} = 1$, we have

$$F_{21} = 1 - F_{23} = 1 - 0.05 = 0.95$$

$$A_1 = \pi d h = \pi \times 0.025 \times 0.1$$

$$= 0.00785\ \text{m}^2$$

$$A_2=\frac{\pi}{4}d^2=\frac{\pi}{4}(0.025)^2=0.000491\ \text{m}^2$$

Then from reciprocity theorem :

$$A_1 F_{12} = A_2 F_{21}$$

$$F_{12}=\frac{A_2}{A_1}F_{21}$$

$$=\frac{0.000491}{0.00785}\times 0.95=0.0594$$

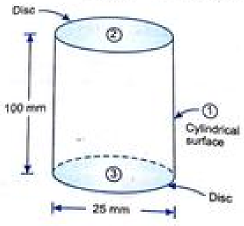


Fig. 8.34.

There is no net heat flow from surface 2 to 3 or from surface 3 to 2 as $Q_{23} = Q_{32}$. Further because of symmetry $Q_{12} = Q_{13}$ and

$$Q_{12} = (F_g)_{12}\, A_1\, \sigma_b (T_1^4 - T_2^4)$$

$$\text{where}\quad (F_g)_{12}=\frac{1}{\left(\frac{1}{\epsilon_1}-1\right)+\frac{1}{F_{12}}+\left(\frac{1}{\epsilon_2}-1\right)\frac{A_1}{A_2}}$$

$$=\frac{1}{\left(\frac{1}{0.75}-1\right)+\frac{1}{0.0594}+\left(\frac{1}{0.8}-1\right)\times\frac{0.00785}{0.000491}}$$

$$=\frac{1}{0.333+16.835+3.997}$$

$$=\frac{1}{21.165}=0.0472$$

$$\therefore\quad Q_{12} = 0.0472\times 0.00785\times 5.67\times 10^{-8}(800^4-700^4)$$

$$= 3.561\ \text{W}$$

EXAMPLE 8.35

A steel rod of 20 mm diameter has been mounted axially in a heat treatment muffle furnace of inside diameter 160 mm. The inside surface temperature of the muffle is at 1360 K and has an emissivity of 0.85, while the emissivity of the surface of the rod is 0.6. Find the time required to heat the rod from 700 K to 800 K assuming that it occupies full length of the furnace. For the rod material, take specific heat as 0.65 kJ/kg K and the density as 7840 kg/m^3.

Solution : The rate of radiant interchange between the steel rod and the furnace is given by:

$$Q_{12} = (F_g)_{12}\, A_1\, \sigma_b (T_1^4 - T_2^4)$$

where the suffix 1 and 2 denote the conditions at the steel rod and the furnace respectively.

The configuration corresponds to a relatively large body complety enclosed, the inner body sees only the outer surface and not itself. Therefore $F_{12} = 1$ and hence

$$(F_g)_{12}=\frac{1}{\frac{1-\epsilon_1}{\epsilon_1}+\frac{1}{F_{12}}+\frac{1-\epsilon_2}{\epsilon_2}\times\frac{A_1}{A_2}}$$

$$=\frac{1}{\left(\frac{1}{\epsilon_1}-1\right)+1+\left(\frac{1}{\epsilon_2}-1\right)\times\frac{A_1}{A_2}}$$

$$=\frac{1}{\frac{1}{\epsilon_1}+\left(\frac{1}{\epsilon_2}-1\right)\times\frac{A_1}{A_2}}$$

For concentric cylinders of equal length, the ratio of areas A_1/A_2 is that of the diameters d_1/d_2.

$$(F_g)_{12}=\frac{1}{\frac{1}{0.6}+\left(\frac{1}{0.86}-1\right)\times\frac{20}{160}}=0.592$$

Therefore for unit length of the rod and the furnace

$$Q_{12} = 0.592\times(\pi\times 0.02\times 1)\times 5.67\times 10^{-8}\times(T_1^4-T_2^4)$$

$$=0.211\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]$$

At the beginning of the heating period $T_2 = 700$ K and at the end $T_2 = 800$ K. Therefore

Initially Q_{12}

$$= 0.181\left[\left(\frac{1360}{100}\right)^4 - \left(\frac{700}{100}\right)^4\right]$$

$$= 6711.78 \text{ W}$$

Finally,

$$Q_{12} = 0.181\left[\left(\frac{1360}{100}\right)^4 - \left(\frac{800}{100}\right)^4\right]$$

$$= 6353.90 \text{ W}$$

$\therefore$ Mean rate of heating per unit length

$$= \frac{1}{2}(6711.78 + 6353.90)$$

$$= 6532.84 \text{ W} = 6.53 \text{ kJ/s}$$

Heat required to raise the temperature of rod from 700 K to 800 K

$$= m\, c_p\, \Delta t$$

$$= \left[\frac{\pi}{4}(0.02)^2 \times 1 \times 7840\right] \times 0.65 \times (800 - 700)$$

$$= 159.98 \text{ kJ}$$

$\therefore$ Time required for the desired heating

$$= \frac{159.98}{6.53} = 24.5 \text{ sec}$$

8.5. RADIATION SHIELDS

Many situations are encountered where it is desired to reduce the overall heat transfer between two radiating surfaces. The task is accomplished by placing radiation shields between the emitting surfaces.

The shields are thin opaque partitions arranged in the direction perpendicular to the propagation of radiated heat, and made of materials of very low absorptivity and high reflectivity (thin sheets of aluminium, copper etc.) The shields introduce a sort of additional resistance in the heat flow path and accordingly the net heat flux is reduced.

(*i*) With no radiation shields, the net heat exchange between the infinite parallel planes is given by :

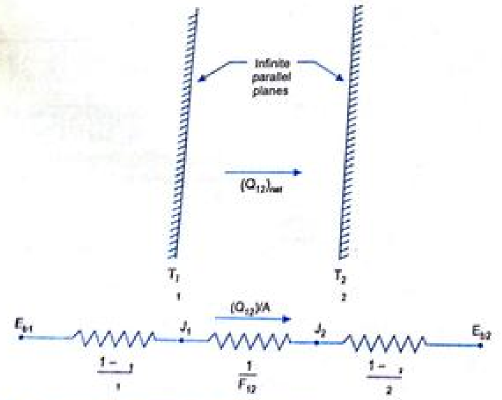


Fig. 8.35. Heat exchange between two infinite parallel planes without radiation shields

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

For the given cofiguration of parallel planes which see each other and nothing else,

$$F_{12} = 1 \quad \text{and} \quad A_1 = A_2 = A$$

$$(F_g)_{12} = \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + \dfrac{1}{F_{12}} + \dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}}$$

$$= \frac{1}{\left(\dfrac{1}{\epsilon_1} - 1\right) + 1 + \left(\dfrac{1}{\epsilon_2} - 1\right)}$$

$$= \frac{1}{\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 1}$$

$$\therefore \quad Q_{12} = \frac{A\sigma_b (T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 1} \quad \text{...(8.28)}$$

(*ii*) The placement of a radiation shield between these two planes would neither remove nor add any heat to the system. Under steady state conditions, the screens attain a uniform temperature of T_3, and heat transfer between plane 1 and the shield $(Q_{13})_{net}$

= heat transfer between the shield and the plane 2 $(Q_{32})_{net}$

$$\frac{A\sigma_b (T_1^4 - T_3^4)}{\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_3} - 1} = \frac{A\sigma_b (T_3^4 - T_2^4)}{\dfrac{1}{\epsilon_3} + \dfrac{1}{\epsilon_2} - 1} \quad \text{...(8.29)}$$

Both faces of the radiation shield have been assumed to have the same emissivity. Simplification of expression 8.29 yields :

$$T_3^4 = \frac{T_1^4\left(\dfrac{1}{\epsilon_3} + \dfrac{1}{\epsilon_2} - 1\right) + T_2^4\left(\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_3} - 1\right)}{\left(\dfrac{1}{\epsilon_3} + \dfrac{1}{\epsilon_2} - 1\right) + \left(\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_3} - 1\right)} \quad \text{...(8.30)}$$

Each side of expression 8.29 represents the heat flow through the system. Substituting the value of T_3 in left hand side of expression 8.29, we obtain

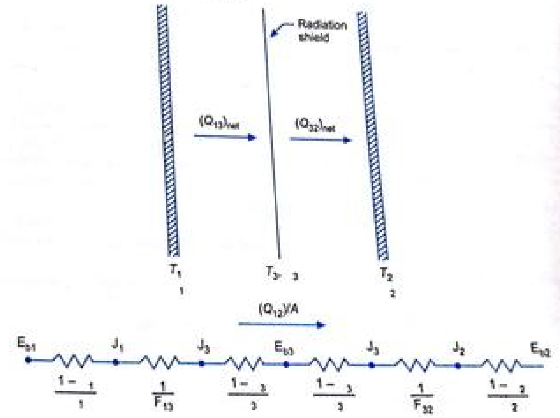


Fig. 8.36. Heat exchange between two infinite parallel planes with radiation shield

$$(Q_{12})_{net} = \frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1\right) + \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1\right)} \qquad ...(8.31)$$

The ratio of radiant energy transfer with one shield to that without any shield is obtained from expressions 8.28 and 8.31. That gives,

Radiant energy transfer

$$\frac{\text{With shield}}{\text{Without shield}} = \frac{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1\right) + \left(\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1\right)} \qquad ...(8.32)$$

When $\epsilon_1 = \epsilon_2 = \epsilon_3$, the above fraction takes the value 1/2. Thus by inserting one shield, the heat transfer rate is reduced to one-half of the original value. The corresponding temperature T_3 of the shield attains the value :

$$T_3^4 = \frac{1}{2}\left(T_1^4 + T_2^4\right) \qquad ...(8.33)$$

The electrical net works for the system with and without a radiation shield have been indicated in Fig. 8.34 and 8.35.

Heat exchange without any shield

$$= \frac{A\left(E_{b_1} - E_{b_2}\right)}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2}}$$

$$= \frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\epsilon_1} - 1\right) + 1 + \left(\frac{1}{\epsilon_2} - 1\right)}$$

$$= \frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\frac{2}{\epsilon} - 1} \qquad ...(a)$$

Heat exchange with one shield

$$= \frac{A\left(E_{b_1} - E_{b_2}\right)}{\left[\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{13}} + \frac{1-\epsilon_3}{\epsilon_3}\right] + \left[\frac{1-\epsilon_3}{\epsilon_3} + \frac{1}{F_{32}} + \frac{1-\epsilon_2}{\epsilon_2}\right]}$$

$$= \frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\epsilon_1 - 1}\right) + 1 + \left(\frac{1}{\epsilon_3 - 1}\right) + \left(\frac{1}{\epsilon_3 - 1}\right) + 1 + \left(\frac{1}{\epsilon_2 - 1}\right)}$$

$$= \frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \frac{2}{\epsilon_3} + \frac{1}{\epsilon_2} - 2}$$

$$= \frac{1}{2}\frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\left(\frac{2}{\epsilon} - 1\right)} \qquad ...(b)$$

Comparison of expressions (*a*) and (*b*) also shows that the heat flow ratio with a radiation shield becomes just half of what it would have been without the radiation shield.

If *n*-radiation shields are inserted between the two planes, then

(*i*) there will be two surface resistances for each radiation shield, and one for each radiating plane. When the emissivity of all the surfaces are equal, then all the $(2n + 2)$ surface resistances will have the same value $(1 - \epsilon)/\epsilon$.

(*ii*) there would be $(n + 1)$ space resistance and the configuration factor for each will be unity.

Obviously the total resistance of the physical system will be :

R (n – shields)

$$= (2n + 2)\frac{1-\epsilon}{\epsilon} + (n + 1) \times 1$$

$$= (n+1)\left(\frac{2}{\epsilon} - 1\right)$$

and therefore the heat exchange is given by :

$$= \frac{1}{n+1}\frac{A\,\sigma_b\left(T_1^4 - T_2^4\right)}{\left(\frac{2}{\epsilon} - 1\right)} \qquad ...(c)$$

A comparison of expression (*a*) and (*c*) does indicate that the presence of *n*-shields reduces the radiant heat transfer by a factor of $(n + 1)$.

EXAMPLE 8.36

Two parallel square plates, each 4 m² area, are large compared to a gap of 5 mm separating them. One plate has a temperature of 800 K and surface emissivity of 0.6, while the other has a temperature of 300 K and a surface emissivity of 0.9. Find the net energy exchange by radiation between the plates.

If a thin polished metal sheet of surface emissivity 0.1 on both sides is now located centrally between the two plates, what will be its steady state temperature ? How the heat transfer would be altered ? Neglect the convection and edge effects if any. Comment upon the significance of this exercise.

Solution : The rate of heat interchange between the two plates is given by :

$$Q_{12} = \left(F_g\right)_{12} A_1\,\sigma_b\left(T_1^4 - T_2^4\right)$$

For infinite long parallel planes which see each other and nothing else, $F_{12} = 1$ and $A_1 = A_2$.

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$= \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.6} + \frac{1}{0.9} - 1} = 0.562$$

$$\therefore \quad Q_{12} = 0.562 \times 4 \times 5.67 \times 10^8\left(800^4 - 300^4\right)$$

$$= 51175.8 \text{ Watts or } 51.176 \text{ kW}$$

(*ii*) Let suffix 3 designate the sheet which has been inserted between the two plates.

Heat flow from plate 1 to sheet,

$$Q_{13} = \left(F_g\right)_{13} A_1\,\sigma_b\left(T_1^4 - T_3^4\right)$$

Now,

$$(F_g)_{13} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1}$$

$$= \frac{1}{\frac{1}{0.06} + \frac{1}{0.1} - 1} = 0.09374$$

$$\therefore \quad Q_{13} = 0.09374\,A_1\,\sigma_b\left(800^4 - T_3^4\right) \qquad ...(a)$$

Heat flow from sheet to plate 2,

$$Q_{32} = (F_g)_{32}\,A_3\,\sigma_b\left(T_3^4 - T_2^4\right)$$

Now,

$$(F_g)_{32} = \frac{1}{\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.1} + \frac{1}{0.9} - 1} = 0.0989$$

$$\therefore \quad Q_{32} = 0.0989\,A_3\,\sigma_b\left(T_3^4 - 300^4\right)$$

Under steady state conditions,

$$0.09374\,A_1\,\sigma_b\left(800^4 - T_3^4\right) = 0.0989\,A_3\,\sigma_b\left(T_3^4 - 300^4\right)$$

Recognising that $A_1 = A_3$, we get

$$0.09374\left(800^4 - T_3^4\right) = 0.0989\left(T_3^4 - 300^4\right)$$

$$T_3^4 = \frac{0.09374 \times 800^4 + 0.0989 \times 300^4}{0.09374 + 0.0989}$$

$$T_3 = 671.65°K$$

Therefore equilibrium temperature T_3 of the shield is 671.65°K

Any one of the expressions (*a*) and (*b*) can now be used to workout the heat interchange between the plates:

$\therefore$ Heat flow through the system,

$$= 0.09374 \times 4 \times \left(5.67 \times 10^{-8}\right) \times \left(800^4 - 671.65^4\right)$$

$$= 4381.66 \text{ Watts or } 4.382 \text{ kW}$$

The placement of a radiation shield reduces the radiant heat transfer by a factor of 51.176/4.382 = 11.678 times. Screens are thus placed between surfaces to cut down the radiation loss in heat insulation structure members. For example, the bulb of a thermometer or a thermocouple junction is often shielded in order to reduce radiation effects to a minimum in the act of temperature measurement of a fluid.

EXAMPLE 8.37

Two large parallel planes with emissivity 0.4 are maintained at different temperatures and exchange heat only by radiation. What percentage change in net radiative heat transfer would occur if two equally large radiation shields with surface emissivity 0.04 are introduced in parallel to the plates ?

Solution : Case (*i*) When shields are not used.

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

$$= \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

$$= \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

as $A_1 = A_2$ and $F_{12} = 1$

$$= \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\frac{1}{0.4} + \frac{1}{0.4} - 1} = 0.2\,C$$

where $C = A_1 \sigma_b (T_1^4 - T_2^4)$

Case (*ii*) When two radiation shields are used

$$Q_{12} = \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left\{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{1s}} + \left(\frac{1}{\epsilon_s} - 1\right)\frac{A_1}{A_s}\right\} + \left\{\left(\frac{1}{\epsilon_s} - 1\right) + \frac{1}{F_{ss}} + \left(\frac{1}{\epsilon_S} - 1\right)\frac{A_S}{A_S}\right\} + \left(\frac{1}{\epsilon_s} - 1\right) + \frac{1}{F_{s2}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_S}{A_2}}$$

As $A_1 = A_2 = A_s$
and $F_{1s} = F_{ss} = F_{s2} = 1$, we get

$$Q_{12} = \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_s} - 1\right) + \left(\frac{1}{\epsilon_s} + \frac{1}{\epsilon_s} - 1\right) + \left(\frac{1}{\epsilon_s} + \frac{1}{\epsilon_2} - 1\right)}$$

$$= \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{4}{\epsilon_s} - 3}$$

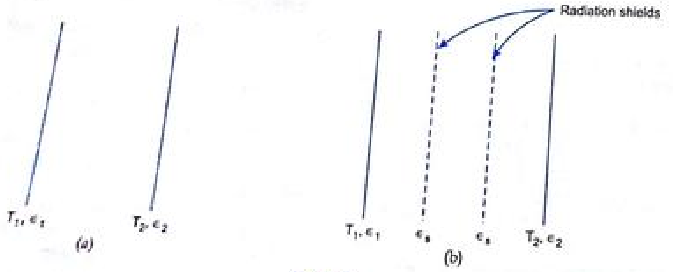


Fig. 8.37.

$$= \frac{C}{\frac{1}{0.4} + \frac{1}{0.4} + \frac{4}{0.04} - 3}$$

$$= \frac{C}{102} = 0.0098\,C$$

∴ Percentage change in heat radiant heat exchange

$$= \frac{0.2C - 0.0098C}{0.2C} \times 100 = 95.1\%$$

EXAMPLE 8.38

Determine the net radiant heat exchange per m² area for two infinite parallel plates held at temperature of 800 K and 500 K respectively. Take emissivity as 0.6 for the hot plate and 0.4 for the cold plate.

What should be the emissivity of a polished aluminium shield placed between them if heat flow is to be reduced to 40 per cent of its original value ? Proceed to calculate the equilibrium temperature of the shield.

Solution : Refer Fig. 8.35. The rate of heat interchange between the two plates is

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

where the gray body factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

For infinite long parallel plates which see each other and nothing else, $F_{12} = 1$ and $A_1 = A_2$

That gives :

$$(F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.6} + \frac{1}{0.4} - 1}$$

$$= \frac{1}{1.67 + 2.5 - 1} = 0.315$$

$$\therefore \quad Q_{12} = 0.315 \times 1 \times (5.67 \times 10^{-8}) \times (800^4 - 500^4)$$

$$= 6200\ \text{W/m}^2$$

(*ii*) Refer Fig. 8.36. When a radiation shield with emissivity ϵ_3 on both sides is placed between the two plates, then

$$(F_g)_{12} = \frac{1}{\left[\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{13}} + \left(\frac{1}{\epsilon_3} - 1\right)\frac{A_1}{A_3}\right] + \left[\left(\frac{1}{\epsilon_3} - 1\right) + \frac{1}{F_{32}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_3}{A_2}\right]}$$

For the given arrangement

$A_1 = A_2 = A_3$ and $F_{13} = F_{32}$

That gives

$$\frac{1}{(F_g)_{12}} = \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2}{\epsilon_3} - 2$$

Since the radiant heat interchange is limited to 40% of original value,

$$\frac{40}{100} \times 6200 = (F_g)_{12} \times 1 \times (5.67 \times 10^{-8}) \times (800^4 - 500^4)$$

or $(F_g)_{12} = 0.126$ or $\frac{1}{(F_g)_{12}} = 7.936$

Accordingly :

$$\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2}{\epsilon_3} - 2 = 7.936$$

or $$\frac{1}{0.6} + \frac{1}{0.4} + \frac{2}{\epsilon_3} - 2 = 7.936$$

or $$1.67 + 2.5 + \frac{2}{\epsilon_3} - 2 = 7.936$$

or $$\frac{2}{\epsilon_3} = 7.936 + 2 - 1.67 - 2.5$$

$$= 5.766$$

∴ Desired emissivity of screen ϵ_3

$$= \frac{2}{5.766} = 0.347$$

(*iii*) Under steady state conditions

$$Q_{12} = Q_{13} = Q_{32}$$

or $$\frac{40}{100} \times 6200 = \frac{A \sigma_b (T_1^4 - T_3^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1}$$

$$= \frac{1 \times 5.67 \times 10^{-8}\left(800^4 - T_3^4\right)}{\frac{1}{0.6} + \frac{1}{0.347} - 1}$$

$$= \frac{5.67 \times 10^{-8}\left(800^4 - T_3^4\right)}{1.67 + 2.88 - 1}$$

$$= \frac{5.67 \times 10^{-8}\left(800^4 - T_3^4\right)}{3.55}$$

or $800^4 - T_3^4 = \frac{40}{100} \times \frac{6200 \times 3.55}{5.67 \times 10^{-8}}$

$= 1550 \times 10^8$

or $T_3^4 = 4096 \times 10^8 - 1550 \times 10^8$

$= 2546 \times 10^8$

$\therefore \quad T_3 = \left(2546 \times 10^8\right)^{1/4} = 710$ **K**

EXAMPLE 8.39

Two large parallel planes with emissivity 0.8 are exchanging heat by radiation. One plane has a temperature of 1000 K while the other plane is at 400 K temperature. It is proposed to interpose a radiation shield with emissivity value of 0.05 on one side and 0.6 on the other side. The design conditions stipulate that the low emissivity side should face the hotter place. How would the shield temperature be affected if during installation, a mistake occurs and the higher emissivity side is placed facing the hot place ?

Solution : Heat interchange as per the desired set up

$$Q_{12} = \left(F_g\right)_{12} A_1 \sigma_b \left(T_1^4 - T_2^4\right)$$

$$= \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\left[\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{1s}} + \left(\frac{1}{\epsilon_{s1}} - 1\right)\frac{A_1}{A_s}\right] + \left[\left(\frac{1}{\epsilon_{s2}} - 1\right) + \frac{1}{F_{s2}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_s}{A_2}\right]}$$

For infinite long parallel planes and the radiation shield placed in between

$F_{1s} = F_{s2} = 1$

and $A_1 = A_s = A_2$

$$\therefore Q_{12} = \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\left[\frac{1}{\epsilon_1} + \frac{1}{\epsilon_{s1}} - 1\right] + \left[\frac{1}{\epsilon_{s2}} + \frac{1}{\epsilon_2} - 1\right]}$$

Considering unit area and substituting the relevant values,

$$Q_{12} = \frac{1 \times 5.67 \times 10^{-8}\left(1000^4 - 400^4\right)}{\left(\frac{1}{0.8} + \frac{1}{0.05} - 1\right) + \left(\frac{1}{0.6} + \frac{1}{0.8} - 1\right)}$$

$$= \frac{55248}{(1.25 + 20 - 1) + (1.67 + 1.25 - 1)}$$

$$= \frac{55248}{22.17} = 2492 \text{ W}$$

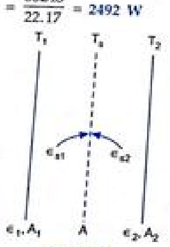


Fig. 8.38.

Under steady state conditions, the heat interchange between hot plane (1) and the shield (*s*) equals the heat interchange between the hot and cold planes. Therefore

$$Q_{12} = \frac{A_1 \sigma_b \left(T_1^4 - T_s^4\right)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_{s1}} - 1}$$

$$\therefore \quad 2492 = \frac{1 \times 5.67 \times 10^{-8}\left(1000^4 - T_s^4\right)}{\frac{1}{0.8} + \frac{1}{0.05} - 1}$$

$$= \frac{5.67 \times \left[10^4 - \left(\frac{T_s}{100}\right)^4\right]}{20.25}$$

or $\left(\frac{T_s}{100}\right)^4 = 10^4 - \frac{2492 \times 20.25}{5.67} = 1100$

$\therefore$ Shield temperature T_s

$= (1100)^{\frac{1}{4}} \times 100 = 575.9$ **K**

When the shield gets placed wrongly during installation, the total resistance would remain the same as there is only a shift in the location of thermal resistances. Accordingly heat interchange between the two planes would remain same. However, the temperature of the shield would be different and it can be worked out from the identity,

$$2492 = \frac{1 \times 5.67 \times 10^{-8}\left(1000^4 - T_s^4\right)}{\frac{1}{0.8} + \frac{1}{0.06} - 1}$$

$$= \frac{5.67 \times \left[10^4 - \left(\frac{T_s}{100}\right)^4\right]}{1.92}$$

or $\left(\frac{T_s}{100}\right)^4 = 10^4 - \frac{2492 \times 1.92}{5.67} = 9156$

$\therefore$ Shield temperature T_s

$= (9156)^{\frac{1}{4}} \times 100 = 978$ **K**

Comment : The temperature of the shield becomes much higher due to wrong installation.

EXAMPLE 8.40

The furnace of boiler is laid from fire clay brick with outside lagging from plate steel; the distance between the two is quite small compared with the size of the furnace. The brick setting is at an average temperature of 365 K whilst the steel lagging is at 290 K. Calculate the radiant heat flux. Assume the following emissivity values :

$\epsilon_{(brick)} = 0.85$ *and* $\epsilon_{(steel)} = 0.65$

(*b*) *What will be the reduction in heat loss if a steel screen having an emissivity value of 0.6 on both sides is placed between the brick and steel setting ? Also calculate the desired emissivity of screen if the radiation loss is to be limited to 100 W/m².*

Solution : Refer Fig. 8.35. The rate of heat interchange between the brick setting (suffix 1) and steel lagging (suffix 2) is

$$Q_{12} = (F_g)_{12} A_1 \sigma_b \left(T_1^4 - T_2^4\right)$$

The gray body factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

For infinite long parallel plates which see each other and nothing else,

$F_{12} = 1$ and $A_1 = A_2$

Then:

$$(F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{1}{\frac{1}{0.85} + \frac{1}{0.65} - 1} = 0.583$$

$\therefore \quad Q_{12} = 0.583 \times 1 \times (5.67 \times 10^{-8}) \times (365^4 - 290^4)$

$= 352.9$ **W/m²**

(*b*) Refer Fig. 8.36. When a screen (suffix 3) is inserted between the brick (suffix 1) and steel lagging (suffix 2).

$$(F_g)_{12} = \frac{1}{\left[\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{13}} + \frac{1-\epsilon_3}{\epsilon_3} \times \frac{A_1}{A_3}\right] + \left[\frac{1-\epsilon_3}{\epsilon_3} + \frac{1}{F_{32}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_3}{A_2}\right]}$$

For the given arrangement

$A_1 = A_2 = A_3$ and $F_{13} = F_{32} = 1$

Then:

$$(F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2}{\epsilon_3} - 2}$$

$$= \frac{1}{\frac{1}{0.85} + \frac{1}{0.65} + \frac{2}{0.6} - 2} = 0.247$$

$\therefore \quad Q_{12} = 0.247 \times 1 \times (5.67 \times 10^{-8}) \times (365^4 - 290^4)$

$= 149.51$ **W/m²**

The placement of radiation shield reduces the radiant heat transfer by a factor of

$$\frac{352.9}{149.51} = 2.36$$

If radiant heat loss is to be limited to 100 W/m², then

$$100 = (F_g)_{12} \times 1 \times (5.67 \times 10^{-8}) \times (365^4 - 290^4)$$

That gives :

$$(F_g)_{12} = 0.165 \quad \text{or} \quad \frac{1}{(F_g)_{12}} = 6.06$$

$$\therefore \quad \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2}{\epsilon_3} - 2 = 6.06$$

$$\text{or} \quad \frac{1}{0.85} + \frac{1}{0.65} + \frac{2}{\epsilon_3} - 2 = 6.06$$

$$\text{or } 1.176 + 1.538 + \frac{2}{\epsilon_3} - 2 = 6.06$$

$$\text{or} \quad \frac{2}{\epsilon_3} = 6.06 + 2 - 1.176 - 1.538 = 5.346$$

$\therefore$ Desired emissivity of the screen, ϵ_3

$$= \frac{2}{5.346} = 0.374$$

EXAMPLE 8.41

Consider radiative heat transfer between two large parallel planes of surface emissivities 0.8. How many thin radiation shields of emissivity 0.05 be placed between the surfaces to reduce the radiation heat transfer by a factor of 75 ?

Solution : The net radiation heat transfer between two parallel planes (called 1 and 2) with one radiation shield (called 3) inserted between them is,

$$(Q_{12})_{\text{one shield}} = \frac{A\sigma_b\left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1\right) + \left(\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1\right)}$$

$$= \frac{A\sigma_b\left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} + \frac{2}{\epsilon_3} - 2}$$

Generalising the above equation for a system of two parallel plates separated by n-radiation shields of emissivity $\epsilon_{s_1}, \epsilon_{s_2}, \ldots\ldots \epsilon_{s_n}$, we get

$$(Q_{12})_{n\text{-shields}} = \frac{A\sigma_b\left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + 2\sum_{i=1}^{n} \frac{1}{\epsilon_{s_i}} - (n+1)}$$

For $n = 0$ (no shield)

$$(Q_{12})_{\text{no-shield}} = \frac{A\sigma_b\left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

As per the given condition,

$$\frac{(Q_{12}) \text{ with no shield}}{(Q_{12}) \text{ with } n\text{-shields}} = 75$$

$$\therefore \quad \frac{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2n}{\epsilon_s} - (n+1)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} = 75$$

$$\text{or} \quad \frac{\frac{1}{0.8} + \frac{1}{0.8} + \frac{2n}{0.05} - (n+1)}{\frac{1}{0.8} + \frac{1}{0.8} - 1} = 75$$

$$\text{or} \quad \frac{1.25 + 1.25 + 40n - (n+1)}{1.25 + 1.25 - 1} = 75$$

$$\text{or} \quad \frac{39n + 1.5}{1.5} = 75$$

$$\therefore \quad n = \frac{(75 \times 1.5) - 1.5}{39} = 2.846$$

Hence three screens are required to reduce the heat exchange by a factor 75.

EXAMPLE 8.42

A 10 mm outside diameter pipe carries a cryogenic fluid at 100 K temperature. Another pipe of 13 mm outside diameter and at 280 K surrounds it coaxially and the space between the pipes is completely evacuated. Determine the radiant heat flow for 3 m length of pipe if the surface emissivity for both surfaces is 0.2. Proceed to calculate percentage reduction in heat flow if a shield of 11.5 mm diameter and 0.05 surface emissivity is placed between the pipes.

Solution : The rate of heat interchange between the inner (suffix 1) and outer (suffix 2) pipes is given by

$$Q_{12} = A_1 \sigma_b \left(T_1^4 - T_2^4\right)$$

The gray body factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

For the given configuration : $F_{12} = 1$ and therefore

$$Q_{12} = \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

For 3 m length of pipe, the pipe areas are

$A_1 = \pi d l = \pi \times 0.01 \times 3 = 0.0942$ m²

$A_2 = \pi \times 0.013 \times 3 = 0.1225$ m²

$$\frac{A_1}{A_2} = \frac{0.0942}{0.1225} = 0.769$$

$$\therefore Q_{12} = \frac{0.0942 \times \left(5.67 \times 10^{-8}\right) \times \left(100^4 - 280^4\right)}{\frac{1}{0.2} + \left(\frac{1}{0.2} - 1\right) \times 0.769}$$

$$= \frac{0.534\left(1^4 - 2.8^4\right)}{5 + 3.076} = -3.998 \text{ W}$$

The negative sign implies that heat flows from outer to inner pipe.

(*ii*) When a radiation shield (suffix s) is placed between the two pipes

$$(F_g)_{12} = \frac{1}{\left\{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{1s}} + \left(\frac{1}{\epsilon_s} - 1\right)\frac{A_1}{A_s}\right\} + \left(\frac{1}{\epsilon_s} - 1\right) + \frac{1}{F_{s2}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_s}{A_2}}$$

Here, $F_{1s} = F_{s2} = 1$, and therefore

$$(F_g)_{12} = \frac{1}{\frac{1}{\epsilon_1} + \left(\frac{1}{\epsilon_s} - 1\right)\frac{A_1}{A_s} + \frac{1}{\epsilon_s} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_s}{A_2}}$$

A_s (shield pipe area)

$$= \pi \times 0.0115 \times 3 = 0.1084 \text{ m}^2$$

$$\frac{A_1}{A_s} = \frac{0.0942}{0.1084} = 0.869 \; ;$$

$$\frac{A_s}{A_2} = \frac{0.1084}{0.1225} = 0.885$$

$$\therefore \quad Q_{12} = \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\frac{1}{\epsilon_1} + \left(\frac{1}{\epsilon_s} - 1\right)\frac{A_1}{A_s} + \frac{1}{\epsilon_s} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_s}{A_2}}$$

$$= \frac{0.0942 \times \left(5.67 \times 10^{-8}\right) \times \left(100^4 - 280^4\right)}{\frac{1}{0.2} + \left(\frac{1}{0.05} - 1\right) \times 0.869 + \frac{1}{0.05} + \left(\frac{1}{0.2} - 1\right) \times 0.885}$$

$$= \frac{0.534\left(1^4 - 2.8^4\right)}{5 + 16.51 + 20 + 3.54}$$

$$= -0.716 \text{ W}$$

Hence percentage raduction in heat flow

$$= \frac{3.998 - 0.716}{3.998} \times 100 = 82.09\%$$

8.6. ADIABATIC, RERADIATING SURFACES

Often encountered in engineering practice are the adiabatic walls (surfaces) which are thermally insulated so that they have no net gain or loss of thermal energy. Such surfaces interact radiately with other surfaces of an enclosure ; absorbing and reflecting incident radiation and subsequently remitting all the absorbed energy. An adiabatic surface (referred to as reradiating surface) attains an equilibrium temperature depending upon its absorptivity

and emissivity and the temperatures of the heat exchanging surfaces. Common examples of reradiating surfaces are the radiation shields and the refractory walls in a furnace. The refractory walls serve to reflect or absorb and reradiate energy from the fire.

Even though an adiabatic surface is non-conducting (as far as heat flow to it is concerned), it does not influence the heat flow between the other two surfaces because it absorbs and radiates energy to them.

8.6.1. Heat Exchange Between Two Black Surfaces Enclosed by an Insulated (Adiabatic) Surface

Consider an enclosure made of three black surfaces comprising :

(i) a heat source (furnace, combustion chamber or fuel bed in a boiler) maintained at a specific temperature T_1.

(ii) a heat sink (water tubes in a boiler) maintained at specific temperature T_2.

(iii) an adiabatic surface (refractory wall) which prevents the escape of heat. The net loss or gain of heat through a refractory wall is negligible. It radiates energy at the same rate at which it receives energy from the fuel bed and attains a temperatures T_3 which lies between T_1 and T_2. The net heat exchange between the fuel bed and the water tube is not only direct but also through the refractory wall.

The heat source (surface 1) exchanges heat with heat sink (surface 2) and the refractory wall (surface 3). The net radiative heat transfer from surface 1 may be written as

$$Q_1 = Q_{12} + Q_{13}$$
$$= A_1 F_{12} \sigma_b \left(T_1^4 - T_2^4\right) + A_1 F_{13} \sigma_b \left(T_1^4 - T_3^4\right)$$

Likewise

$$Q_2 = A_2 F_{21} \sigma_b \left(T_2^4 - T_1^4\right) + A_2 F_{23} \sigma_b \left(T_2^4 - T_3^4\right)$$
$$Q_3 = A_3 F_{31} \sigma_b \left(T_3^4 - T_1^4\right) + A_3 F_{32} \sigma_b \left(T_3^4 - T_2^4\right)$$

Since there is no heat loss through the refractory lining, $Q_3 = 0$ and that gives

$$T_3 = \left[\frac{F_{32} T_2^4 + F_{31} T_1^4}{F_{32} + F_{31}}\right]^{1/4}$$

Accordingly :

$$Q_1 = A_1 F_{12} \sigma_b \left(T_1^4 - T_2^4\right) + A_1 F_{13} \sigma_b \left[T_1^4 - \frac{F_{32} T_2^4 + F_{31} T_1^4}{F_{32} + F_{31}}\right]$$
$$= A_1 F_{12} \sigma_b \left(T_1^4 - T_2^4\right) + A_1 F_{13} \sigma_b \left[\frac{F_{32} T_1^4 - F_{31} T_3^4}{F_{32} + F_{31}}\right]$$

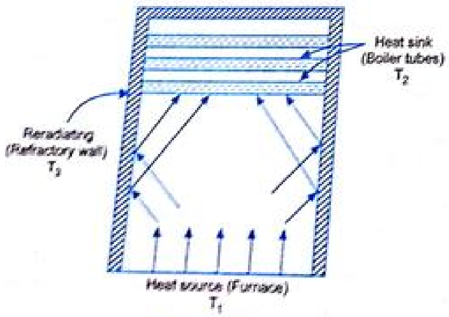


Fig. 8.39.

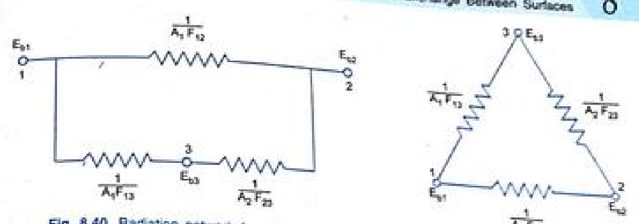


Fig. 8.40. Radiation network for two black surfaces connected by a reradiating surface

$$= \sigma_b \left(T_1^4 - T_2^4\right)\left[A_1 F_{12} + \frac{A_1 F_{13} F_{32}}{F_{32} + F_{31}}\right] \quad ...(8.34)$$

From reciprocity theorem

$A_3 F_{31} = A_1 F_{13}$ and $A_3 F_{32} = A_2 F_{23}$

The equation 8.34 may then be rewritten as

$$Q_1 = \sigma_b \left(T_1^4 - T_2^4\right) \times \left[A_1 F_{12} + \frac{1}{\frac{1}{A_1 F_{13}} + \frac{1}{A_2 F_{23}}}\right] \quad ...(8.35)$$

The radiant network for the surface 1 and 2 exchanging radiation vide a third reradiating surface 3 is shown in Fig. 8.40. The emissive power E_{b3} of the reradiating surface floats between E_{b1} and E_{b2}, and its value depends upon the magnitude of E_{b1} and E_{b2} and the shape resistances between E_{b3} and E_{b1}, E_{b2}. Obviously the system can be considered as series-parallel circuit.

The total thermal resistance R_t of the circuit is

$$\frac{1}{R_t} = \frac{1}{\frac{1}{A_1 F_{12}}} + \frac{1}{\frac{1}{A_1 F_{13}} + \frac{1}{A_2 F_{23}}}$$
$$= A_1 F_{12} + \frac{1}{\frac{1}{A_1 F_{13}} + \frac{1}{A_2 F_{23}}}$$

Now,

$$Q_1 = \frac{E_{b1} - E_{b2}}{R_t}$$
$$= \sigma_b \left(T_1^4 - T_2^4\right)\left[A_1 F_{12} + \frac{1}{\frac{1}{A_1 F_{13}} + \frac{1}{A_2 F_{23}}}\right]$$

which is same as equation 8.35 derived above.

The term $\left[\frac{1}{\frac{1}{A_1 F_{13}} + \frac{1}{A_2 F_{23}}}\right]$ is due to reradiating surface and could be an appreciable addition to net radiation.

Since only two surfaces are involved, the net radiant heat exchange rate is equal in magnitude to the net radiative heat transfer from either surface. Thus

$$Q_{12} = Q_1 = -Q_2$$

The net radiative heat transfer Q_{12} as prescribed by equation 8.35 could be written more simply as

$$Q_{12} = A_1 \bar{F}_{12} \sigma_b \left(T_1^4 - T_2^4\right) \quad ...(8.36)$$

where $\bar{F}_{12}$ is a **modified shape factor** which includes the effect of a reradiating surface between two black surfaces which are exchanging radiant energy. Comparison between identities 8.35 and 8.36 yields :

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$$A_1 \bar{F}_{12} = A_1 F_{12} + \frac{1}{\dfrac{1}{A_1 F_{13}} + \dfrac{1}{A_2 F_{23}}} \quad ...(8.37)$$

Since surfaces 1 and 2 are not concave, their shape factors with respect to themselves will be zero. That is

$$F_{12} = 0 \quad \text{and} \quad F_{22} = 0$$

Then by shape factor algebra

$$F_{12} + F_{13} = 1 \quad \text{and} \quad F_{21} + F_{23} = 1$$

Upon substitution of these identities into equation 8.37,

$$A_1 \bar{F}_{12} = A_1 F_{12} + \frac{1}{\dfrac{1}{A_1 (1 - F_{12})} + \dfrac{1}{A_2 (1 - F_{21})}}$$

Using reciprocity relation $A_1 F_{12} = A_2 F_{21}$ and simplifying.

$$A_1 \bar{F}_{12} = \frac{A_1 \left(A_2 - A_1 F_{12}^2\right)}{A_1 + A_2 - 2 A_1 F_{12}}$$

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2 A_1 F_{12}} \quad ...(8.38)$$

The net radiative heat transfer Q_{12} may then be rewritten as

$$Q_{12} = A_1 \bar{F}_{12} \sigma_b \left(T_1^4 - T_2^4\right)$$

$$= A_1 \left[\frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2 A_1 F_{12}}\right] \sigma_b \left(T_1^4 - T_2^4\right) \quad ...(8.39)$$

It is worthwhile to remark that the above analysis can be readily extended to an enclosure having more than three surfaces.

8.6.2. Heat Exchange Between Two Gray Surfaces Enclosed by an Insulated (Adiabatic Surface)

The radiation network for heat exchange between two gray surfaces enclosed by an insulated surface is shown in Fig. 8.41. It is to be noted that for the insulated surface $E_{b_3} = J_3$ as this surface does not exchange any heat energy.

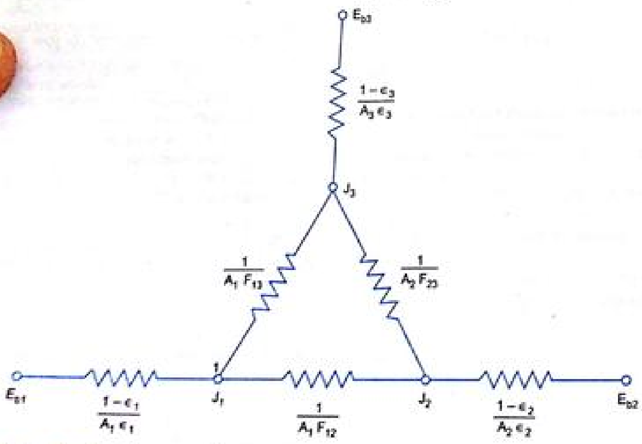

Fig. 8.41. Radiation network for two gray surfaces connected by a raradiating surface

The total thermal resistance R_t between nodes E_{b1} and E_{b2} is given by

$$R_t = \frac{1-\epsilon}{A_1 \epsilon_1} + \left[\frac{1}{\dfrac{1}{A_1 F_{12}} + \left\{\dfrac{1}{\left(\dfrac{1}{A_1 F_{12}} + \dfrac{1}{A_2 F_{23}}\right)}\right\}}\right] + \frac{1-\epsilon_2}{A_2 \epsilon_2}$$

With surfaces 1 and 2 as convex or flat,

$$F_{11} = 0 \quad \text{and} \quad F_{22} = 0$$

Then by shape factor algebra

$$F_{12} + F_{13} = 1 \quad \text{or} \quad F_{13} = 1 - F_{12}$$

$$F_{21} + F_{23} = 1 \quad \text{or} \quad F_{23} = 1 - F_{21}$$

With substitution of these identities, invoking reciprocity theorem $A_1 F_{12} = A_2 F_{21}$ and upon simplification, the middle term of above relation takes the form

$$= \frac{1}{A_1}\left(\frac{A_1 + A_2 - 2A_1 F_{12}}{A_2 - A_1 F_{12}^2}\right)$$

$$= \frac{1}{A_1 \bar{F}_{12}} \quad \text{(refer equation 8.38)}$$

$$\therefore \quad R_t = \frac{1-\epsilon_1}{A_1 \epsilon_1} + \frac{1}{A_1}\left(\frac{A_1 + A_2 - 2A_1 F_{12}}{A_2 - A_1 F_{12}^2}\right) + \frac{1-\epsilon_2}{A_2 \epsilon_2}$$

$$= \frac{1-\epsilon_1}{A_1 \epsilon_1} + \frac{1}{A_1 \bar{F}_{12}} + \frac{1-\epsilon_1}{A_2 \epsilon_2} \quad ...(8.40)$$

and as such

$$Q_{12} = \frac{E_{b1} - E_{b2}}{R_t}$$

$$= \frac{\sigma_b \left(T_1^4 - T_2^4\right)}{\dfrac{1-\epsilon_1}{A_1 \epsilon_1} + \left\{\dfrac{1}{A_1}\left(A_1 + A_2 - \dfrac{2 A_1 F_{12}}{A_2} - A_1 F_{12}^2\right)\right\} + \dfrac{1-\epsilon_2}{A_2 \epsilon_2}}$$

$$= \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\dfrac{1-\epsilon_1}{\epsilon_1} + \left[\left(A_1 + A_2 - \dfrac{2 A_1 F_{12}}{A_2 - A_1 F_{12}^2}\right)\right] + \dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}}$$

$$= \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\dfrac{1-\epsilon_1}{\epsilon_1} + \dfrac{1}{\bar{F}_{12}} + \dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}} \quad ...(8.41)$$

If $A_1 = A_2 = A$, then

$$\bar{F}_{12} = \frac{1 + F_{12}}{2}$$

$$Q_{12} = \frac{A \sigma_b \left(T_1^4 - T_2^4\right)}{\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 2 + \dfrac{2}{1 + F_{12}}} \quad ...(8.42)$$

8.6.3. Radiation Heat Exchange for Three Gray Surfaces

The radiation network for three gray surfaces which see each other and nothing else is shown in Fig. 8.42. Each surface exchanges heat with the other two surfaces.

The sum of energy exchange at each of the interior nodes are :

Node J_1 :

$$\frac{E_{b1} - J_1}{\dfrac{1-\epsilon_1}{A_1 \epsilon_1}} = \frac{J_1 - J_2}{\dfrac{1}{A_1 F_{12}}} + \frac{J_1 - J_3}{\dfrac{1}{A_1 F_{13}}}$$

Node J_2 :

$$\frac{E_{b2} - J_2}{\dfrac{1-\epsilon_2}{A_2 \epsilon_2}} = \frac{J_2 - J_1}{\dfrac{1}{A_1 F_{12}}} + \frac{J_2 - J_3}{\dfrac{1}{A_2 F_{23}}}$$

Node J_3 :

$$\frac{E_{b3} - J_3}{\dfrac{1-\epsilon_1}{A_3 \epsilon_3}} = \frac{J_3 - J_1}{\dfrac{1}{A_1 F_{13}}} + \frac{J_3 - J_2}{\dfrac{1}{A_2 F_{23}}}$$

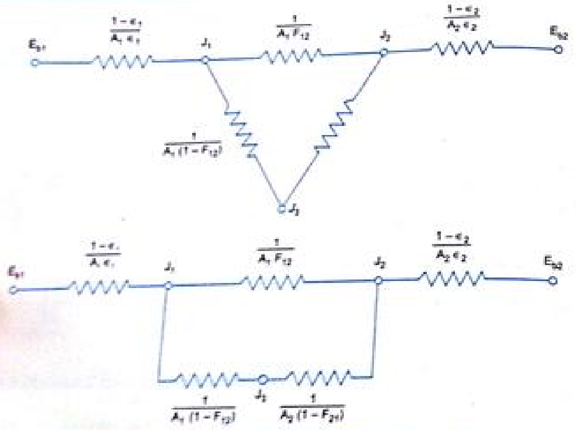

Fig. 8.42 Radiation heat exchange for three grey surfaces

The above identities are solved simultaneously for the three unknowns J_1, J_2 and J_3. Subsequently computations are made for the net heat transfer to or from any body (say a) as a result of all surrounding bodies (i) :

$$Q_{net} = \sum_i \frac{J_i - J_a}{\frac{1}{A_a F_{ai}}} \qquad \text{...(8.43)}$$

EXAMPLE 8.43

An enclosure consists of a rectangular parallel piped 1 m × 2 m × 4 m. One of the 1 m × 2 m surfaces acts as a black surface at 475 K and the other acts as black surface at 375 K. If the other four surfaces of the enclosure act as reradiating surfaces, find the equilibrium temperature of the reradiating surfaces and the net radiative heat transfer between the two active surfaces.

How would this heat exchange be affected if the 475 K rectangle is gray with emissivity = 0.6 and the 375 K rectangle is gray with emissivity 0.8 ? All other data remain the same.

Solution : The net radiative heat transfer between two black surfaces connected by a reradiating surface is given by

$$Q_{12} = A_1 \bar{F}_{12} \sigma_b \left(T_1^4 - T_2^4\right)$$

where the modified shape factor $\bar{F}_{12}$ is

$$= \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}$$

With $A_1 = A_2$

$$\bar{F}_{12} = \frac{1 - F_{12}^2}{2(1 - F_{12})}$$

From Fig 8.2 with $X/L = 2/4 = 0.5$ and $Y/Z = 1/4 = 0.25$; $F_{12} = 0.035$

$$\therefore \quad \bar{F}_{12} = \frac{1 - 0.035^2}{2(1 - 0.035)} = 0.517$$

and $Q_{12} = (1 \times 2) \times 0.517 \times (5.67 \times 10^{-8}) \times (475^4 - 375^4)$

$= \mathbf{1825.15\ W}$

The equilibrium temperature of the reradiating (adiabatic) walls may be worked from the fact that there is no heat loss though the reradiating surfaces. That is

$$Q_3 = A_3 F_{31} \sigma_b \left(T_3^4 - T_1^4\right) + A_3 F_{32} \sigma_b \left(T_3^4 - T_2^4\right) = 0$$

or $A_1 F_{13} \sigma_b \left(T_1^4 - T_3^4\right)$

$$= A_3 F_{32} \sigma_b \left(T_3^4 - T_2^4\right)$$

$$= A_2 F_{23} \sigma_b \left(T_3^4 - T_2^4\right)$$

$$= A_2 (1 - F_{21}) \sigma_b \left(T_3^4 - T_2^4\right) \qquad \text{...}(\because F_{21} + F_{23} = 1)$$

$$= (A_2 - A_1 F_{12}) \sigma_b \left(T_3^4 - T_2^4\right) \qquad \text{...}(\because A_2F_{21} = A_1F_{12} = 1)$$

$$\therefore \frac{T_1^4 - T_3^4}{T_3^4 - T_2^4} = \frac{(A_2 / A_1) - F_{12}}{1 - F_{12}}$$

With $A_1 = A_2$,

$$\frac{T_1^4 - T_3^4}{T_3^4 - T_2^4} = \frac{1 - F_{12}}{1 - F_{12}} = 1$$

or $$T_3^4 = \frac{T_1^4 + T_2^4}{2}$$

$$= \frac{475^4 + 375^4}{2} \text{ or } T_3 = \mathbf{433.6\ K}$$

(*b*) For two gray surfaces connected by a reradiating surfaces

$$Q_{12} = \frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{\bar{F}_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}$$

With $A_1 = A_2 = A$,

$$\bar{F}_{12} = \frac{1 + F_{12}}{2}$$

$$Q_{12} = \frac{A \sigma_b \left(T_1^4 - T_2^4\right)}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right) - 2 + \frac{2}{1 + F_{12}}}$$

Substituting the relevant values, we get

$$Q_{12} = \frac{(1 \times 2) \times \left(5.67 \times 10^{-8}\right) \times \left(475^4 - 375^4\right)}{\left(\frac{1}{0.6} + \frac{1}{0.8}\right) - 2 + \frac{2}{1 + 0.035}}$$

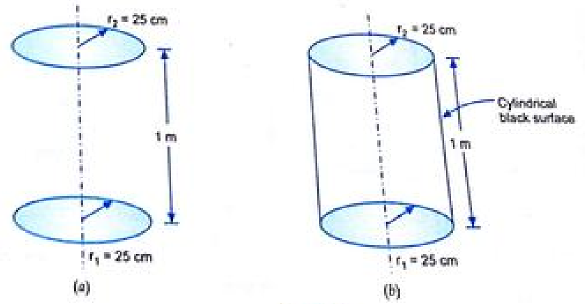

Fig. 8.43.

$$= \frac{2 \times 5.67 \times 10^{-8} \times (475^4 - 375^4)}{(1.47 + 1.25) - 2 \times 1.93}$$

$$= \frac{2 \times 5.67 \times (509.06 - 197.75)}{2.85}$$

$= 1238.68$ W

EXAMPLE 8.44

Two black discs each of diameter 50 cm are placed parallel to each other concentrically at a distance of one meter. The discs are maintained at 1000 K and 500 K respectively. Calculate the heat flow between the discs

(i) when no other surface is present,

(ii) when the disks are connected by a cylindrical black no flux surface.

Solution : Case (i) The heat flow between two discs in the absence of third body is

$$Q_{12} = (F_g)_{12}\, A_1\, \sigma_b \left(T_1^4 - T_2^4\right)$$

The gray body factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

For the given configuration ;

$\epsilon_1 = \epsilon_2 = 1$ and $A_1 = A_2$ and so $(F_g)_{12} = F_{12}$

Corresponding to $\frac{L}{r_1} = \frac{1}{0.25} = 4$ and $\frac{r_2}{L} = \frac{0.25}{1} = 0.25$, the shape factor F_{12} is read as 0.04.

$$\therefore \quad Q_{12} = 0.04 \times \pi\,(0.25)^2 \times 5.67 \times 10^{-8}\,(1000^4 - 500^4)$$

$= 417.27$ W

Case (ii) : The heat transfer between the discs when these are connected by a refractory surface is

$$Q_{12} = (F_g)_{12}\, A_1\, \sigma_b \left(T_1^4 - T_2^4\right)$$

where the factor $(F_g)_{12}$ is

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2} + \frac{1}{\bar{F}_{12}}}$$

and $\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2 A_1 F_{12}}$

For the given configuration,

$A_1 = A_2 = \pi\,(0.25^2)$
$= 0.196$ m^2 and $\epsilon_1 = \epsilon_2 = 1$

That gives

$$\bar{F}_{12} = \frac{A_1\left(1 - F_{12}^2\right)}{2 A_1 (1 - F_{12})}$$

$$= \frac{1 + F_{12}}{2} = \frac{1 + 0.04}{2} = 0.52$$

and $(F_g)_{12} = \bar{F}_{12} = 0.52$

$$\therefore \quad Q_{12} = 0.52 \times 0.196 \times 5.67 \times 10^{-8}\,(1000^4 - 500^4)$$

$= 5417.68$ W

EXAMPLE 8.45

A square room 4 m × 4 m and height 3.0 m has all its walls perfectly insulated. The floor and ceiling are maintained at 300 K and 280 K respectively. Assuming an emissivity value 0.75 for all the surfaces, make calculations for the wall temperature and net heat interchange between the floor and ceiling.

Take floor to ceiling shape factor as 0.28

Solution : From the configuration of the room depicted in Fig. 8.44, it is apparent that

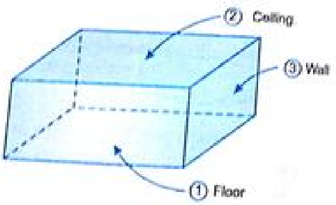

Fig. 8.44.

$F_{11} = F_{22} = 0$

$F_{12} = F_{21}$
$= 0.28$ because of symmetry

Further;

$F_{11} + F_{12} + F_{13} = 1$

or $F_{13} = 1 - F_{11} - F_{12}$
$= 1 - 0 - 0.25 = 0.75$

and $F_{21} + F_{22} + F_{23} = 1$

or $F_{23} = 1 - F_{21} - F_{22}$
$= 1 - 0.25 - 0 = 0.75$

The heat flow from the ceiling to the floor is given by

$$Q_{12} = (F_g)_{12}\, A_1\, \sigma_b \left(T_1^4 - T_2^4\right)$$

The gray body factor is

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{\epsilon_1} - 1\right) + \left(\frac{1}{\epsilon_1} - 1\right)\frac{A_1}{A_2} + \frac{1}{\bar{F}_{12}}}$$

and $\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2 A_1 F_{12}}$

For the given configuration

$A_1 = A_2 = 16$ m^2

and $\epsilon_1 = \epsilon_2 = 0.75$

That gives

$$\bar{F}_{12} = \frac{A_1 - \left(A_1 F_{12}^2\right)}{2 A_1 (1 - F_{12})}$$

$$= \frac{1 + F_{12}}{2} = \frac{1 + 0.28}{2} = 0.64$$

$$(F_g)_{12} = \frac{1}{\left(\frac{1}{0.75} - 1\right) + \left(\frac{1}{0.75} - 1\right) + \frac{1}{0.64}}$$

$$= \frac{1}{0.333 + 0.333 + 1.562} = 0.449$$

$$\therefore \quad Q_{12} = 0.449 \times 16 \times 5.67 \times 10^{-8}\,(300^4 - 280^4)$$

$= 795.7$ W

Obviously the heat is gained by the ceiling

The wall temperature is obtained by using the relation

$$T_3^4 = \frac{1}{2}\left(T_1^4 + T_2^4\right) = \frac{1}{2}\left[300^4 + 280^4\right]$$

$= 71.23 \times 10^8$

$$\therefore \quad T_3 = \left(71.23 \times 10^8\right)^{1/4} = 290.5 \text{ K}$$

EXAMPLE 8.46

Consider two surfaces which lie parallel to each other with distance 5 cm between them. The surface 1 is a circular plane of radius 20 cm, emissivity 0.8 and is maintained at 2000 K temperature. The surface 2 is also a circular plane of same radius but has emissivity 0.5 and temperature 500 K. A third surface which is a reradiating surface joins these surfaces to form a short cylinderical enclosure. Make calculations for the net radiant heat transfer between surfaces 1 and 2.

The following relation may be used to find the shape factor between two circular, coaxial and parallel planes :

$$F_{12} = \frac{1}{2B^2}\left[X - \sqrt{X^2 - 4B^2C^2}\right]$$

where $B = \frac{R_1}{H}$; $C = \frac{R_2}{H}$ *and* $X = 1 + B^2 + C^2$.

Here R_1 and R_2 are radii of circular plates and H is the distance between them.

Solution : Refer Fig. 8.45 for the configuration and the nomenclature

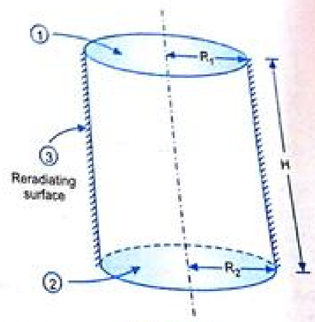

Fig. 8.45.

$B = C = \frac{20}{5} = 4$

$X = 1 + B^2 + C^2 = 1 + 4^2 + 4^2 = 3$

$$\bar{F}_{12} = \frac{1}{2R^2}\left[X - \sqrt{X^2 - 4R^2C^2}\right]$$

$$= \frac{1}{2 \times 4^2}\left[[illegible] - \sqrt{[illegible]}\right]$$

$$= 0.79$$

The net radiative heat transfer Q_{12} can be worked out from the relation

$$Q_{12} = \frac{A_1 \sigma_b (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{\bar{F}_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

where $\bar{F}_{12}$ is a modified shape factor which includes the effect of a reradiating surfaces between two surfaces which are exchanging heat:

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}$$

$$= \frac{1 - F_{12}^2}{2(1 - F_{12})} \quad (\because A_1 = A_2)$$

$$= \frac{1 - 0.79^2}{2(1 - 0.79)} = \frac{0.3958}{0.44} = 0.89$$

$\therefore Q_{12} =$

$$\frac{(\pi \times 0.2)^2 \times 5.67 \times 10^{-8} \times (2000^4 - 500^4)}{\left(\frac{1}{0.8} - 1\right) + \frac{1}{0.89} + \left(\frac{1}{0.5} - 1\right) \times \left(\frac{\pi \times 0.2^2}{\pi \times 0.2^2}\right)}$$

$$= \frac{113499}{0.25 + 1.123 + 1} = \frac{113499}{2.373}$$

$$= 47829 \text{ W}$$

EXAMPLE 8.47

A blind cylindrical hole of 2 cm diameter and 3 cm length is drilled into a metal slab having emissivity 0.7. If the metal slab is maintained at 650 K, make calculations for the radiation heat escape from the hole.

Solution : The shape factor of a cavity with itself is given by

$$F_{11} = \frac{4h}{4h + d} = \frac{4 \times 3}{4 \times 3 + 2} = 0.857$$

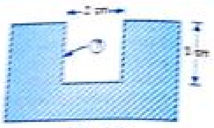

Fig. 8.46.

The heat loss from a cylindrical hole is prescribed by the relation

$$Q = \epsilon_1 A_1 \sigma_b T_1^4 \left[\frac{1 - F_{11}}{1 - (1 - \epsilon_1) F_{11}}\right]$$

Here $A_1 = \pi d h + \frac{\pi}{4} d^2$

$$= \pi \times 0.02 \times 0.03 + \frac{\pi}{4}(0.02)^2$$

$$= 0.001884 + 0.000314$$

$$= 0.0022 \text{ m}^2$$

$$\therefore \quad Q = 0.7 \times 0.0022 \times 5.67 \times 10^{-8}$$

$$\times 650^4 \left[\frac{1 - 0.857}{1 - (1 - 0.7) \times 0.857}\right]$$

$$= 15.287 \times \left[\frac{0.143}{0.7429}\right] = 3 \text{ W}$$

EXAMPLE 8.48

A cavity in the shape of a frustrum of a cone has diameters 30 cm and 60 cm and the height is 80 cm. If the cavity is maintained at temperatures of 800 K, determine the heat loss from the cavity when the smaller diameter is at the bottom. How this heat loss would be affected if the cavity is positioned with bigger diameter at the base. Assume the cavity to behave as a black body.

Solution : Refer Fig. 8.47 for the cavity configuration and nomenclature.

The heat flow from the cavity to the surroundings is given by

$$Q = A_1 \epsilon_1 \sigma_b T_1^4 \left[\frac{1 - F_{11}}{1 - (1 - \epsilon_1) F_{11}}\right] \quad ...(i)$$

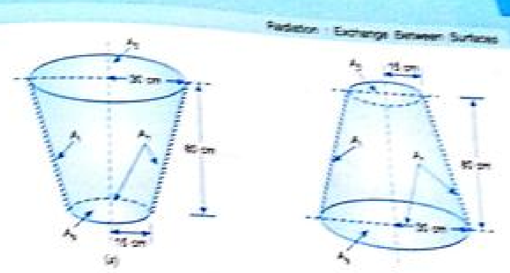

Fig. 8.47.

where A_1 = area of bottom surface + area of inclined or curved surface

$$= \pi r_1^2 + \pi (r_1 + r_2)\sqrt{(r_2 - r_1)^2 + h^2}$$

$$= \pi (0.15)^2 + \pi (0.15 + 0.3)$$

$$\times \sqrt{(0.3 - 0.15)^2 + 0.8^2}$$

$$= 0.07065 + 1.15 = 1.2206 \text{ m}^2$$

and $F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{\pi \times 0.3^2}{1.2206} = 0.768$

Substituting the relevant data in expression (*i*), we get

$$Q = 1.2206 \times 1 \times 5.67 \times 10^{-8}$$

$$\times (800)^4 \left[\frac{1 - 0.768}{1 - (1 - 1) \times 0.768}\right]$$

$$= 6577 \text{ W}$$

(*ii*) $A_1 = \pi \times (0.3)^2 + \pi (0.15 + 0.3)$

$$+ \sqrt{(0.3 - 0.15)^2 + 0.8^2}$$

$$= 0.2826 + 1.15 = 1.4326 \text{ m}^2$$

$$A_2 = \pi \times (0.15)^2 = 0.07065 \text{ m}^2$$

$$F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{0.07065}{1.4326} = 0.9507$$

$$\therefore \quad Q = 1.4326 \times 1 \times 5.67 \times 10^{-8}$$

$$\times (800)^4 \left[\frac{1 - 0.9507}{1 - (1 - 1) \times 0.9507}\right]$$

$$= 1640 \text{ W}$$

Percentage change in heat flow,

$$= \frac{6577 - 1640}{6577} \times 100$$

$$= 75.06\% \text{ (decrease)}$$

EXAMPLE 8.49

For a cavity of configuration depicted in the adjoining figure, calculate the rate of radiant heat emission from the cavity to the surroundings. The cavity has a surface temperature of 1200 K and its emissivity is 0.6.

Solution : The heat flow from the cavity to the surroundings is given by

Fig. 8.48.

$$Q = A_1 \epsilon_1 \sigma_b T_1^4 \left[\frac{1-F_{11}}{1-(1-\epsilon_1)F_{11}}\right] \quad ...(i)$$

where A_1 = area of cavity
= area of frustrum + area of half sphere

$$= \pi(r_1 + r_2)\sqrt{(r_2 + r_1)^2 + h^2} + \frac{1}{2} \times 4\pi r^2$$

$$= \pi(2.5 + 5)\sqrt{(5 - 2.5)^2 + 4^2} + \frac{1}{2} \times 4\pi \times 5^2$$

$$= 268 \text{ cm}^2 = 0.0268 \text{ m}^2$$

$$F_{11} = 1 - \frac{A_2}{A_1}$$

$$= 1 - \frac{\text{area of opening of cavity}}{\text{area of cavity}}$$

$$= 1 - \frac{\frac{\pi}{4}(0.05)^2}{0.0268} = 0.927$$

Substituting the relevant data in expression (i), we get

$$Q = 0.0268 \times 0.6 \times (5.67 \times 10^{-8}) \times (1200)^4 \left[\frac{1-0.927}{1-(1-0.6)\times 0.927}\right]$$

$$= 1890.57 \times \left[\frac{0.073}{1-0.3708}\right]$$

$$= \mathbf{219.34\ W}$$

EXAMPLE 8.50

A thin copper sphere of 25 cm diameter has its internal surface highly oxidised. What smallest size of lid be made in the sphere to give an opening that will have an absorptivity of 92 percent ?

Solution : Refer Fig. 8.49 for the configuration of the arrangement

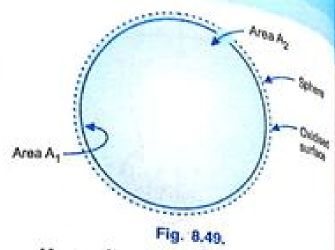

Fig. 8.49.

Heat radiated from the oxidised surface,

$$Q = \epsilon \sigma_b A_1 T_1^4$$

Heat leaving the hole,

$$Q' = \epsilon \sigma_b A_1 T_1^4 \left[\frac{1-F_{11}}{1-(1-\epsilon_1)F_{11}}\right]$$

Fraction of heat leaving from the hole

$$\frac{Q'}{Q} = \frac{1-F_{11}}{1-(1-\epsilon_1)F_{11}}$$

$$= (1 - 0.92) = 0.08 \text{ (Given)}$$

Further assuming that emissivity equals the absorptivity,

$$\epsilon_1 = \alpha = 0.92$$

$$\therefore \quad \frac{1-F_{11}}{1-(1-0.92)F_{11}} = 0.08$$

or $1 - F_{11} = 0.08\,[1 - 0.08\, F_{11}]$

$$= 0.08 - 0.0064\, F_{11}$$

$$\therefore \quad F_{11} = \frac{1-0.08}{1-0.0064} = 0.926$$

Now, $F_{11} = 1 - \frac{A_2}{A_1}$, $\quad \therefore \quad 0.926 = 1 - \frac{A_2}{A_1}$

$$\frac{A_2}{A_1} = 0.074\ ; \quad \frac{A_2}{A_s - A_2} = 0.074$$

where A_s = complete surface area

$$= 4\pi \times (12.5)^2 = 1962.5 \text{ cm}^2$$

$$\therefore \quad \frac{A_2}{1962.5 - A_2} = 0.074\ ;$$

$$A_2 = 145.22 - 0.074\, A_2$$

or Area of lid A_2

$$= \frac{145.22}{1.074} = \mathbf{135.2\ cm^2}$$

EXAMPLE 8.51

A conical cavity of base diameter 15 cm and height 20 cm has inside surface temperature 650 K. If emissivity of each surface is 0.85, determine the net radiative heat transfer from the cavity.

How this heat transfer would be affected if this conical cavity is replaced by a cylindrical cavity of 15 cm diameter and 20 cm height ?

Solution : The net radiative heat transfer from a cavity is prescribed by the relation :

$$Q_1 = \epsilon_1 \sigma_b A_1 \left[\frac{1-F_{11}}{1-(1-\epsilon_1)F_{11}}\right] T_1^4$$

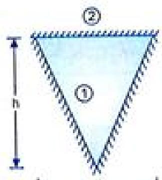

Fig. 8.50.

(i) For a conical cavity,

$$F_{11} = 1 - \frac{d}{\sqrt{4h^2 + d^2}}$$

$$= 1 - \frac{0.15}{\sqrt{4 \times 0.2^2 + 0.15^2}}$$

$$= 1 - \frac{0.15}{0.4272} = 0.649$$

$$A_1 = \frac{\pi d \times \text{slant theight}}{2}$$

$$= \frac{\pi d}{2}\sqrt{h^2 + (d/2)^2}$$

$$= \frac{\pi \times 0.15}{2}\sqrt{0.2^2 + (0.15/2)^2}$$

$$= 0.0503 \text{ m}^2$$

$$\frac{1-F_{11}}{1-(1-\epsilon_1)F_{11}} = \frac{1-0.649}{1-(1-0.85)\times 0.649}$$

$$= \frac{0.351}{0.9026} = 0.389$$

$$\therefore \quad Q_1 = 0.85 \times (5.67 \times 10^{-8}) \times 0.0503 \times 0.389 \times (650)^4$$

$$= \mathbf{168.3\ W}$$

(ii) For a cylindrical cavity

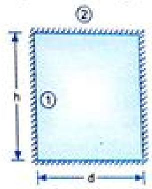

Fig. 8.51.

$$F_{11} = \frac{4h}{4h+d} = \frac{4 \times 0.2}{4 \times 0.2 + 0.15}$$

$$= 0.842$$

$$A_1 = \frac{\pi}{4}d^2 + \pi d h$$

$$= \frac{\pi}{4}(0.15)^2 \times \pi \times 0.15 \times 0.2$$

$$= 0.11186 \text{ m}^2$$

$$\frac{1-F_{11}}{1-(1-\epsilon_1)F_{11}} = \frac{1-0.842}{1-(1-0.85)\times 0.842}$$

$$= \frac{0.158}{0.8737} = 0.1808$$

$$\therefore \quad Q_2 = 0.85 \times (5.67 \times 10^{-8}) \times 0.11186 \times 0.1808 \times (650)^4$$

$$= \mathbf{174\ W}$$

Percentage change in heat transfer

$$= \frac{174 - 168.3}{168.3} \times 100$$

$$= \mathbf{3.387\%\ (increase)}$$

EXAMPLE 8.52

A prismatic cavity with square base of 5 cm side and 10 cm depth exists in a very large flat surface.

Make calculations for the percentage of heat lost by the cavity to the surroundings.

Solution : Refer Fig. 8.52 for the configuration of prismatic cavity.

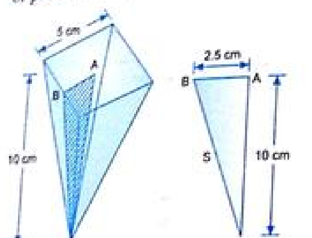

Fig. 8.52.

Slant height of the cavity, S

$$= \sqrt{2.5^2 + 10^2} = 10.31 \text{ cm}$$

Area of cavity A = area of 4-triangles

$$= 4 \times \left(\frac{1}{2} \times 5 \times 10.31\right) = 103.1 \text{ cm}^2$$

Heat radiated from the cavity, Q

$$= \epsilon \sigma_b A T^4$$

Heat leaving the cavity,

$$Q' = \epsilon \sigma_b A T^4 \left[\frac{1 - F_{11}}{1 - (1 - \epsilon_1) F_{11}}\right]$$

∴ Fraction of heat lost from the cavity

$$= \frac{Q'}{Q} = \frac{1 - F_{12}}{1 - (1 - \epsilon_1) F_{12}}$$

Now,

$$F_{11} = 1 - \frac{A_2}{A_1} \text{ where}$$

$$\frac{A_2}{A_1} = \frac{\text{top surface of cavity}}{\text{surface area of cavity}} = \frac{5 \times 5}{103.1} = 0.2425$$

$$F_{11} = 1 - 0.2425 = 0.7575$$

Let it be presumed that surface of cavity has a black body behaviour, *i.e.* $\epsilon_1 = 1$. Then

$$\frac{Q'}{Q} = 1 - F_{11} = 1 - 0.7575 = 0.2425 \text{ or } 24.25\%$$

That is 24.25 percent heat is lost by the cavity to the surroundings.

8.7. COEFFICIENT OF RADIANT HEAT TRANSFER AND RADIATION COMBINED WITH CONVECTION

The radiative heat exchange between two systems (sufaces) is generally calculated from the simplified equation

$$Q = h_r A (T_1 - T_2)$$

rather than from the relation

$$Q = f_{12} F_{12} A \sigma_b (T_1^4 - T_2^4)$$

Comparing these identities

$$h_r = f_{12} F_{12} \sigma_b \frac{(T_1^4 - T_2^4)}{T_1 - T_2}$$

$$= f_{12} F_{12} \sigma_b (T_1 + T_2)(T_1^2 + T_2^2) \quad ...(8.44)$$

The factor h_r is called the ***coefficient of radiant heat transfer*** from solid to solid and is expressed in W/m²-deg temperature difference between the enclosed and enclosing surfaces.

The value of h_r can be calculated from the heat flux equation for any configuration. For example, the value of h_r for the case of two large parallel plates would be

$$\frac{Q}{A} = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} = h_r (T_1 - T_2)$$

$$\text{or} \quad h_r = \frac{\sigma (T_1 + T_2)(T_1^2 + T_2^2)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} \quad ...(8.45)$$

There occurs simultaneous heat exchange due to radiation and convection in many situations of engineering importance such as :

(*i*) the heat loss from a hot steam pipe passing through a room,

(*ii*) the heat loss from hot combustion products when they pass through a cooled duct.

The total heat transfer by both convection and radiation is then obtained by the linear superposition of heat fluxes due to these modes. That is :

$$q = q_c + q_r$$

For a hot gas at temperature t_g passing through a duct with wall temperature t_w, we may write

$$q = h_c (t_g - t_w) + h_r (t_g - t_w)$$
$$= (h_c + h_r)(t_g - t_w) \quad ...(8.46)$$

The radiation heat transfer coefficient is a strong function of temperature in contrast to convective heat transfer coefficient.

EXAMPLE 8.53

A pipe with a surface temperature of 480 K is kept within a large enclosure whose walls are at 380 K. Presuming the pipe surface to be black, calculate the coefficient of radiant heat transfer. If the heat transfer coefficient including the effect of radiation and convection is 34.9 W/m²-deg, find the convective heat transfer coefficient.

Solution : The rate of radiant interchange between the pipe and the walls of the enclosure is

$$Q = \sigma_b A (T_1^4 - T_2^4)$$
$$= 5.67 \times 10^{-8} \times 1 \times (480^4 - 380^4)$$
$$= 1827.6 \text{ W}$$

If h_r denotes the coefficient of radiant heat transfer, then

$$Q = h_r A (T_1 - T_2)$$
$$1827.6 = h_r \times 1 \times (480 - 380)$$

$$\text{or} \quad h_r = \frac{1827.6}{100} = \mathbf{18.28 \ W/m^2\text{-deg}}$$

Further,

$$h_{total} = h_{rad} + h_{con}$$

$$\therefore \quad h_{con} = 34.9 - 18.28 = \mathbf{16.62 \ W/m^2\text{-deg}}$$

EXAMPLE 8.54

A hot water radiator of overall dimensions 2.5 m × 1.25 m × 0.25 m and having surface temperature of 335 K is being used to maintain the room temperature at 290 K. Calculate the coefficient of radiant heat transfer, convective heat transfer coefficient and the heat loss from the radiator due to both convection and radiation.

The radiations are considered close to that of a black body and the convective heat transfer coefficient is prescribed by the relation

$$h_{con} = 1.32 \, (\Delta T)^{0.33} \text{ W/m}^2\text{-deg}$$

For convection, the actual surface area of the radiator is twice the area of its envelope.

Solution : Area of the radiator

$$= 2 (ab + bc + ca)$$
$$= 2 (2.5 \times 1.25 + 1.25 \times 0.25 + 0.25 \times 2.5)$$
$$= 8.105 \text{ m}^2$$

Radiant heat loss from the radiator to the room,

$$Q_{rad} = \sigma_b A (T_1^4 - T_2^4)$$

In terms of the coefficient of radiant heat transfer,

$$h_{rad} A (T_1 - T_2) = \sigma_b A (T_1^4 - T_2^4)$$

$$\text{or} \quad h_{rad} = \sigma_b (T_1 + T_2)(T_1^2 + T_2^2)$$
$$= 5.67 \times 10^{-8} \times (335 + 290)(335^2 + 290^2)$$
$$= 6.96 \text{ W/mK}$$

Radiant heat loss from the radiator

$$= \sigma_b A (T_1^4 - T_2^4)$$
$$= h_{rad} A (T_1 - T_2)$$
$$= 6.96 \times 8.105 \times (335 - 290)$$
$$= 2538 \text{ W}$$

Convective heat transfer coefficient

$$h_{con} = 1.32 \, (\Delta t)^{0.33}$$
$$= 1.32 \, (335 - 290)^{0.33}$$
$$= 4.63 \text{ W/m}^2\text{K}$$

Convective heat loss from the radiator,

$$h_{con} A \, \Delta t = 4.63 \times (2 \times 8.105) \times (335 - 290)$$
$$= 3377 \text{ W}$$

Heat loss due to both convection and radiation,

$$= 3377 + 2538 = \mathbf{5914\ W}$$

EXAMPLE 8.55

A 18 cm diameter heating pipe ($\epsilon = 0.8$) at surface temperature 475 K has been placed centrally in a brick duct ($\epsilon = 0.9$) of 25 cm side square section and at 300 K temperature. Determine the radiation heat loss from each metre of the heating pipe.

Assuming the system to be in steady state condition, calculate the surface heat transfer coefficient of the brick duct. The surroundings of the duct are at 285 K temperature.

Solution : The rate of radiant interchange between the steam pipe and the brick duct is given by:

$$Q_{12} = (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

where the suffixes 1 and 2 denote the conditions at the heat pipe and the brick duct respectively.

The configuration corresponds to a relatively large body completely enclosed; the inner body cannot see itself, $F_{12} = 1$ and hence

$$(F_g)_{12} = \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + \dfrac{1}{F_{12}} + \dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}}$$

$$= \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + 1 + \dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}}$$

$$= \frac{1}{\dfrac{1}{\epsilon_1} + \left(\dfrac{1}{\epsilon_2} - 1\right) \times \dfrac{A_1}{A_2}}$$

Surface area per metre length of the pipe,

$$A_1 = \pi d l = \pi \times 0.18 \times 1 = 0.565\ m^2$$

Surface area per metre length of duct,

$$A_2 = 4 \times (0.25 \times 1) = 1\ m^2$$

$$\therefore (F_g)_{12} = \frac{1}{\dfrac{1}{0.8} + \left(\dfrac{1}{0.9} - 1\right) \times \dfrac{0.565}{1}}$$

$$= \frac{1}{1.25 + 0.06277} = 0.7616$$

and $Q_{12} = 0.7616 \times 0.565 \times 5.67 \times 10^{-8} (475^4 - 300^4)$

$$= 1044.4\ W/m \text{ length}$$

Let T_3 denote the atmospheric temperature. Then under steady state conditions :

$$Q = A_2 h (T_2 - T_3)$$

$$1044.4 = 1 \times h (300 - 285)$$

or $h = 69.27$

$\therefore$ Heat transfer coefficient h

$$= \mathbf{69.627\ W/m^2\ K}$$

EXAMPLE 8.56

A thermocouple is used to measure the temperature of hot gases flowing through a pipeline which is exposed to surroundings at 625 K and convective film coefficient 46.5 W/m²-deg. The thermocouple indicates a temperature of 725 K and it is so arranged that heat transfer by conduction along the wires is negligible. If emissivity of the thermocouple material is 0.85, make calculations for the true gas temperature.

Take radiation constant for black body $\sigma_b = 5.67 \times 10^{-8}\ W/m^2K^4$

Solution : The arrangement of thermocouple for measuring the temperature of gas flowing a pipe line is indicated in Fig. 8.53. There is convective heat flow from gas to thermocouple but a part of this heat is lost by radiation to tube wall. As such, the tmperature indicated by a thermocouple is laways less than the actual gas temperature.

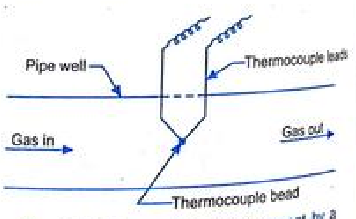

Fig. 8.53. Temperature measurement by a thermocouple

Convective heat flow from gas to thermocouple,

$$= h A (T_{gas} - T_{couple})$$

$$= 46.5\ A (T_{gas} - T_{couple})$$

Heat radiated by thermocouple to pipe wall

$$= F_g A \sigma_b \left(T_{couple}^4 - T_{wall}^4\right)$$

The configuration factor F_g corresponds to a completely enclosed body (thermocouple), and small compared with the enclosing body (pipe line,), *i.e.*, $F_{12} = 1$ and $A_1 <<< A_2$

$$\therefore \quad F_g = \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + \dfrac{1}{F_{12}} + \dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}}$$

$$= \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + 1 + 0} = \epsilon_1 = 0.85$$

Under steady state conditions, there exists an equilibrium between the convective heat flow from gas to thermocouple, and the heat radiated by thermocouple to the pipe wall. Thus

$$= 46.5\ A (T_{gas} - T_{couple})$$

$$= 0.85\ A \sigma_b \left(T_{couple}^4 - T_{wall}^4\right)$$

Inserting the appropriate values,

$T_{gas} - 725$

$$= \frac{0.85 \times 5.67 \times 10^{-8}}{46.5} (725^4 - 625^4)$$

$$= 127.74$$

$\therefore$ True gas temperature $T_g = 852.74$ K

Note: The heat radiated by the gas to the thermocouple is very small as compared with convective heat flow and hence neglected.

EXAMPLE 8.57

In a natural-cum-radiation experiment, a small electrically heated element of surface area 30 cm² is enclosed in a large vessel. The following observations were recorded at the steady state conditions :

Voltage across heaing element = 7.5 V
Current through the element = 0.7 A
Surface temperature of element = 360 K

Presuming that the surface of the element has a unit emissivity and that the temperature of the vessel is equal to the surrounding air temperaature of 300 K, workout the heat exchange by radiation and natural convection between the element and the vessel. Also obtain the corresponding natural convection heat transfer coefficient.

Solution : The arrangement of the heated element and the vessel is shown in Fig. 8.54.

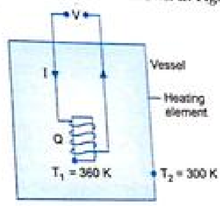

Fig. 8.54.

Power generated at the heated element

$$VI = 7.5 \times 0.7 = 5.25\ W$$

This equals the total heat exchange by radiation and natural convection between the element and the vessel

Heat radiated by element to vessel

$$= (F_g)_{12} A_1 \sigma_b (T_1^4 - T_2^4)$$

The configuration factor $(F_g)_{12}$ corresponds to a completely enclosed body (heated element), and small compared with the enclosing body (vessel), *i.e.*, $F_{12} = 1$ and $A_1 <<< A_2$

$$\therefore (F_g)_{12} = \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + \dfrac{1}{F_{12}} + \left[\dfrac{1-\epsilon_2}{\epsilon_2} \times \dfrac{A_1}{A_2}\right]}$$

$$= \frac{1}{\dfrac{1-\epsilon_1}{\epsilon_1} + 1 + 0} = \epsilon_1 = 1$$

$$\therefore \quad Q_{rad} = \epsilon_1 A_1 \sigma_b (T_1^4 - T_2^4)$$

$$= 1 \times (30 \times 10^{-4}) \times (5.67 \times 10^{-8}) \times (360^4 - 300^4)$$

$$= \mathbf{1.48\ W}$$

Heat exchange by natural convection,

$$Q_{con} = 5.25 - 1.48 = 3.77 \text{ W}$$

Natural convection heat transfer coefficient,

$$h_{con} = \frac{Q_{con}}{A_1 \, \Delta t}$$

$$= \frac{3.77}{\left(30 \times 10^{-4}\right)(360 - 300)}$$

$$= \mathbf{20.94 \ W/m^2\text{-deg}}$$

EXAMPLE 8.58

The inside and outside surfaces of a hollow brick lining of a furnace are to be kept at 780 K respectively by the placement of radiation shields in between the hollow space. Determine the number of radiation shields if the heat loss to the furnace surroundings at 300 K is both by natural convection and radiation. The surface heat transfer coefficient has been prescribed by the relation $h = 1.45 \, (\Delta t)^{0.33}$ W/m²-deg, and the emissivity of the wall and the shields are all equal to 0.86.

Solution : Under steady state conditions, the radiation heat loss to the outside surface of the brick lining equals the heat transfer from the brick lining to the surroundings by natural convection and radiation.

(*i*) Convective heat loss Q_{conv} to surroundings.

$$Q_{conv} = h_{conv} \, (T_2 - T_3) \quad \text{per unit area}$$

$$= 1.45 \, (T_2 - T_3)^{0.33} \times (T_2 - T_3)$$

$$= 4.45 \, (380 - 300)^{1.33}$$

$$= 492.57 \text{ W/m}^2$$

(*ii*) Radiation heat loss Q_{rad} to surroundings,

$$Q_{rad} = (F_g)_{23} \, \sigma_b \left(T_1^4 - T_2^4\right) \quad \text{per unit area}$$

where factor $(F_g)_{23}$ =

$$\frac{1}{\dfrac{1-\epsilon_2}{\epsilon_2} + \dfrac{1}{F_{23}} + \dfrac{1-\epsilon_3}{\epsilon_3} \times \dfrac{A_2}{A_3}}$$

For the give arrangement $A_2 <<< A_3$ and $F_{23} = 1$ and therefore,

$$(F_g)_{23} = \epsilon_2 = 0.86$$

$$\therefore \quad Q_{rad} = 0.86 \times (5.67 \times 10^{-8}) \times (380^4 - 300^4)$$

$$= 621.76 \text{ W/m}^2$$

(*iii*) With n-radiation shields of equal emissivities, the heat flow from internal surface of the hollow lining to the external surface is:

$$Q_{12} = \frac{1}{n+1} \times \frac{\sigma_b \left(T_1^4 - T_2^4\right)}{(2/\epsilon - 1)}$$

$$= \frac{1}{n+1} \times \frac{5.67 \times 10^{-8} \left(780^4 - 300^4\right)}{(2/0.86 - 1)}$$

$$= \frac{14940.8}{n+1} \text{ W/m}^2$$

From energy balance :

$$\frac{14940.8}{n+1} = (492.57 + 621.76)$$

or $n + 1 = 13.41$

$\therefore$ Radiation shields required are $n = 13$

EXAMPLE 8.59

Dry standard steam at 5 bar is carried within a steel pipe of 10 cm outside diameter and lagged with 5 cm thick insulation. The conductivity and emissivity of the insulating material are 0.14 W/m-deg and 0.9 respectively. If the surroundings are at 25°C temperature and the heat transfer coefficient on the outer surface of the lagged pipe is 11.5 W/m²-deg, make calculations to work out (i) conduction and radiation heat loss from one metre length of the pipe (ii) radiation and overall heat transfer coefficient and (iii) rate of condensation of steam from 40 metre length of the pipe.

Solution : (*i*) The heat loss by conduction and convection from one metre length of the pipe can be worked out by using the relation :

$$Q_c = \frac{t_1 - t_3}{\dfrac{1}{2\pi k l} \log_e \dfrac{r_2}{r_1} + \dfrac{1}{2\pi r_2 h_0}}$$

where t_1 is the temperature of dry saturated steam at 5 bar pressure. From steam tables:

$$t_1 = 151.8°\text{C}$$

$$Q_c = \frac{151.8 - 25}{\dfrac{1}{2\pi \times 0.14 \times 1} \log_e \dfrac{10}{5} + \dfrac{1}{2\pi \times 0.12 \times 11.5}}$$

$$= \frac{151.8 - 25}{0.788 + 0.138}$$

$$= \mathbf{136.86 \ W/m \ length \ of \ pipe}$$

For finding the temperature t_2 on the outer surface of the insulation

$$Q_c = \frac{t_1 - t_2}{\dfrac{1}{2\pi k l} \log_e \dfrac{r_2}{r_1}}$$

$$136.86 = \frac{151.8 - t_2}{0.788} \; ; \quad t_2 = 43.95°\text{C}$$

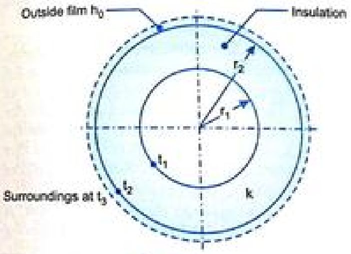

Fig. 8.55.

The radiation heat loss from the outside surface of the pipe to the surroundings is given by the relation :

$$Q_r = \left(F_g\right)_{23} A_2 \, \sigma_b \left(T_1^4 - T_2^4\right)$$

where the factor $(F_g)_{23}$ =

$$\frac{1}{\dfrac{1-\epsilon_2}{\epsilon_2} + \dfrac{1}{F_{23}} + \dfrac{1-\epsilon_3}{\epsilon_3} \times \dfrac{A_2}{A_3}}$$

The arrangement (location of pipe in the atmosphere) corresponds to the situation where $F_{23} = 1$ and $A_2 <<< A_3$. Therefore,

$$(F_g)_{23} = \frac{1}{\dfrac{1-\epsilon_2}{\epsilon_2} + 1} = \epsilon_2 = 0.9$$

$$Q_r = 0.9 \times (\pi \times 0.2 \times 1) \times 5.67 \times 10^{-8} \, (316.95^4 - 298^4)$$

$$= \mathbf{70.68 \ W/m \ length}$$

(*ii*) Let h_r be the radiation loss coefficent. Then

$$Q_r = h_r \, A_2 \, (t_2 - t_3)$$

$$70.68 = h_r \times (\pi \times 0.2 \times 1) \, (43.95 - 25)$$

$$\therefore \quad h_r = 5.939 \text{ W/m}^2\text{-deg}$$

Overall heat transfer coefficient.

$$= h_0 + h_r = 11.5 + 5.939$$

$$= \mathbf{17.439 \ W/m^2\text{-deg}}$$

(*iii*) The loss of heat from 40 m length of pipe

$$= (Q_c + Q_r) \, l$$

$$= (136.86 + 70.68) \, 40$$

$$= 8301.6 \text{ W} = 8301 \text{ J/s}$$

Latent heat h_{fg} of steam at 5 bar pressure

$$= 2107 \text{ kJ/kg} = 2107 \times 10^3 \text{ J/kg}$$

$\therefore$ Condensation rate of steam

$$= \frac{8302}{2107 \times 10^3}$$

$$= 3.94 \times 10^{-3} \text{ kg/s}$$

$$= \mathbf{14.18 \ kg/hr}$$

EXAMPLE 8.60

Dry steam at 300°C is flowing through a steel pipe (k = 46 W/m-deg) 10 cm inner diameter and 1 cm thick. The pipe is located in a large room and is covered with 5 cm asbestos insulation of thermal conductivity 0.073 W/m-deg. The emissivity of the outer surface is 0.88 and the surroundings are at 10°C. The convective heat transfer co-efficient at the inner and outer surfaces of the composite system are 581.5 and 8.14 W/m²-deg respectively, and the outer surface coefficient includes the effects of both convection and radiation. Make calculations for the (a) heat transferred per metre length of the pipe, (b) equivalent radiation film coefficient, and (c) convective heat transfer coefficient for the outer surface.

Solution : The geometrical dimensions of the lagged steam pipe are :

$r_1 = 5$ cm ;

$r_2 = 5 + 1 = 6$ cm ;

$r_3 = 6 + 5 = 11$ cm

Considering 1 m length of pipe, the various thermal resistances to flow of heat are offered by :

(i) Inner steam film

$$= \frac{1}{h_i A_i}$$

$$= \frac{1}{581.5 \times (2\pi \times 0.05 \times 1)}$$

$$= 0.00548$$

(ii) Pipe material : The resistance offered due to steel may be neglected as thermal conductivity of steel is quite high and the pipe thickness is small.

(iii) Asbestos insulation

$$= \frac{\log_e \frac{r_3}{r_2}}{2\pi k l}$$

$$= \frac{\log_e \frac{11}{6}}{2\pi \times 0.073 \times 1} = 1.32$$

(iv) Outside air film

$$= \frac{1}{h_o A_o}$$

$$= \frac{1}{8.14 \times (2\pi \times 0.11 \times 1)} = 0.178$$

The resistance of a series path is equal to the sum of individual resistances.

$$\therefore \sum R_t = 0.00548 + 1.32 + 0.178$$

$$= 1.503 \text{ deg/W}$$

Therefore heat lost through conduction,

$$Q = \frac{\Delta t}{\sum R_t} = \frac{300 - 20}{1.503} = 186.29 \text{ W}$$

The same heat passes through each layer of the composite system. Then for the outside air film,

$$186.29 = \frac{t_0 - 20}{0.178}$$

where t_0 is the temperature on the outer surface of asbestos insulation

$$t_o = 186.29 \times 0.178 + 20$$

$$= 53.16°\text{C} = 326.16 \text{ K}$$

In terms of coefficient of radiation heat transfer,

$$h_r A (T_o - T_a) = \epsilon A \sigma_b (T_o^4 - T_a^4)$$

$$h_r = \epsilon \sigma_b (T_0 + T_a)(T_0^2 + T_a^2)$$

$$= 0.88 \times (5.67 \times 10^{-8})$$

$$\times (326.16 + 293)(326.16^2 + 293^2)$$

$$= 5.938 \text{ W/m}^2\text{-deg}$$

$\therefore$ Convective coefficient for outer surface

$$h_{con} = h_o - h_r$$

$$= 8.14 - 5.938 = 2.202 \text{ W/m}^2\text{-deg}$$

EXAMPLE 8.61

Two infinitely long cylinders are placed concentrically. The inner cylinder has an outside diameter of 7.5 cm, and emissivity of 0.1 and is maintained at 1050 K. The outer cylinder has an inside diameter of 22.5 cm, an emissivity of 0.2 and is maintained at 315 K. A cylindrical shield (very thin so that its temperature is the same on both sides) is placed concentrically midway between the two cylinders and allowed to come to thermal equilibrium; it has an emissivity of 0.3 on both sides. Assuming that vacuum exists between the annular spaces, find the steady state temperature attained by the cylindrical shield. Also calculate the heat loss per metre length of the composite cylinder and the convective heat transfer coefficient based on outside surface of the cylinder. The ambient air is at 280 K temperature.

Solution : Let the suffix 1, 2 and 3 refer to the inner cylinder, outer cylinder and cylinderical shield placed midway between the two cylinders. Then

$$d_3 = \frac{d_1 + d_2}{2} = \frac{7.5 + 22.5}{2} = 15 \text{ cm}$$

Heat flow from inner cylinder to shield

$$Q_{13} = (F_g)_{13} A_1 \sigma_b (T_1^4 - T_3^4)$$

$$(F_g)_{13} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{13}} + \frac{1-\epsilon_3}{\epsilon_3} \times \frac{A_1}{A_3}}$$

The inner cylinder is completely enclosed by the shield and accordingly $F_{13} = 1$

$$\therefore (F_g)_{13} = \frac{1}{\frac{1}{\epsilon_1} + \left(\frac{1}{\epsilon_2} - 1\right) \times \frac{A_1}{A_3}}$$

For concentric cylinders of equal length,

$$\frac{A_1}{A_3} = \frac{\pi d_1 l}{\pi d_3 l} = \frac{d_1}{d_3} = \frac{7.5}{15.0} = \frac{1}{2}$$

$$\therefore (F_g)_{13} = \frac{1}{\frac{1}{0.1} + \left(\frac{1}{0.3} - 1\right) \times \frac{1}{2}}$$

$$= 0.0895$$

and $Q_{13} = 0.0895 \ A_1 \sigma_b (1050^4 - T_3^4)$ ...(i)

Likewise, the heat flow from the shield to the outer cylinder is

$$Q_{32} = (F_g)_{32} A_3 \sigma_b (T_3^4 - T_2^4)$$

$$(F_g)_{32} = \frac{1}{\frac{1}{\epsilon_3} + \left(\frac{1}{\epsilon_2} - 1\right) \times \frac{A_3}{A_2}}$$

$$\frac{A_3}{A_2} = \frac{d_3}{d_2} = \frac{15}{22.5} = \frac{2}{3}$$

$$\therefore (F_g)_{32} = \frac{1}{\frac{1}{0.3} + \left(\frac{1}{0.2} - 1\right) \times \frac{2}{3}} = 0.167$$

and $Q_{32} = 0.167 \ A_3 \sigma_b (T_3^4 - 315^4)$ ...(ii)

Under steady state conditions, $Q_{13} = Q_{32}$

$$\therefore 0.0895 A_1 \sigma_b (1050^4 - T_3^4)$$

$$= 0.167 A_3 \sigma_b (T_3^4 - 315^4)$$

$$\frac{A_3}{A_1} = \frac{d_3}{d_1} = \frac{15.0}{7.5} = 2$$

$$1050^4 - T_3^4 = \frac{0.167}{0.0895} \times 2 (T_3^4 - 315^4)$$

$$= 3.73 T_3^4 - 3.73 \times 315^4$$

$$\therefore \quad T_3 = \left[\frac{1050^4 + 3.73 \times 315^4}{3.73 + 1}\right]^{1/4}$$

$$= 717°\text{C}$$

Heat loss per metre length of the composite cylinder can be calculated from either of the expressions (i) or (ii)

$$Q = 0.0895 \times (\pi d_1 l) \times \sigma_b \times (1050^4 - 717^4)$$

$$= 0.0895 \times (\pi \times 0.075 \times 1) \times (5.67 \times 10^{-8}) \times (1050^4 - 717^4)$$

$$= 1136 \text{ W}$$

For steady heat flow, this heat is lost to the surroundings both by convection and radiation form the outer surface of the outside cylinder. That is

$$1136 = h_c A_2 (T_2 - T_a) + \sigma_b \epsilon A_2 (T_2^4 - T_a^4)$$

$$= h_c \times (\pi \times 0.225 \times 1)(315 - 280) + 5.67 \times 10^{-8} \times 0.3 \times (\pi \times 0.225 \times 1) \times (315^4 - 280^4)$$

$$= 24.3 \ h_c + 29.64$$

$\therefore$ Convective heat transfer coefficient,

$$h_c = \frac{1136 - 29.64}{24.73}$$

$$= 44.73 \text{ W/m}^2\text{-deg}$$

8.8. GASEOUS RADIATION

Some of the salient features of radiation from gases, vapours and flames are :

1. Most solids and liquids have a continuous spectrum, *i.e.*, they emit and absorb radiant energy of all wavelengths. In contrast, gases and vapours emit and absorb radiant energy in definite parts of the spectrum called bands, and are commonly known as selective absorbers and emitters. The number of bands and their widths, and the monochromatic emissive power of a gas at each wavelength within the bands vary for different gases and vapours. These characteristics also vary with temperature, pressure, thickness of gas volume and atomicity. The total emissive power equals

the summation of the areas within the selective bands of wavelength.

2. Only thin layers (1 micron to 1 mm thick) participate in the process of thermal radiation through solids and liquids. Gases possess a much small radiating power and all their volumes participate in the radiation.

3. Monoatomic gases such as orgon and helium, and diatomic gases such as oxygen, nitrogen are extremely inert to thermal radiation. Their emissive power and absorptivity is so small that they are considered practically diathermaneous, *i.e.*, transparent to thermal radiation in the temperature ranges of common engineering interest.

4. Polyatomic gases such ammonia, carbon dioxide, methane, sulphur dioxide, many of the hydrocarbons and water vapours are fairly good absorbers and radiators over certain wavelength ranges.

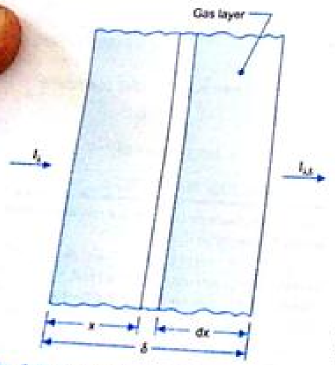

Fig. 8.56. Monochromatic radiation through an absorbing gas

Consider a beam of monochromatic radiation at wavelength λ that enters a layer of absorbing gas (Fig. 8.56). As the beam passes through the gas layer, its intensity gets reduced and the decrease is given by :

$$dI_{\lambda x} = -k_\lambda I_{\lambda x}\, dx \qquad \text{...(8.47)}$$

where $I_{\lambda x}$ is the monochromatic intensity at depth x. The proportionality constant k_λ, called the ***monochromatic absorption coefficient***, depends upon the state of gas (its temperature and pressure) and the wavelength. If the gas scatters radiation,

$$dI_{\lambda x} = -\gamma_\lambda I_{\lambda x}\, dx$$

where γ_λ is the ***monochromatic scattering coefficient***. In case, the gas both absorbs and scatters radiation :

$$dI_{\lambda x} = -\beta_\lambda I_{\lambda x}\, dx$$

where $\beta_\lambda = (k_\lambda + \gamma_\lambda)$ is the ***monochromatic extinction coefficient***

Integrating equation 8.47 from x

$$= 0\left(I_\lambda = I_{\lambda o}\right) \text{ to } x = \delta\ (I_\lambda = I_{\lambda \delta})$$

$$I_{\lambda\delta} = I_{\lambda o} \exp(-k_\lambda\, \delta) \qquad \text{...(8.48)}$$

The ratio $I_{\lambda\delta}/I_{\lambda o}$ is the ***monochromatic transmissivity*** τ_λ of the gas.

In general, the gases do not reflect radiant energy, *i.e.*, their reflectivity is zero, therefore

$$\alpha_\lambda + \tau_\lambda = 1$$

or $\quad \alpha_\lambda = 1 - \tau_\lambda = 1 - \exp(-k_\lambda\, \delta)$

Obviously the quantity $[1 - \exp(-k_\lambda\, \delta)]$ represents the monochormatic absorptivity of the gas; it also represents monochromatic emissivity in accordance with Kirchoff's law

$$\epsilon_\lambda = 1 - \exp(-k_\lambda\, \delta)$$

Thus if thickness δ of the gas layer is very large

$$\alpha_\lambda = \epsilon_l = 1$$

Apparently for very thick layers, gas radiation approaches black body radiation wihin the wavelength of the band. The absorptivity of a gas is a fairly complicated function of pressure, temperature, size and configuration of the gaseous region.

5. The following empirical relations have been suggested to workout the emissive power of CO_2 and water vapours

$$E_{CO_2} = 3.5\,(pl)^{0.33}\left(\frac{T}{100}\right)^{3.5}$$

$$E_{H_2o} = 3.5p^{0.8}l^{0.6}\left(\frac{T}{100}\right)^3$$

where p is the partial pressure and l takes into consideration both the size and configuration of the gaseous region; it is defined as

$$l = 4 \times \frac{\text{volume of gas}}{\text{area of the enclosure (wall)}}$$

Thus for two infinite parallel plates a distance x apart, $l = 4 \times Ax/2A = 2x$. The data on the beam length for gaseous bodies of various geometrical shapes have been compiled by Hottel and the relevant graphs are given in hand books on heat transfer.

The empirical relations listed above suggest that the emissive power of gases and vapours deviates considerably from Stefan Boltzman's law which stipulates that emissive power is proportional to the fourth power of the absolute temperatre.

CO_2 and H_2O vapour are somewhat opaque to each other and as such when both are present, the total radiation becomes less than the sum of their separate individual effects.

6. The radiant heat exchange between a gas at temperature T_g and a black surface of finite area A at temperature T_b is worked out from the relation:

$$Q = \sigma_b A\left(\epsilon_g T_g^4 - \alpha_g T_b^4\right) \qquad \text{...(8.49)}$$

The emissivity ϵ_g of the gas is evaluated at the temperature of the gas, and its absorprivity α_g at the temperature of the surface

If inside surface of the enclosure is not black, then computations of the net rate of heat transfer from the gas to the walls are made from the relation

$$Q = \alpha A\,\bar{\epsilon}_w\left(\epsilon_g T_g^4 - \alpha_g T_w^4\right) \qquad \text{...(8.50)}$$

where $\bar{\epsilon}_w$ is the effective surface emissivity; its approximate value is taken to be the mean of ϵ_w and 1. *i.e.*, $\bar{\epsilon}_w = (\epsilon_w + 1)/2$ where ϵ_w is the emissivity of the gas at the wall temperature T_w.

The average value accounts for the beams of radiation from the gas to the walls which are reflected and then absorbed by the gas.

7. When a gas or vapour is in the process of oxidation or combustion, it is called a ***flame***. Radiation heat exchange from flames is of common occurrence in furnaces, jet engine burners and heaters. Most of the flames are luminous; and the firing of pulverised fuel in central station steam generators is a good example of luminous flame. Luminosity stems from the presence of glowing particles of carbon and ash, or from pure chemical reactions which occur at elevated temperatures. The net interchange of energy between a flame and its enclosure is given by :

$$Q_{net} = \sigma A_f F_{fw} \epsilon_f \epsilon_w \left(T_f^4 - T_w^4\right) \qquad \text{...(8.51)}$$

where the subscript f corresponds to flames and w to wall of the enclosure; A_f is the area of the flame envelope.

The flames produced in the household stoves using kerosene or gaseous fuel for heating purposes are non-luminous in nature. The emittance of energy by non-luminous flames is at certain discrete wavelengths; obviously such flames cannot be treated either black or gray surfaces.

SALIENT POINTS

1. The fraction of rate of energy leaving area A_1 and impinging on A_2 is given by

$$\frac{Q_{12}}{Q_1} = \frac{1}{A_1}\int_{A_1}\int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2}\, dA_1\, dA_2$$

This fraction is known as shape factor, view factor or configuration factor of surface A_1 with respect to surface A_2; it is designated as F_{12}. That gives

$$Q_{12} = A_1 F_{12} \sigma_1 T_1^4$$

Likewise

$$Q_{21} = A_2 F_{21} \sigma_2 T_2^4$$

$$\therefore (Q_{12})_{net} = A_1 F_{12} \sigma_1 T_1^4 - A_2 F_{21} \sigma_2 T_2^4$$

When both the surfaces are black ($\sigma_1 = \sigma_2 = \sigma_b$) and are maintained at the same temperatures ($T_1 = T_2 = T$) there will be no heat exchange.

$$0 = (A_1 F_{12} - A_2 F_{21}) \sigma_b T^4$$

Since σ_b and T are both non-zero quantities, we get

$$A_1 F_{12} - A_2 F_{21} \text{ or } A_1 F_{12} = A_2 F_{21}$$

This result is known as reciprocity theorem.

2. The shape factor depends only on the geometry of the emitter and that of the collection surfaces.
3. In an enclosure of n surfaces, the shape factor satisfies the condition

$$\sum_{j=1}^{n} F_{ij} = 1$$

$$F_{11} + F_{12} + F_{13} + \ldots F_{1n} = 1$$

If the surface j is subdivided into m sub-surfaces, then

$$F_{ij} = \sum_{k=1}^{m} F_{ik}$$

(i) F_{11} is called the self-viewing factor and it is non-zero only for a concave surface.

(ii) For a flat or convex surface, the shape factor with respect to itself is zero

4. If the two surfaces A_1 and A_2 are parallel and large, the radiation occurs across the gap between them so that $A_1 = A_2$, and all radiation emitted by one falls on the other, then

$$F_{12} = F_{21} = 1$$

5. The inter-relationship between different shape factor is called shape factor algebra.
The desired shape factors are worked out by invoking the summation rule, the reciprocity ttheorem and from inspection of geometry.
6. The net heat interchange between non-black bodies at temperatures T_1 and T_2 is given by

$$Q_{net} = f_{12} F_{12} \sigma_b A_1 \left(T_1^4 - T_2^4\right)$$

The factor f_{12} is called the interchange factor for the radiation from surface 1 to surface 2.

(i) For infinite parallel planes with emissivities ϵ_1 and ϵ_2,

$$f_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} \quad \text{and} \quad F_{12} = 1$$

(ii) For radiant heat exchabge concentric spheres or infinitely long cylinders

$$f_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\epsilon_2} - 1\right)} \quad \text{and} \quad F_{12} = 1$$

(iii) For radiant heat exchange between body 1 enclosed by body 2 (body 1 is small)

$$f_{12} = 1 \quad \text{and} \quad F_{12} = 1$$

(iv) For radiant heat exchange between body 1 enclosed by body 2 (body 1 is large)

$$f_{12} = \frac{1}{\frac{1}{\epsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\epsilon_2} - 1\right)} \quad \text{and} \quad F_{12} = 1$$

The interchange factor is also called equivalent emissivity.

7. Irradiation (G) is defined as the total radiation incident upon a surface per unit time per unit area; some of it may be reflected to become a part of the radiosity of the surface.
Radiosity (J) is the term used to indicate the total radiation leaving a surface per unit time per unit aeea. It comprises the original emittance from the surface plus the reflected portion of radiation incident upon it.
8. Radiation shields are thin opaque partitions made of materials of very low absorptivity and high reflectivity (aluminimum or copper). They are arranged in the direction perpendicular to the propagation of heat between two radiating surfaces. These shields introduce a sort of additional resistance in the heat flow path and accordingly the net heat flux is reduced.
When the radiating surfaces and the radiation shields are of same emissivity ϵ, then
Heat exchange without any shield

$$= \frac{A \sigma_b \left(T_1^4 - T_2^4\right)}{\frac{2}{\epsilon} - 1}$$

Heat exchange with n-radiation shields inserted between the two planes

$$= \frac{1}{n+1} \frac{A \sigma_b \left(T_1^4 - T_2^4\right)}{\frac{2}{\epsilon} - 1}$$

Apparently the presence of n-shields reduces the radiant heat transfer by a factor of $(n + 1)$.

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. Derive a general relation for the radiation shape factor in case of radiation between two surfaces.
2. Define shape factor and state its physical significance. What are the other names for shape factor?
3. Two facing parallel plates radiating only from their facing sides see only each other but the two rectangular plates meeting at right angles do not radiate solely to each other. How do you account for the variation in shape factor?
4. State and explain the reciprocity theorem.
5. What would be the shape factor of a concave, convex and flat surface with respect to itself?
6. Show that the quantity of radiant heat interchange between two gray surfaces can be expressed by the relation

$$(Q_{ij})_{net} = F_{ij} f_{ij} A_i \sigma_b \left(T_i^4 - T_j^4\right)$$

The notations have their usual meainings. State all the assumptions made in the derivation of this relation.
How the relation gets modified when the surfaces are parallel to each other ?

7. Using the definition of radiosity and irradiation, prove that the radiant interchange between two gray bodies is given by relation $Q_{net} =$

$$\frac{A_1 \sigma_b \left(T_1^4 - T_2^4\right)}{(1-\epsilon_1)/\epsilon_1 + 1/F_{12} + (1-\epsilon_2)/\epsilon_2 \times A_1/A_2}$$

The notations have their usual meanings
Give two examples of practical interest where surface 1 sees only another surface 2. How the above relation gets modified in that case?

8. Write down the values of F_{12} for the following cases:
   (i) black surfaces
   (ii) infinite parallel planes
   (iii) two long concentric cylinders
   (iv) two concentric spheres, and
   (v) small body enclosed within a large body
9. When a small body of emissivity ϵ at temperature T is placed in a large enclosure at temperature T_0, the net radiation heat loss is $\epsilon \sigma_b \left(T^4 - T_0^4\right)$. Express this result in the form of Newton-Rikhman law and obtain the value of radiative heat transfer coefficient
10. Workout the net rate of radiant energy from a pond of water (10 m × 10 m) at 5°C on a clear night in winter when the sky radiation temperatue is of the order of 230 K. Take emissivity of the water surface ϵ = 0.95 and the black body radiation constant

$$\sigma_b = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$$

(Ans. 17.12 kW)

11. A metal storage tank of external area 0.5 m² contains water at 65°C and is located inside a large enclosure with walls at 10°C. The tank is painted black on the outside and its emissivity is approximated to 0.95. How much reduction in heat loss would occur if the outside surface was coated with aluminium paint of emissivity 0.55 ?
Take radiation constant σ_b

$$= 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$$

(Ans. 276.6 kJ/hr)

12. A steel tube 10 cm in diameter and with the surface temperature 130°C is placed inside a brick duct. The duct has a cross-section 0.28 m × 0.35 and its walls are at 35°C temperature.
Determine the hourly loss of heat by radiation from one metre length of this tube.

(Ans. 225 W/m)

13. A stainless steel plate (ϵ = 0.6) at 100°C faces a brick wall (ϵ = 0.75) at 500°C. Estimate the heat flux and the radiant heat transfer coefficient.

(Ans. 9573 W/m², 24 W/m²-deg)

14. A steam radiator with the enveloping radiating surface 1.5 m long, 0.6 m high and 0.3 m deep is supporting itself on the floor of a large room. The radiator surface has been painted with a lacquer containing 10% aluminium (ϵ = 0.55). If the radiator and the surface are at 370 K and 300 K respectively, estimate the rate of heat interchange between them. Take radiation constant

$$\sigma_b = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$$

(Ans. 3674 kJ/h

15. A large tank forming the hot well for a boil plant contains water at 330 K. It has superficial area of 46.5 m² and sits in spa

in a large enclosure, the walls of which are at 285 K. If emissivity of the outside surface of the tank is 0.85, find the rate of heat loss by radiation from the tank. Neglect loss by convection to the atmospheric air. Take Stefan Boltzman constant

$$\sigma_b = 0.87 \times 10^{-8}\ W/m^2K^4$$

(Ans. 11.79 kW)

16. *A domestic hot water tank, 0.5 m diameter and 1 m high, is kept in a large space that effectively forms black surroundings. If the tank surface is oxidised copper with an emissivity of 0.8, make calculations for the radiation heat flow. It may be presumed that estimation of heat loss is to be based on a tank surface temperature of 80°C and an ambient temperature of 25°C.*

What would be the reduction in heat loss if the tank surface is given a coating of aluminium paint with emissivity 0.52 ? Take radiation constant $\sigma_b = 5.67 \times 10^{-8}\ W/m^2K^4$.

Hint: Area $= 2\ (\pi r^2) + 2\ \pi\ r\ h$

(Ans. 0.66 kW, 0.43 kW)

17. *During emergency after an accident, the naked body of a person is placed in an aluminium coated plastic bag for protection against heat loss until help arrives. The bag surrounds the body completely and has a temperature of 10°C and an emissivity 0.2. The skin tempeature of the victim is 37°C, the skin emissivity is 0.8 and the effective radiating areas of both skin and bag are 1.1 m².*

How much heat will the victim lose by radiation ? If the body can sustain a loss of 20 W, at what temperature will the bag give adequate protection ?

18. *A dead black cylinder of emissivity 0.95 is kept at 95°C in a large enclosure at 10°C. Find the radiation heat loss per square meter of its surface. What would the radiation loss become if the cylinder were surrounded by a concentric cylinder with its inner surface of a brightly polished metal of emissivity 0.10 ? Take radiation constant $\sigma_b = 5.67 \times 10^{-8}\ W/m^2K^4$.*

(Ans. 556 and 245 kJ/m²-hr)

19. *Two concentric pipes 20 cm and 30 cm in diameter, with the space between them evacuated are, used to store liquid air (boiling point − 150°C) in room at 25°C. The surface of both the pipes are flushed with aluminium for which emissivity ϵ = 0.03. Estimate the* rate of heat flow by radiation to the air and the rate of its evaporation. Given that the latent heat of vaporisation of liquid air is

208 kJ/kg

(Ans. 18 kJ/hr, 0.0863 kg/hr)

20. During an experiment to measure the temperature of combustion gases flowing through a large duct with a thermocouple located at the centre of the duct, the following data were noted under steady state conditions.
Temperature of the duct walls = 20°C, temperature indicated by the theermocouple = 500°C, emissivity of thermocouple material = 0.6, convective film coefficient between the thermocouple surface and gas stream

$= 200\ W/m^2\ K$

Estimate the error between the actual gas stream temperature and the temperature indicated by the thermocouple.

(Ans. 59.6°C)

21. Two infinite plates maintained at constant temperatures T_1 and T_2 with emissivities ϵ_1 and ϵ_2 are placed parallel to each other. Another thin metallic plate with emissivity ϵ_3 on both sides is placed parallel between them. Obtain an expression to determine the change in radiation heat transfer when $\epsilon_1 = \epsilon_2$ and $\epsilon_3 = \epsilon_1/2 = \epsilon_2/2$. What will be the change when $\epsilon_1 = \epsilon_2 = \epsilon_3$?

22. A thin shield of emissivity ϵ_s (on both sides) is placed between two infinite parallel plates of emissivities ϵ_1 and ϵ_2, and temperature T_1 and T_2 respectively. If $\epsilon_1 = \epsilon_2 = \epsilon_s$, show that temperature of the shield is given by :

$$\left(\frac{T_1^4 + T_2^4}{2}\right)^{1/4}$$

23. In steelmaking apparatus two parallel plates, large compared to the distance between them are at 280°C and 10°C.
   (*a*) Calculate the radiation heat exchange per m² of area if $\alpha_1 = \alpha_2 = 1$.
   (*b*) A third parallel black plate is located between two plates. What is its steady state temperature? How is the heat transfer altered ?
   (*c*) Suppose the hotter of the two plates is oxidised steel (ϵ = 0.80) while the cooler oxidised aluminium (ϵ = 0.25). What is the rate of heat transfer and what is the temperature of the black plate between them ?

Discuss briefly the significance of this exercise.

24. Explain the utility of radiation shields.
Two large parallel planes having emissivities 0.3 and 0.5 are maintained at temperatures of 900°C and 400°C respectively. A radiation shield having an emissivity of 0.05 is placed between the two planes. Workout :
   (*a*) heat exchange per m² of area if the shield was not present,
   (*b*) temperature of the shield, and
   (*c*) heat exchange per m² area when the shield is present.

25. Air flows between two concentric cylindrical gray surfaces; the relevant geometrical and thermodynamic parameters are :-
diameters : 40 mm and 100 mm
absorptivities : 0.75 and 0.85
temperatures : 440 K and 870 K
At a given point, the temperature of air is 750 K. Compare the rate of radiant heat transfer to the inner surface with the convective heat transport at that point. Presume that convective coefficient
$h = 125.6\ kJ/m^2$-hr-deg and radiation constant $\sigma_b = 5.67 \times 10^{-8}\ W/m^2K^4$

$$\left(\text{Ans. } \frac{Q_r}{Q_c} = 1.997\right)$$

26. Explain the meaning of the terms radiosity and irradiation.

27. Explain the special features of radiation from gases.
"Radiation from thick layers of polytropic gases approaches black body radiation within the wavelength of its bands." Comment upon the validity of this statement.

B. Fill in the blanks with appropriate word/words :

1. The fraction of radiative energy that is diffused from one surface element and strikes the other surface directly with no intervening reflections is called the ....
2. The value of shape factor depends on the .... and .... of surfaces with respect to each other.
3. For a flat or convex surface, the shape factor with respect to itself is ....
4. The shape factor is equal to .... for infinite parallel planes radiating to each other.
5. The interchange factor is a function of the .... of the surfaces.
6. Radiation shields provide .... heat flow path between two surfaces and that .... the overall rate of heat transfer.
7. .... are made of materials of very .... absorptivity and .... reflectivity.
8. With insertion of *n*-shields, the radiation heat transfer is reduced by a factor of ....
9. .... denotes the total radiation incident upon a surface per unit time and per unit surface area.
10. The radiosity is the sum total of the radiation emitted, reflected and ....

Answers : 1. shape factor, view factor or configuration factor; 2. geometry, orientation; 3. zero; 4. one; 5. emissivities; 6. additional resistance reduces; 7. low, high; 8. (*n* + 1); 9. Irradiation; 10. transmitted.

C. Multiple choice questions :

1. For the same type of shapes, the value of radiation shape factor will be higher when
   (*a*) surfaces are more closer
   (*b*) surfaces are moved further apart
   (*c*) surfaces are smaller and held closer
   (*d*) surfaces are larger and held closer
2. Which of the followings is a wrong statement? The shape factor is equal to one
   (*a*) for any surface completely enclosed by another surface
   (*b*) for infinite parallel planes radiating only to each other
   (*c*) for a flat or convex surface with respect to itself
   (*d*) inner cylinder to outer cylinder of a long co-axial cylinder
3. The reciprocity theorem states that
   (*a*) $F_{12} = F_{21}$ (*b*) $A_1 F_{12} = A_2 F_{21}$
   (*c*) $A_1 F_{12} = A_2 F_{21}$ (*d*) $A_2 F_{12} = A_1 F_{21}$
   where the symbols have their usual meanings
4. Two radiating surface $A_1 = 6\ m^2$ and $A_2 = 4\ m^2$ have shape factor $F_{1\text{-}2} = 0.1$. Then the shape factor $F_{2\text{-}1}$ will be
   (*a*) 0.18 (*b*) 0.15
   (*c*) 0.12 (*d*) 0.10

5. What is the value of shape factor for two infinite parallel surfaces separated by a distance x

(a) 0 (b) ∞

(c) 1 (d) x

6. A hemispherical surface 1 lies over horizontal plane surface 2 such that convex portion of hemisphere is facing sky.

Fig. 8.57.

What is the value of the geometrical shape factor F_{12}?

(a) 1/4 (b) 1/2

(c) 3/4 (d) 1/8

7. A small sphere of outer area 0.6 m² is totally enclosed by a large cubical hall. The shape factor of hall with respect to sphere is 0.004? What is the measure of the internal side of the hall?

(a) 4 m (b) 5 m

(c) 6 m (d) 10 m

8. What will be the view factor F_{21} for the geometry as shown below (sphere within a cube)?

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{e}$

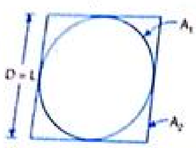

Fig. 8.58.

9. For infinite parallel planes with emissivities ϵ_1 and ϵ_2, the interchange factor for radiation from surface 1 to surface 2 is

(a) $\epsilon_1\,\epsilon_2$ (b) $\epsilon_1 + \epsilon_1$

(c) $\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}$ (d) $\frac{\epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2}$

10. Two plane parallel grey surfaces having [illegible] emissivity are maintained at 400 K and 300 K. The radiative heat transfer rate per unit area of these surfaces is about

(a) 992 W/m² (b) 812 W/m²

(c) 567 W/m² (d) 464 W/m²

11. The heat exchange between a small body having emissivity ε_1 and area A_1, and a large enclosure having emissivity ε_2 and area A_2 is given by

$$Q_{1-2} = A_1\,\varepsilon_1\,\sigma\,(T_1^4 - T_2^4)$$

What is the assumption for this relation?

(a) $\varepsilon_2 = 1$

(b) A_1 is very small as compared to A_2

(c) $\varepsilon_2 = 0$

(d) small body is at the centre

12. What is the equivalent emmissivity for radiant heat exchange between a small body (emissivity = 0.4) in a very large enclosure (emissivity = 0.5)?

(a) 0.5 (b) 0.4

(c) 0.2 (d) 0.1

13. A radiation shield should

(a) have high transmissivity

(b) absorb all the radiations

(c) have high reflective power

(d) partly absorb and partly transmit the incident radiation

14. Two long parallel plates of same emissivity 0.5 are maintained at different temperatures and have radiation heat exchange between them. A radiation shield of emissivity 0.25 placed in the middle will reduce radiation heat exchange to

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{3}{10}$ (d) $\frac{3}{5}$

15. A thin shield of emssivity ϵ_3 (on both sides) is placed between two infinite parallel plates of emissivities ϵ_1 and ϵ_2 and temperatures T_1 and T_2 respectively. If $\epsilon_1 = \epsilon_2 = \epsilon_3$, then the fraction radiant energy transfer without shield/with shield takes the value

(a) 0.25 (b) 0.50

(c) 0.75 (d) 1.25

16. Two long parallel surfaces, each of emissivity 0.7 are maintained at different temperatures and accordingly have radiation exchange between them. It is desired to reduce 75% of this radiant heat transfer by inserting thin parallel shields of equal emissivity 0.7 on both sides. What should be the number of shields?

(a) 1 (b) 2

(c) 3 (d) 4

17. The grey body shape factor for radiant heat exchange between a small body (emissivity 0.4) in a large enclosure (emissivity 0.5) is

(a) 0.1 (b) 0.2

(c) 0.4 (d) 0.5

18. An enclosure consists of four surfaces 1, 2, 3 and 4. The view factors for radiation heat transfer are:

$F_{11} = 0.1$; $F_{12} = 0.4$ and $F_{13} = 0.25$

The surface areas A_1 and A_4 are 4 m² and 2 m² respectively. The view factor F_{41} is

(a) 0.75 (b) 0.50

(c) 0.25 (d) 0.1

Answers :

1. (d)	2. (c)	3. (b)	4. (b)	5. (c)
6. (b)	7. (b)	8. (d)	9. (d)	10. (b)
11. (b)	12. (b)	13. (c)	14. (c)	15. (b)
16. (c)	17. (c)	18. (b)		

HINTS AND COMMENTS

2(c):

For a flat or convex surface, the shape factor with respect to itself is zero. This aspect stems from the fact that for any part of flat or convex surface, one cannot see any other part of the same surface.

15(b):

The ratio of radiant energy transfer without and with shield is given by

$$\frac{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right)}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_3} - 1\right) + \left(\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2} - 1\right)}$$

When $\epsilon_1 = \epsilon_2 = \epsilon_3$, the above fraction takes the value $\frac{1}{2}$.

16(c):

Let N be the required number of shields. When emissivities of the main radiating surfaces and those of parallel radiation shields are equal, then the rates of heat tramsfer with and without shields are prescribed by the relation

$$\frac{\text{without shields}}{\text{with shields}} = \frac{1}{N+1}$$

We are given that,

$$(Q)_{\text{shielded}} = (1 - 0.75)\ (Q)_{\text{unshielded}}$$

or $\frac{1}{N+1} = 0.25$ or $N = 3$

17(c):

$$(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2} \times \frac{A_1}{A_2}}$$

The configuration corresponds to a completely enclosed body, and small compared with the enclosing body. That is

$A_1 <<< A_2$ and $F_{12} = 1$

Hence, $(F_g)_{12} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1 + 0} = \epsilon_1$

$= 0.4$

18(b):

$F_{11} + F_{12} + F_{13} + F_{14} = 1$

or $F_{14} = 1 - (F_{11} + F_{12} + F_{13})$

$= 1 - (0.1 + 0.4 + 0.25)$

$= 0.25$

Invoking reciprocity theorem,

$A_1\,F_{14} = A_4\,F_{41}$

$\therefore \quad F_{41} = \frac{A_1}{A_4} F_{14}$

$= \frac{4}{2} \times 0.25 = 0.5$

❑❑❑

Chapter

9 Convection : Processes and Properties

Learning objectives : After study of this chapter, the reader will be able to
- *understand free and forced convection*
- *write the convective heat rate equation, and define the convective heat transfer coefficient*
- *define Nusselt modulus, state its physical significance and list the methods used for its estimation*

Thermal convection occurs when a temperature difference exists between a solid surface and a fluid flowing past it. Convection is essentially a process of energy transport affected by the circulation or mixing of a fluid medium which may be a gas, a liquid or a powdery substance. The transport of heat energy during convection is directly linked with the transport of medium itself, and as such convection presents a combined problem of conduction, fluid flow and mixing. This chapter is concerned with brief introduction to free and forced convection; the convection rate equation, the convection film coefficient and the different parameters affecting the convection phenomenon.

9.1. FREE AND FORCED CONVECTION

With respect to the cause of fluid circulation or flow, two types of convection are distinguished:

Free Convection : Circulation of bulk fluid motion is caused by changes in fluid density resulting from temperature gradients between the solid surface and the main mass of fluid. The stagnant layer of fluid in the immediate vicinity of the hot body gets thermal energy by conduction. The energy thus transferred serves to increase the temperature and internal energy of fluid particles. Because of temperature rise, these particles become less dense and hence lighter than the surrounding fluid particles. The lighter fluid particles move upwards to a region of low temperature where they mix with and transfer a part of their energy to the cold particles. Simultaneously the cool heavier particles descend downwards to fill the space vacated by the warm fluid particles. The circulation pattern, upward movement of the warm fluid and downward movement of cool fluid, is called ***convection currents***. These currents are setup naturally due to gravity alone and are responsible for heat convection.

Designers of furnaces, house heating systems, architectural projects, roads and concrete structures will be concerned with free convection. Since there are no density forces (no gravitational field) in the orbiting satellites, the space vehicles with a zero gravity trajectory, free convection would be non-existent in such vehicles.

Forced Convection : Flow of fluid is caused by a pump, a fan or by the atmospheric winds. These mechanical devices provide a definite circuit for the circulating currents and that speeds up the heat transfer rate. Examples of forced convection are cooling of internal combustion engines, air conditioning installations and nuclear reactors, condenser tubes and other heat exchange equipment.

9.2. LAMINAR AND TURBULENT FLOW

The convection heat is affected to an appreciable extent by the nature of fluid flow. In the realms of fluid mechanics, essentially two types of fluid flow are characterised :

Laminar flow : The fluid particles move in flat or curved un-mixing layers or streams and follow a smooth continuous path. The paths of fluid movement are well-defined and the fluid particles retain their relative positions at successive cross-sections of the flow passage. There is no transverse displacement of fluid particles; the particles remain in an orderly sequence in each layer. Soldiers on a parade provide a somewhat crude analogy to laminar flow.

Turbulent flow : The motion of fluid particles is irregular, and it proceeds along erratic and unpredictable paths. The stream lines are interwined and they change in position from instant to instant. Fluctuating transverse velocity components are superimposed on the main flow, and the velocity of individual fluid elements fluctuate both along the direction of flow and perpendicular to it. Obviously a turbulent flow is eddying and sinuous rather than rectilinear in character. The turbulent flow resembles a crowd of commuters in a rail road station during the rush hour.

Osborne Reynolds, an English scientist, confirmed the existence of these two regimes experimentally and postulated that under certain conditions there could be transition from laminar to turbulent flow and vice versa. His investigations revealed that the nature of fluid flow is governed by the following parameters :

- mean flow velocity V
- density of fluid ρ
- dynamic viscosity of the fluid μ
- characteristic dimension of the flow passage, for example the diameter d of the pipe.

A grouping of these variables results into a dimensionless quantity, $R_e = Vd\rho/\mu$, called the Reynolds number. This number represents the ratio of inertia to viscous forces. At low Reynolds number, the viscous forces predominate and the flow is laminar. At high values of Reynolds number, the inertia forces overcome the viscous friction forces and consequently the fluid layers break up into a turbulent flow.

For fluid flow through a pipe, low Reynolds number upto 2300 is indicative of laminar flow. From $R_e = 2300$ to 6000, the laminar flow begins a transition to turbulent flow. Usually the flow is completely turbulent at $R_e = 6000$.

In many flow situations, the duct is not circular but is rectangular, trapezoidal or even an annulus formed by a tube within another tube. In that case, the characteristic dimension d in the relation $R_e = Vd\rho/\mu$ is the equivalent (hydraulic) diameter which is defined as four times the corss-sectional flow area divided by the wetted perimeter.

Equivalent diameter d_e

$$= 4 \times \frac{\text{cross-sectional flow area}}{\text{wetted perimeter}} \quad ...(9.1)$$

Thus for a duct of rectangular cross-section with length l and breadth b

$$d_e = 4\frac{l \times b}{2l + 2b} = \frac{2lb}{l + b} \quad ...(9.2)$$

If the annulus (a flow passage formed by a tube within a tube) has an inner diameter (outer diameter of inner tube) of d_1 and an outer diameter (inner diameter of outer tube) of d_2 then the equivalent diameter is

$$d_e = 4\frac{\frac{\pi}{4}\left(d_2^2 - d_1^2\right)}{\pi(d_1 + d_2)} = d_2 - d_1 \quad ...(9$$

9.3. NEWTON–RIKHMAN LAW : CONVECTION RATE EQUATION

Regardless of the particular nature (free or forced), the appropriate rate equation for the convective heat transfer between a surface and an adjacent fluid is prescribed by Newton's law of cooling

$$Q = hA\,(t_s - t_f) \quad \text{...(9.4)}$$

where Q is the convective heat flow rate, A is the surface area exposed to heat transfer, t_s is the surface temperature of the solid, and t_f is the undisturbed temperature of the fluid. The constant of proportionality h relates the heat transfer per unit time and unit area to the overall temperature difference. The unit of h are W/m²-deg, and it is referred to as convective heat transfer coefficient, the surface conductance or the film coefficient.

The value of film coefficient is dependent upon :

- surface conditions : roughness and cleanliness
- geometry and orientation of the surface : plate, tube and cylinder placed vertically or horizontally
- thermo-physical properties of the fluid : density, viscosity, specific heat, coefficient of expansion and thermal conductivity
- nature of fluid flow : laminar or turbulent
- boundary layer configuration
- prevailing thermal conditions

Typical values of convective coefficient are given below in Table 9.1.

Convection mechanisms involving phase changes lead to important field of boiling (evaporation) and condensation. The convection coefficients for boiling and condensation lie in the range 2500 - 100,000 W/m²K.

EXAMPLE 9.1

A motor cycle cylinder consists of ten fins, each 150 mm outside diameter and 75 mm inside diameter. The average fin temperature is 500°C and the surrounding air is at 20°C temperature. Make calculations for the rate of heat dissipation from the cylinder fins by convection when (i) motor cycle is stationary and convection coefficient h = 6 W/m²K (ii) motor cycle is moving at 60 km/hr and h = 75 W/m²K.

Solution : Since both sides of each fin are exposed to the surroundings air, the surface area for convective heat transfer is :

$$A = 10 \times \left[2 \times \frac{\pi}{4}(0.15^2 - 0.075^2)\right]$$
$$= 0.265 \text{ m}^2$$

The heat flow from the surface to the fluid is given by Newton's law of cooling :

$$Q = hA\,(t_s - t_f)$$

Therefore, when the motor cyle is stationary

$$Q = 6 \times 0.265\,(500 - 20) = 763.2 \text{ W}$$

and when the motor cycle is moving,

$$Q = 75 \times 0.265\,(500 - 20)$$
$$= 9540 \text{ W}$$

EXAMPLE 9.2

Forced air flows over a convective heat exchanger in a room heater, resulting in a convective heat transfer coefficient 1.136 kW/m²K. The surface temperature of heat exchanger may be considered constant at 65°C, and the air is at 20°C. Determine the heat exchanger surface area required for 8.8 kW of heating.

Table 9.1. Typical values of convective coefficient

Free convection		Forced Convection	
(i) air	3 - 7 W/m²K	(i) air and superheated steam	30-300 W/m²K
(ii) gases	2 - 20	(ii) oil	60 - 3000
(iii) liquids	30 - 300	(iii) water	300 - 10,000

Solution : The convective heat flow from a solid surface to the surrounding fluid is given by Newton's law of cooling

$$Q = hA\,(t_s - t_f)$$

Substituting the given data, we obtain

$$8.8 = 1.136A\,(65 - 20)$$

Therefore heat exchanger surface area,

$$A = \frac{8.8}{1.136 \times 45} = 0.172 \text{ m}^2$$

9.4. NUSSELT NUMBER

Consider a heated wall surface at temperature t_s over which a fluid is flowing with undisturbed velocity U_∞ and temperature t_∞. The particles of fluid in intimate contact with the plate tend to adhere to it, and a region of variable velocity builds up between the plate surface and the free fluid stream as indicated in Fig. 9.1.

The fluid velocity decreases as it approaches the solid surface, reaching to zero (no slip condition) in the fluid layer immediately next to the surface. This thin layer of stagnated fluid has been called the *hydrodynamic boundary layer*. The quantity of heat transferred is highly dependent upon the fluid motion within this boundary layer, being determined chiefly by the thickness of the layer. The boundary layer thickness δ is arbitrarily defined as the distance y from the plate surface at which the velocity approaches 99% of free stream velocity.

Likewise a region of fluid motion near the plate in which temperature gradients exist is the thermal boundary layer and its thickness δ_t is defined as the value of transverse distance y from the plate surface at which

$$\frac{t_s - t}{t_s - t_\infty} = 0.99$$

At the plate surface, there is no fluid motion and the energy transport can occur only by conduction. From energy balance, this heat transport must equal the heat transferred by convection into the rest of the fluid. Thus

$$Q = -kA\left(\frac{\partial t}{\partial y}\right)_{y=0}$$
$$= hA\,(t_s - t_\infty) \quad \text{...(9.5)}$$

Heat flow rate is thus dependent upon temperature gradient at the wall, and the temperature gradient is influenced by the fluid velocity; high temperature gradients are associated with the higher velocities.

If temperature field of the fluid varies only in the direction of the coordinate normal to the plate surface, then

$$h = -\frac{k}{(t_s - t_\infty)}\left(\frac{dt}{dy}\right)_{y=0} \quad \text{...(9.6)}$$

Thus, the convective coefficient h can be evaluated from a knowledge of fluid temperature distribution in the neighbourhood of the surface.

Introducing a characteristic dimension l, the equation 9.6 can be recast as

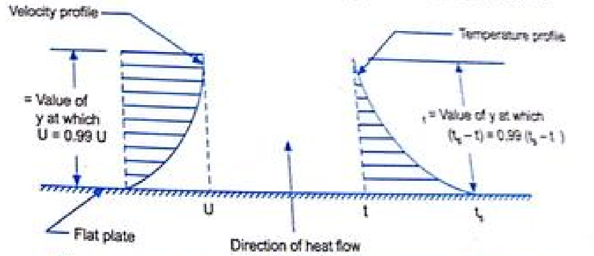

Fig. 9.1. Velocity and temperature profiles in convective heat transfer

$$\frac{hl}{k} = \frac{-(dt/dy)_{y=0}}{(t_s - t_\infty)/l}$$

$$= \left[\frac{d\{(t_s - t)/(t_s - t_\infty)\}}{d(y/l)}\right]_{y=0} \quad ...(9.7)$$

The dimensionless parameter hl/k is called ***Nusselt number***. Apparently the Nusselt number may be interpreted as the ratio of temperature gradient at the surface to an overall or reference temperature gradient. The parameter

$$\left[\frac{d\{(t_s - t)/(t_s - t_\infty)\}}{d(y/l)}\right]_{y=0}$$

represents the dimensionless slope of the temperature distribution curve at the surface.

The Nusselt number is a convenient measure of the convective heat transfer coefficient. For a given value of Nusselt number, the convective surface coefficient h is directly proportional to thermal conductivity k of the fluid, and inversely proportional to the significant length l.

EXAMPLE 9.3

The temperature profile at a particular location in a thermal boundary layer is prescribed by an expression of the form :

$$t(y) = A - By + Cy^2$$

where A, B and C are constants. Set up an expression for the corresponding heat transfer coefficient.

Solution : The heat transfer coefficient is given by the expression

$$h = -\frac{k}{(t_s - t_\infty)}\left(\frac{dt}{dy}\right)_{y=0}$$

From the given boundary layer temperature profile

$$t = A - By + Cy^2;$$

$$\frac{dt}{dy} = -B + 2Cy$$

$$\left(\frac{dt}{dy}\right)_{y=0} = -B$$

$$\therefore \quad h = \frac{-k}{(t_s - t_\infty)}(-B) = \frac{Bk}{(t_s - t_\infty)}$$

EXAMPLE 9.4

The temperature profile at a particular location on the surface of plate is prescribed by the identities :

(i) $\dfrac{t_s - t}{t_s - t_\infty} = \sin\left(\dfrac{\pi y}{0.015}\right)$

(ii) $\dfrac{t_s - t}{t_s - t_\infty} = \dfrac{1}{2}\left(\dfrac{y}{0.0075}\right)^3 + \dfrac{3}{2}\left(\dfrac{y}{0.0075}\right)$

If thermal conductivity of air is stated to be 0.03 W/m-deg, determine the value of convective heat transfer coefficient in each case.

Solution : The convective heat transfer coefficient is prescribed by the relation

$$h = -\frac{k}{(t_s - t_\infty)}\left(\frac{dt}{dy}\right)_{y=0}$$

(i) $$\frac{d}{dy}\left[\frac{t_s - t}{t_s - t_\infty}\right] = d\left[\sin\left(\frac{\pi y}{0.015}\right)\right]$$

$$= \frac{\pi}{0.015}\cos\left(\frac{\pi y}{0.015}\right)$$

or $$\left(\frac{dt}{dy}\right)_{y=0} = -(t_s - t_\infty)\frac{\pi}{0.015}(\cos 0)$$

$$= -\frac{\pi}{0.015}(t_s - t_\infty)$$

$$\therefore h = -\frac{k}{(t_s - t_\infty)} \times -\left\{\frac{\pi}{0.015}(t_s - t_\infty)\right\}$$

$$\frac{\pi k}{0.015} = \frac{\pi \times 0.03}{0.015} = \mathbf{6.28\ W/m^2K}$$

(ii) $$\frac{d}{dy}\left(\frac{t_s - t}{t_s - t_\infty}\right) =$$

$$\frac{d}{dy}\left[\frac{1}{2}\left(\frac{y}{0.0075}\right)^3 + \frac{3}{2}\left(\frac{y}{0.0075}\right)\right]$$

$$= \frac{3}{2}\left(\frac{y}{0.0075}\right)^2 \times \frac{1}{0.0075} + \frac{3}{2} \times \frac{1}{0.0075}$$

or $$\left(\frac{dt}{dy}\right)_{y=0} = -(t_s - t_\infty) \times \left(\frac{3}{2 \times 0.0075}\right)$$

$$\therefore \quad h = -\frac{k}{(t_s - t_\infty)} \times \left[-(t_s - t_\infty) \times \frac{3}{2 \times 0.0075}\right]$$

$$= \frac{3k}{2 \times 0.0075}$$

$$= \frac{3 \times 0.03}{2 \times 0.0075} = \mathbf{6\ W/m^2K}$$

EXAMPLE 9.5

Air at 20°C flows over a flat plate maintained at 75°C. Measurements show that temperature at a distance of 0.5 mm from the surface of plate is 50°C. Presuming thermal conductivity of air as 0.0266 W/m-deg, estimate the value of local heat transfer coefficient.

Solution : The heat transfer coefficient is prescribed by the relation

$$h = -\frac{k}{(t_s - t_\infty)}\left(\frac{dt}{dy}\right)_{y=0}$$

Assuming linear variation of temperature,

$$\frac{dt}{dy} = \frac{50 - 75}{0.0005} = -50 \times 10^3\ °C/m$$

$$\therefore \quad h = -\frac{0.0266}{75 - 20} \times (-50 \times 10^3)$$

$$= \mathbf{24.18\ W/m^2\text{-deg}}$$

EXAMPLE 9.6

Air at 20°C flows over a flat surface maintained at 80°C. Estimate the value of local heat transfer coefficient if the local heat flow at a point was measured as 1250 W/m². Proceed to calculate the temperature gradient at the surface and the temperature at a distance of 0.5 mm from the surface. Take thermal conductivity of air as

0.028 W/m-deg

Solution : The convective heat flow is prescribed by the relation

$$Q = hA\,(t_s - t_\infty)$$

or $1250 = h \times 1 \times (80 - 20)$

$\therefore \quad h = 20.83\ W/m^2\text{-deg}$

Then from the relation :

$$h = -\frac{k}{(t_s - t_\infty)}\left(\frac{dt}{dy}\right)_{y=0}$$

Temperature gradient at the surface,

$$\left(\frac{dt}{dy}\right)_{y=0} = -\frac{h}{k}(t_s - t_\infty)$$

$$= -\frac{20.83}{0.028} \times (80 - 20)$$

$$= -44636\ °C/m$$

Temperature at 0.5 mm from the surface is

$$= 80 + \left(\frac{dt}{dy}\right)_{y=0} \times 0.0005$$

$$= 80 + (-44636) \times 0.0005$$

$$= 80 - 22.318 = 57.682\ °C$$

EXAMPLE 9.7

Air enters a rectangular duct measuring 30 cm × 40 cm with a velocity of 8.5 m/s and a temperature of 40°C. The flowing air has a thermal conductivity 0.028 W/m-deg, kinematic viscosity 16.95 × 10⁻⁶ m²/s and from empirical correlations the Nusselt number has been approximated to be 425. Work out the equivalent diameter of the flow passage, the flow Reynolds number and the convective heat flow coefficient.

Solution : For a duct of rectangular cross section with length l and breadth b, the equivalent or hydraulic diameter is given by

$$d_e = \frac{2lb}{l + b}$$

$$= \frac{2 \times 0.4 \times 0.3}{0.4 + 0.3} = 0.3428\ m$$

(ii) Reynolds number R_e is equal to

$$\frac{V d_e \rho}{\mu} = \frac{V d_e}{\nu}$$

$$= \frac{8.5 \times 0.3428}{16.95 \times 10^{-6}} = 0.1719$$

Clearly the flow is turbulent in character.

(*iii*) The Nusselt number and the convective film coefficient are related to each other by the expression

$$N_u = \frac{h\,l}{k} = \frac{h d_e}{k}$$

$\therefore$ Convective coefficient h

$$= \frac{N_u k}{d_e} = \frac{425 \times 0.028}{0.3428}$$

$$= 34.71 \text{ W/m}^2\text{K}$$

9.5. DETERMINATION OF NUSSELT NUMBER

The determination of the value of Nusselt number or the convective film coefficient forms a basis for the computation of heat transfer by convection. Towards that end, the following approaches have been suggested :

(*a*) Dimensional analysis and experimental correlations

(*b*) Hydrodynamic concept of velocity boundary layer and the analogous concept of temperature boundary layer

(*c*) Reynolds similarity between the mechanism of fluid friction in the boundary layer and the transfer of heat by convection.

A basic knowledge of each of these methods is essential for an engineer engaged in the problems connected with heat exchange. Considerations would be given in the subsequent sections on the fundamental principles of these techniques of convective heat flow. Convection mechanism involving phase change (boiling and condensation) and the basis of heat exchange equipment have also been dealt with.

SALIENT POINTS

1. *Convection is the transfer of heat within a fluid by mixing of one portion of the fluid with another. Convection is possible only in a fluid medium and is directly linked with the transport of medium itself.*
2. *The effectiveness of heat transfer by convection depends largely upon the mixing motion of the fluid.*
3. *Depending upon the nature of forces which cause mixing, convective heat transfer is classified as*
   (*i*) **Free or natural convection:** *Temperature gradient in the fluid causes density gradient which sets the fluid into motion.*
   (*ii*) **Forced convection:** *Motion of fluid is caused by an external agent such as fan or pump.*
4. *The rate of heat transfer by convection from the surface of a conducting body to the surrounding fluid is given by*

   $$Q = hA\,(t_s - t_o)$$

   where t_s is the temperature of the surface, t_o is the temperature of fluid and A is the surface area of the body.

   The parameter h is the convective heat transfer coefficient having units W/m^2-deg. The convective coefficient is a complicated function of
   (*i*) geometry of the system
   (*ii*) thermal properties of the fluid
   (*iii*) characteristics of fluid flow
5. At the interface of solid body, heat flows by conduction and therefore

   $$-kA\left(\frac{\partial t}{\partial y}\right)_{y=0} = hA\,(t_s - t_o)$$

   That gives:

   $$\frac{hl}{k} = \frac{-\left(\dfrac{\partial t}{\partial y}\right)_{y=0}}{\dfrac{t_s - t_o}{l}}$$

   The length parameter l specifies the geometry of the solid body. The non-dimension parameter $\frac{hl}{k}$ is called Nusselt number or Nusselt modulus Nu and it physically signifies the ratio of the temperature gradient at the surface to a reference temperature gradient.
6. The following methods have been suggested for estimating the Nusselt number which is a convenient measure of convective heat transfer coefficient:
   - Dimensional analysis coupled with experimental correlations
   - Approximate analysis or exact numerical analysis of boundary layer equations
   - Analogy between mass, momentum and heat transfer.

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. (*i*) What is the difference between a laminar flow and a turbulent flow.
   (*ii*) Define Reynolds number and mention its physical significance.
   (*iii*) What are the generally accepted values of critical Reynolds number for flow over a flat plate and flow in a tube.
2. Differentiate between mechanims of heat transfer by free and forced convection. Mention some of the areas where these mechanisms are predominant.
3. Explain the phenomenon of heat transfer by free convection. What forces control the fluid motion? Can free convection occur in space vehicles with a zero's 'g' trajectory.
4. Explain the phenomenon of heat transfer by forced convection. What forces control the fluid motion ? Cite suitable examples to illustrate your answer.
5. Give a general equation for the rate of heat transfer by convection, and hence define the coefficient of heat transfer. List the various factors on which the value of this coefficient depends.
6. The forced convective heat transfer coefficient for a hot fluid flowing over a cool surface has been estimated to be 816 kJ/m^2-deg. The fluid temperature upstream of the cool surface is 120°C, and the surface is held at 10°C. Determine the heat transfer, per unit surface area, from the fluid to the surface.
   (Ans. 89.8 × 10^6 kJ/m^2-hr)
7. A plate 60 cm high and 30 cm wide, having a surface temperature at 35°C, is in contact with air at 20°C. If the observed convective heat transfer rate is 45 W for each side, compute the average convection coefficient.
   (Ans. 16.67 W/m^2 K)
8. An electric resistance heater at 125°C is being cooled by air at 60°C. What is the average convection coefficient if the heat flux at the heater surface is 5600 W/m^2 ? If the heat flux is reduced to 1800 W/m^2, what will be the heater temperature ? The convective film coefficient remains unchanged.
   (Ans. 86.15 W/m^2-deg, 80.89°C)
9. Define the Nusselt number. How it is related to temperature gradient in the fluid immediately in contact with the solid surface? Mention the various approaches which have suggested for estimating the value of Nusselt number.

B. Fill in the blanks with appropriate word/words :

1. .... is the process of energy transfer between a solid surface and a fluid system in motion.
2. The value of Reynolds number at which the flow pattern changes from laminar to turbulent motion is called ....
3. The characteristic dimension used in estimating the Reynolds number is the hydraulic diamater defined as .... times the cross-sectional area divided by the wetted parameter.
4. A region of fluid motion near a plate in which temperature gradients exist is ....
5. The .... physically signifies the ratio of temperature gradient at the surface to a reference temperature gradient.
6. Dimensional analysis coupled with experimental data provide a convenient method for the estimation of .... heat transfer coefficient.
7. For a given value of Nusselt number, the convection surface coefficient is .... proportional to thermal conductivity of the fluid, and .... proportional to the significance length.

Answers : 1. convection; 2. critical Reynolds number; 3. four; 4. thermal boundary layer; 5. Nusselt number; 6. convective; 7. directly, indirectly.

C. Multiple choice questions :

1. Which of the following heat flow situations pertains to free or natural convection ?
   (a) cooling of internal combustion engines
   (b) flow of water inside the condenser tubes
   (c) cooling of billets in atmosphere
   (d) air-conditioning installations and nuclear reactors

2. Mark the system where heat transfer is by forced convection
   (a) chilling effect of cold wind on warm body
   (b) fluid passing through the tubes of a condenser and other heat exchange equipment
   (c) heat flow from a hot pavement to surrounding atmosphere
   (d) heat exchange on the outside of cold and warm pipes

3. Forced convection in a liquid bath is caused by
   (a) density difference brought about by temperature gradients
   (b) molecular energy interaction
   (c) flow of electrons in a random fashion
   (d) intense stirring by an external agency

4. A finned tube hot water radiator with a fan blowing air over it is kept in rooms during winter. The major portion of the heat transfer from the radiator is due to
   (a) better conduction
   (b) convection to the air
   (c) radiation to the surroundings
   (d) combined conduction and radiation

5. A body cooling from 80°C to 70°C takes 10 minutes when left exposed to environmental conditions. If the body is to cool further from 70°C to 60°C under the same external conditions, it will take
   (a) same time of 10 minutes
   (b) more than 10 minutes
   (c) less than 10 minutes
   (d) time will depend upon the environmental conditions

6. A hot metal piece kept in air cools from 80°C to 70°C in t_1 seconds, from 70 to 60°C in t_2 seconds and from 60 to 50°C in t_3 seconds. Then
   (a) $t_1 = t_2 = t_3$
   (b) $t_1 < t_2 < t_3$
   (c) $t_1 > t_2 > t_3$
   (d) the relationship between t_1, t_2 and t_3 will depend upon the material of hot piece

7. A sphere, a cube and a thin circular plate, all made of the same material and having the same mass are initially heated to a temperature of 250°C. When left in air at room temperature, what will be their response to cooling ?
   (a) they will cool at the same rate
   (b) circular plate will cool at the slowest rate
   (c) sphere will cool faster
   (d) cube will cool faster than sphere but slower than the circular plate

8. Heat is being transferred by convection from water at 48°C to glass plate whose surface that is exposed to water is at 40°C.

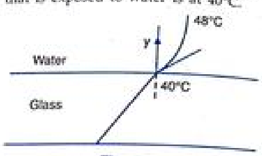

Fig. 9.2.

The thermal conductivity of water is 0.6 W/mK and the thermal conductivity of glass is 1.2 W/mK. The spatial gradient of temperature in the water at the water glass interface is $\frac{dT}{dy} = 1 \times 10^4$ K/m. The heat transfer coefficient h in W/m² K is
   (a) 0.0 (b) 4.8
   (c) 6.0 (d) 750

9. On a summer day, a scooter rider feels more comfortable while on the move than while at a stop light because
   (a) an object in motion captures less radiation
   (b) air is transparent to radiation and hence it is cooler than the body
   (c) air has a low specific heat and hence it is cooler
   (d) more heat is lost by convection and radiation while in motion

10. Choose the wrong statement with respect to Nusselt number and convective heat transfer coefficient
   (a) Nusselt number represents the ratio of temperature gradient at the surface to an overall or reference temperature gradient
   (b) Nusselt number represents the dimensionless slope of the temperature distribution curve at the surface
   (c) The convective coefficient can be evaluated from a knowledge of fluid temperature distribution in the neighbourhood of the surface
   (d) For a given Nusselt number, the convective coefficient is inversely proportional to thermal conductivity of the fluid

Answers :

1. (c)	2. (b)	3. (d)	4. (b)	5. (b)
6. (b)	7. (d)	8. (d)	9. (d)	10. (d)

HINTS AND COMMENTS

5(b), 6(b):

$$Q = hA\,(t_b - t_a)$$

Apparently the cooling depends upon the temperature difference $(t_b - t_a)$ where t_b and t_a represent the temperature of the body and that of surrounding air respectively. Initially this temperature difference $(80 - t_a)$ is more and so the cooling will be faster. There after the difference falls to $(70 - t_a)$ and so more time will be taken to cool further for the same temperature of 10°C.

7(d):

$$Q = hA\,(t_b - t_a)$$

Apparently with identical operating conditions of temperature difference and convective coefficient, the cooling depends upon the surface area A of the body. For the same mass of material used in fabrication, the surface area of the sphere, a cube and a thin plate conform to the relation

$$A_{plate} > A_{cube} > A_{sphere}$$

i.e., the plate has the maximum area and sphere has the minimum area. As such the cube will cool faster than sphere but slower than circular plate.

8(d):

For steady state heat flow,

$$q = h\,\Delta T = k_w \left(\frac{dT}{dy}\right)_w$$

$$= 0.6 \times (1 \times 10^4) = 6000$$

$$\therefore \quad h = \frac{6000}{48 - 40} = 750 \text{ W/m}^2\text{ K}$$

9(d):

The situation corresponds to forced convection when the scooter is in motion and the convective heat transfer coefficient for forced convection is greater than that for free convection (scooter stationary).

Chapter

10

Dimensional Analysis

Learning objectives : After studying the subject matter presented in this chapter, the student should be able to

- *understand the system of dimensions and dimensional homogeneity*
- *specify the different variables affecting the mechanisms of free convection and forced convection*
- *perform dimensional analysis and correlate experimental data in terms of dimensional groups which deal with convective heat transfer*
- *discuss the physical significance of dimensionless groups*
- *know about principle of similarity and model studies*

The subject of dimensional analysis deals with the process whereby all the important variables involved in a physical phenomenon are systematically organised into dimensionless groups which are less numerous than the original variables. For example, a law expressing the relation between fluid density ρ, viscosity μ, velocity V and a length parameter l; is designated by a dimensionless number R_e known as Reynolds number ($R_e = V l \rho/\mu$). The number of unknown quantities is thus reduced, the problem is generalised and the need for specifying a particular system of units is eliminated. Such dimensionless grouping facilitates the interpretation and extend the range of application of experimental data. Correlation of experimental data with the help of dimensional analysis is fruitfully utilised to develop empirical relations describing a particular phenomenon.

This chapter presents the general method of dimensional analysis and illustrates its application to various problems of fluid mechanics and convective heat transfer.

10.1. SYSTEM OF DIMENSIONS

Dimensions refer to the qualitative characteristics for physical qunatities, while units are standards of comparison for the quantitative measure of dimensions. For elaborating the difference, let us consider the distance between two points. The term length when applied to it gives the qualitative concept of this physical quantity. The term unit would, however, indicate the magnitude of the distance. The distance may be quantitatively expressed as a metre or a mile. The metre and mile will then be identical dimensions. However they will be different units as each contains a different quantity of length.

The most common system of dimensioning a physicasl quantity is the mass-length-time referred to as the MLT system of units. There is no direct relation between the quantities mass, length and time. These independent quantities are called ***fundamental quantities***. All other physical quantities such as pressure, velocity and energy etc. are expressed in terms of these fundamental quantities and are called

the ***derived or secondary quantities***. Newton's second law of motion gives a relation between force and mass, and allows force to be expressed dimensionally as :

$$F = \text{mass} \times \text{acceleration}$$
$$= ML/T^2 = MLT^{-2}$$

In the force-length-time (FLT) system of units, the force is considered as a fundamental quantity with mass expressed in terms of force, length and time ($M = FT^2 L^{-1}$) .

In the realms of heat transfer, two more dimensions namely the temperature difference θ and the heat H are also taken as fundamental quantities. Dimensions of some of the parameters relevant to the field of fluid mechanics and heat transfer have been listed in Table 10.1. All the variables are expressed dimensionally as some combination of mass, length, time, temperature and heat. The variables include those describing the system geometry and thermo-physical properties of the fluid.

10.2. DIMENSIONAL HOMOGENEITY AND ITS APPLICATIONS

The fundamental theory of dimensional analysis is based on the following axiom :

"Equations describing a physical phenomena must be dimensionally homogeneous and units therein must be consistent ."

Table 10.1. Dimensions of geometrical and thermo-physical properties

Variables	Symbol	Units		Dimensions	
		MKS	SI	MLTθ H	MLTθ
Mass	M	kg	kg	M	M
Length	L	m	m	L	L
Time	T	s	s	T	T
Temperature	θ	°C	K	θ	θ
Heat	H, Q	cal	J	H	ML^2T^{-2}
Area	A	m^2	m^2	L^2	L^2
Volume	V	m^3	m^3	L^3	L^3
Velocity	V, U	m/s	m/s	LT^{-1}	LT^{-1}
Acceleration	a	m/s^2	m/s^2	LT^{-2}	LT^{-2}
Gravity	g	m/s^2	m/s^2	LT^{-2}	LT^{-2}
Force or resistance	F, R	kgf	N	MLT^{-2}	MLT^{-2}
Density	ρ	kg/m^3	kg/m^3	ML^{-3}	ML^{-3}
Dynamic viscosity	μ	kg/ms	kg/ms	$ML^{-1}T^{-1}$	$ML^{-1}T^{-1}$
Kinematic viscosity	ν	m^2/s	m^2/s	L^2T^{-1}	L^2T^{-1}
Energy, work	E, W	m kg_f	m N	ML^2T^{-2}	$ML^2 T^{-2}$
Convective film coefficient	h	kcal/m^2-hr-deg	W/m^2-deg	$HL^{-2}T^{-1}\theta^{-1}$	$MT^{-3}\theta^{-1}$
Coeffiecient of volumetric expansion	β	per deg	per deg	θ^{-1}	θ^{-1}
Specific heat	C_p	kcal /kg-deg	kJ/kg-deg	$HM^{-1}\theta^{-1}$	$LT^{-2}\theta^{-1}$
Thermal conductivity	k	kcal/m-hr-deg	W/m-deg	$HL^{-1}T^{-1}\theta^{-1}$	$MLT^{-3}\theta^{-1}$

Note: *Heat is energy with dimensions of mkg_f or Nm and therefore it can be expressed in terms of mass, length and time. Obviously dimensionless groupings for heat transfer problems can be and are quite often obtained* by *taking M-L-T-θ as the fundamental quantities and expressing all other variables in terms of MLTθ.*

An equation is said to be dimensionally homogeneous if the dimensions of various terms on the two sides of the equations are identical. A dimensionally homogeneous equation is independent of the fundamental units of measurement if the units therein are consistent. This implies that a length dimension can be added to or subtracted from only a length dimension. Addition of area to volume is meaningless since area and volume are dimensionally incompatible. Consistency of units demands that if one term of an equation is measured in a particular unit of velocity say m/s, then all terms in the equation must be measured in the same units of velocity.

The concept of dimensional homogeneity can be elaborated by considering the time period of oscillation T of a simple pendulum of length l and mass m :

$$T = 2\pi\sqrt{\frac{l}{g}} \quad ...(10.1)$$

Inserting the relevant dimensions on both sides of the equation ,

$$[T] = [1]\left[\frac{L}{LT^{-2}}\right]^{1/2} = [T]$$

The dimension of 2π, being constant , is taken as unity. The equation is thus dimensionally homogeneous because dimensions on both sides of the equation are same. Substituting the value of gravitational acceleration $g = 9.807$ m/s^2 , equation 10.1 is simplified to

$$T = 2\pi\sqrt{\frac{l}{9.807}} = 2.006\sqrt{l}$$

...(10.2)

This modified equation is not dimensionally homogeneous because the right hand side does not have the dimensions of time. Further it is valid only when the time is measured in seconds and length is measured in metres.

Applications : The principle of dimensional homogeneity serves the following useful purposes:

(*i*) It helps to check whether an equation of any physical phenomenon is dimensionally homogeneous or not.

Consider the equation of friction loss h_f in a pipe of length l and diameter d through which fluid flows with velocity V.

$$h_f = 4f\frac{l}{d}\frac{V^2}{2g} \quad ...(10.3)$$

where f is any constant with no dimensions whatsoever. Inserting the relevant dimensions:

$$[L] = [1]\left[\frac{L}{L}\right]\left[\frac{(LT^{-1})^2}{LT^{-2}}\right] = [L]$$

Dimensions on both sides of the equation are same and so the equation is dimensionally homogenous.

Again consider the energy (Bernoulli's) equations for fluid flow along a stream line

$$\frac{p}{w}+\frac{V^2}{2g}+y = \text{constant } H$$

...(10.4)

$$\left[\frac{ML^{-1}T^{-2}}{ML^{-2}T^{-2}}\right]+\left[\frac{(LT^{-1})^2}{LT^{-2}}\right]+[L] = \text{constant } H$$

$$[L] + [L] + [L] = \text{constant } H$$

Each side on the left hand side has identical units of length . Thus if the equation 10.4 is to be rational one, *i.e.*, dimensionally homogeneous, the constant H appearing on the right hand side should also be in legnth units.

(*ii*) It helps to determine the dimensions of a physical quantity.

Consider the hydrostatic equation $p = w h$ where p is the pressure intensity, h is the height of fluid column and w is the specific weight of the fluid.

Dimensionally :

$$p \equiv [ML^{-2}T^{-2}][L] = [ML^{-1}T^{-2}]$$

Further if w is taken in N/m^3 and h in metre of fluid column, then units of pressure intensity would be :

$$p = \left(\frac{N}{m^3}\right)m = \left(\frac{N}{m^2}\right)$$

Again, consider the Fourier' s rate equation,

$$Q = -kA\frac{dt}{dx} \;;$$

$$k = \frac{-Q}{A}\frac{dx}{dt} \quad ...(10.5)$$

where Q is the heat flow rate, k is the thermal conductivity, A is the area, dt is the temperature difference and dx is the thickness of slab through which conduction occurs. Dimensionally

$$[k] = \left[\frac{H}{T}\right]\left[\frac{1}{L^2}\right][L]\left[\frac{1}{\theta}\right]$$

$$= \left[\frac{H}{LT\theta}\right] = [HL^{-1}T^{-1}\theta^{-1}]$$

Further if H is in kJ, L is in metre, T is in hour and θ is in degrees celsius, then units of thermal conductivity would be ;

$$k = \text{kJ/m-hr-deg}$$

(*iii*) Dimensional homogeneity helps to convert the units from one system to another.

Let it be required to convert pressure from kg$_f$/cm^2 to N/m^2 . Recalling that

1 kg$_f$ = 9.807 N ; 1 cm = 0.01 m

Conversion factor

$$= \frac{\text{kg}_f/\text{cm}^2}{\text{N}/\text{m}^2} = \left(\frac{\text{kg}_f}{\text{N}}\right)\left(\frac{\text{m}}{\text{cm}}\right)^2$$

$$= 9.807 \times \left(\frac{1}{0.01}\right)^2 = 9.807 \times 10^4$$

Thus any pressure in kg$_f$/cm^2 can be converted to N/m^2 by multiplying it with the factor 9.807×10^4.

Now let it be required to covert convective film coefficient from kcal/m^2-hr-deg to W/m^2- deg (J/m^2-s-deg). Recalling that

1 kcal = 4.186 kJ = 4186 J ;

1 hr = 3600 sec

Conversion factor

$$= \frac{\text{kcal}/\text{m}^2\text{-hr-deg}}{\text{J}/\text{m}^2\text{-s-deg}}$$

$$= \left(\frac{\text{kcal}}{\text{J}}\right)\left(\frac{\text{sec}}{\text{hr}}\right)$$

$$= 4186 \times \frac{1}{3600} = 1.1627$$

Thus the convective film coefficient in kcal/m^2-hr-deg can be converted to J/m^2-s-deg by multiplying it with the factor 1.1627

(*iv*) The concept of dimensional homogeneity is a step towards dimensional analysis which is fruitfully employed to plan experiments and to present the results meaningfully.

10.3. DIMENSIONAL ANALYSIS : RAYLEIGH'S METHOD

The method involves the following steps :

- Gather all the independent variables which are likely to influence the value of the dependent variable.
- Write the functional relationship, *i.e.*, if the dependent variable y is some function of the independent variables x_1, x_2, x_3........, then

$$y = f(x_1, x_2, x_3 ......) \quad ...(10.6)$$

- Write the above equation in the form

$$y = C\left(x_1^a \times x_2^b \times x_3^c ......\right) \quad ...(10.7)$$

where C is a dimensionless coefficient to be determined either from physical characteristics of the problem or through experimentation .

- Express each of the quantities of equation 10.7 in some fundamental units you choose for solution of the problem
- Utilize the principle of dimensional homogeneity to obtain a set of simultaneous equations involving the exponents a, b, c ....
- Solve the simultaneous equations t obtain the values of exponents a, b, c etc., the number of exponents involved is mo than the number of fundamental units, th the exponents of some of the variables evaluated in terms of other components. T

helps to group the variables into recognised dimensionless parameters .

- Substitute the values of exponents in the main equation, and form the non-dimensional parameters by grouping the variables with like exponents.

EXAMPLE 10.1

Show that the resistance R to the motion of a sphere of diameter D moving with uniform velocity V through a real fluid of density ρ and viscosity μ is given by :

$$R = \rho D^2 V^2 f\left(\frac{\mu}{\rho V D}\right)$$

where f stands for a function of.

Solution : It may be premised that the functional relationship is

$$R = f(D, V, \rho, \mu)$$

or $\quad R = C\,(D^a \times V^b \times \rho^c \times \mu^d) \quad ...(i)$

where C is a dimensionless coefficient. Using MLT- system, the corresponding equation for dimensions is,

$$[MLT^{-2}] = [1]\,[L]^a\,[LT^{-1}]^b\,[ML^{-3}]^c\,[ML^{-1}T^{-1}]^d$$

The dimensions of C being a constant is taken as unity.

For dimensional homogeneity, exponents of M, L and T are equated on both sides. The corresponding equations are :

$M: \quad 1 = c + d$

$L: \quad 1 = a + b - 3c - d$

$T: \quad -2 = -b - d$

We have only three equations to solve for four exponents. Therefore solution can be obtained in terms of one exponent which may be arbitrarily chosen. Experience shows that recognised dimensionaless groups appear when exponents of D, V and ρ are evaluated in terms of other exponents. Thus exponents a, b and c are to be expressed in terms of exponent d. By simple algebraic manipulations, one obtains :

$$c = 1 - d\ ;\ b = 2 - d\ ;\ a = 2 - d$$

Substituting these values of exponents in expression (i) ,

$$R = C\,[D^{2-d} \times V^{2-d} \times \rho^{1-d} \times \mu^d]$$

and collecting like terms ;

$$R = C\rho V^2 D^2 \left(\frac{\mu}{\rho V D}\right)^d$$

$$= \rho V^2 D^2 f\left(\frac{\mu}{\rho V D}\right)$$

10.4. DIMENSIONAL ANALYSIS : BUCKINGHAM'S PI-THEOREM

Rayleigh's method of dimensional analysis becomes increasingly laborious and cumbersome when a large number of physical variables are involved. The difficulty is then circumvented by using Buckingham's π-theorem that states :

" If there are n variables in a dimensionally homogeneous equation and if these variables contain m primary dimensions, then the variables can be grouped into (n − m) non-dimensional parameters." The non-dimensional groups are called *Pi*-terms.

Mathematically : given a physical equation,

$$f(x_1, x_2, x_3, \ldots\ldots x_n) = 0$$

where x's are dimensional physical quantities (such as velocity, density, thermal conductivity and heat transfer coefficient etc.) pertinent to a physical phenomenon. The same phenomenon can be described by (n − m) dimensionless *Pi*-terms.

$$\phi\,[\pi_1, \pi_2, \pi_3, \ldots\ldots\pi_{n-m}] = 0$$

where m represents the fundamental dimensions.

Out of the given physical variables one has to select variables which amongst them contain all the fundamental units of M, L, T, θ and H . These variables are not to form non-dimensional parameters amongst themselves. Experience shows that suitable non-dimensional groups result when geometric property (such as length), a fluid property (such as mass density or viscosity), a flow characteristic (such as velocity) and a thermal property (such as thermal conductivity) are chosen to represent such variables. These variables are known as the ***repeated varaibles*** or the ***core group***.

EXAMPLE 10.2

The resistance R experienced by a partially submerged body depends upon the velocity V, length of the body l, viscosity of the fluid μ, density of the fluid ρ and gravitational acceleration g. Establish a suitable relation involving non-dimensional groups.

Solution : The problem can be expressed as :

$$f(R, V, l, \mu, \rho, g) = 0$$

The quantities with their dimensions in M, L and T units are :

Resistance $\quad R = MLT^{-2}$

Velocity $\quad V = LT^{-1}$

Length $\quad l = L$

Viscosity $\quad \mu = ML^{-1}T^{-1}$

Density $\quad \rho = ML^{-3}$

Gravitational acceleration $g = LT^{-2}$

There are 6 physical quantities and 3 fundamental units, hence (6 - 3) or 3 π-terms. We choose length l, velocity V and density ρ as the three repeating variables (core group), with unknown exponents and establish the π-terms as follows :

$$\pi_1 = l^a\,V^b\,\rho^c\,R \quad ...(i)$$

$$1 = [L]^a\,[LT^{-1}]^b\,[ML^{-3}]\,[MLT^{-2}]$$

Equating exponents of M, L, T repectively :

$0 = c + 1\ ;$

$0 = a + b - 3c + 1\ ;$

$0 = -b - 2$

Solution gives

$$c = -1,\quad b = -2,\quad a = -2$$

Substitution in (i) yields,

$$\pi_1 = l^{-2}V^{-2}\rho^{-1}R = \frac{R}{\rho V^2 l^2}$$

Following the same procedure, one would obtain :

$$\pi_2 = l^a\,V^b\,\rho^c\,\mu\ ;\ \pi_2 = \frac{\mu}{\rho V l}$$

$$\pi_3 = l^a\,V^b\,\rho^c\,g\ ;\ \pi_3 = \frac{lg}{V^2}$$

Thus the functional relationship becomes :

$$\phi\left(\frac{R}{\rho V^2 l^2}, \frac{\mu}{\rho V l}, \frac{lg}{V^2}\right) = 0$$

or $\quad \phi\left(\frac{R}{\rho V^2 l^2}, \frac{\rho V l}{\mu}, \frac{V}{\sqrt{lg}}\right) = 0$

The above step has been worked on the postulates that reciprocal as well as square root of a π-term is non-dimensional .

Thus,

$$R = \rho V^2 l^2 \phi\left[\frac{\rho V l}{\mu}, \frac{V}{\sqrt{lg}}\right]$$

The resistance is thus a function of Reynolds number $(\rho V l/\mu)$ and Froude number $\left(V/\sqrt{lg}\right)$.

EXAMPLE 10.3

Explain in detail the mechanism of forced convection.

Show by dimensional analysis that data for forced convection may be correlated by an equation of the form :

$$Nu = \phi(Re, Pr)$$

where Nusselt number $Nu = (hl/k)$; *Reynolds number* $Re = (Vl\rho/\mu)$ *and Prandtl number* $Pr = (\mu\, c_p/k)$

Solution : The different variables specifying the system behaviour have been indicated in Fig. 10.1 which represents the forced convection of fluid flow over a flat plate. The physical quantities with their dimensions in M-L-T-θ-H system of units are :

Rayleigh's Method : It may be premised that the functional relationship is :

$$h = f(\mu, \rho, k, c_p, \Delta t, V, l)$$

$$= C\,(\mu^a\,\rho^b\,k^c c_p^d\,\Delta t^e\,V^f\,l^g) \quad ...(i)$$

where C is a dimensionless coefficient. The corresponding equation for dimensions is :

$$[HL^{-2}\,T^{-1}\,\theta^{-1}] = 1\,(ML^{-1}\,T^{-1})^a \times (ML^{-3})^b \times (HL^{-1}\,T^{-1}\,\theta^{-1})^c \times (HM^{-1}\,\theta^{-1})^d \times (\theta)^e \times (LT^{-1})^f \times (L)^g$$

Variable	Symbol	Dimension
Fluid viscosity	μ	$ML^{-1}\,T^{-1}$
Fluid density	ρ	ML^{-3}
Fluid thermal conductivity	k	$HL^{-1}\,T^{-1}\theta^{-1}$
Fluid heat capacity	c_p	$HM^{-1}\,\theta^{-1}$
Temperature difference	$\Delta t = (t_s - t_f)$	θ
Flow velocity	V	LT^{-1}
Plate length	l	L
Heat transfer coefficient	h	$HL^{-2}\,T^{-1}\,\theta^{-1}$

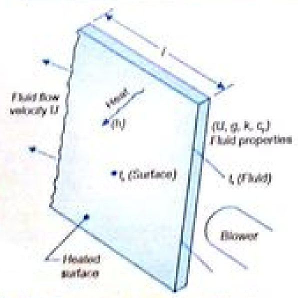

Fig. 10.1. Dimensional analysis variables for forced convection

The dimensions of C being a constant is taken as unity.

For dimensional homogeneity, we equate the exponents of M, L, T, θ and H on both sides:

$M: \quad 0 = a + b - d$

$L: \quad -2 = -a - 3b - c + f + g$

$T: \quad -1 = -a - c - f$

$\theta: \quad -1 = -c - d + e$

$H: \quad 1 = c + d$

We have only five equations to solve for seven exponents. So solution can be obtained in terms of two of exponents which can be arbitrarily chosen. Experience shows that recognised dimensionless groups appear by having velocity in one group and specific heat in the other group. Thus the exponents a, b, c, e and g are to be expressed in terms of d and f. By simple algebraic manipulation, we obtain

$$a = d - f;\quad b = f;\quad c = 1 - d;$$
$$e = 0;\quad g = f - 1$$

Substituting these values of exponents in the functional relationship (*i*),

$$h = C\,(\mu)^{d-f} \times (\rho)^f \times (k)^{1-d} \times (c_p)^d \times (\Delta t)^e \times (V)^f \times (l)^{f-1}$$

and collecting like terms :

$$h = C\left(\frac{k}{l}\right)\left(\frac{\mu c_p}{k}\right)^d\left(\frac{\rho V l}{\mu}\right)^f$$

$$\frac{hl}{k} = \text{constant } C\left(\frac{\mu c_p}{k}\right)^d\left(\frac{\rho V l}{\mu}\right)^f$$

$$= \phi\left(\frac{\mu c_p}{k}, \frac{\rho V l}{\mu}\right)$$

where ϕ stands for a function of.

Buckingham's π-Method. It can be premised that the functional relationship is

$$f\,(\mu, \rho, k, c_p, \Delta t, V, l, h) = 0$$

There are 8 physical quantities and 5 fundamental units, hence (8 - 5) or 3 π-terms. We choose fluid viscosity μ, thermal conductivity *k*, velocity *V* and the characteristic length *l* as the core group (repeated variables) with unknown exponents and establish the π-terms as follows:

$$\pi_1 = \mu^a\,k^b\,V^c\,l^d\,\rho \qquad \text{...(ii)}$$

$$1 = (ML^{-1}\,T^{-1})^a \times (HL^{-1}\,T^{-1}\,\theta^{-1})^b \times (LT^{-1})^c \times (L)^d \times (ML^{-3})$$

Equating the exponents of fundamental dimensions on both sides :

$M: \quad 0 = a + 1$

$L: \quad 0 = -a - b + c + d - 3$

$T: \quad 0 = -a - b - c$

$\theta: \quad 0 = -b$

$H: \quad 0 = b$

Solurion gives :

$$a = -1;\ b = 0;\ c = 1;\ d = 1$$

Substituting these values in relation (*ii*),

$$\pi_1 = \mu^{-1} \times k^0 \times V^1 \times l^1 \times \rho = \frac{\rho V l}{\mu}$$

Following the same procedure, one would obtain :

$$\pi_1 = \mu^a\,k^b\,V^c\,l^d\,c_p;\quad \pi_2 = \frac{\mu c_p}{k}$$

$$\pi_1 = \mu^a\,k^b\,V^c\,l^d\,h;\quad \pi_3 = \frac{hl}{k}$$

Thus the functional relationship becomes :

$$\phi\left(\frac{\rho V l}{\mu}, \frac{\mu c_p}{k}, \frac{hl}{k}\right) = 0$$

or $$\frac{hl}{k} = \phi\left(\frac{\rho V l}{\mu}, \frac{\mu c_p}{k}\right)$$

$$Nu = \phi\,(Re, Pr)$$

where $Nu = (h\,l/k)$, a dimensionless group callerd Nusselt number

$Re = (\rho\,V\,l/\mu)$, a dimensionless group called Reynoles number

$Pr = (\mu\ c_p/k)$, a dimensionless number called Prandtl nunber

It is usual practice to rewrite the above corresleation in the form

$$Nu = C\,(Re)^a\,(Pr)^b$$

The constant C and the exponents *a* and *b* are evaluated through experiments.

Note : If μ, ρ, c_p and V were chosen as the core group (repeated variables), then the analysis would have yielded the following non-dimentional groups :

$$Re = \frac{\rho V l}{\mu};\ Pr = \frac{\mu c_p}{k};\ St = \frac{h}{\rho V c_p}$$

where St is called the Stanton number. Accordingly another correlation for forced convection would be of the form :

$$St = \phi\,(Re, Pr)$$

EXAMPLE 10.4

Explain in detail the machanism of free convection.

Show by dimensional analysis that for problems in heat transfer involving free convection only, the Nusselt number Nu = (h l/k) can be expressed as a function of the Prandtl number

$$Pr = [(\mu\ c_p)/k]$$

and the Grash of number

$$Gr = [(l^3\,\rho^2\,(\beta g\ \Delta t)/\mu^2]$$

Solution : The different variables specifying the system behaviour have been indicated in Fig. 10.2. which represents the free convectiopn of fluid flow over a flat plate. The physical quantites with their dimensions is M-L-T-θ-H system of units are :

Variable	Symbol	Dimension
Fluid viscosity	μ	$ML^{-1}\,T^{-1}$
Fluid density	ρ	ML^{-3}
Fluid thermal conductivity	k	$HL^{-1}\,T^{-1}\theta^{-1}$
Fluid heat capacity	c_p	$HM^{-1}\,\theta^{-1}$
Fluid coefficient of thermal expansion	β	θ^{-1}
Temperature difference	$\Delta t = (t_s - t_f)$	θ
Significant length	l	L
Heat transfer coefficient	h	$HL^{-2}\,T^{-1}\,\theta^{-1}$

The coefficient of thermal expansion, β, is prescribed by the relation;

$$\rho_1 = \rho_2\,[1 + \beta\,(t_2 - t_1)]$$

where ρ_1 and ρ_2 are the fluid densities at temperature t_1 and t_2 respectively,

Buoyant force

$$= (\rho_1 - \rho_2)g$$

$$= \left[\rho_2\left\{1 + \beta(t_2 - t_1)\right\} - \rho_2\right]g = \rho_2\,(\beta g\,\Delta t)$$

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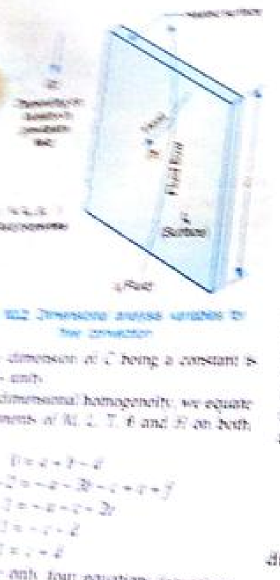

θ : $0 = -b$

H : $0 = -b$

Solution gives :

$a = -1$; $b = 0$; $c = 1/2$ and $d = 3/2$

Substituting these values in relation (ii)

$$\pi_1 = (\mu)^{-1} (k)^0 (\beta g \Delta T)^{1/2} (l)^{3/2} \rho$$

$$= \frac{l^{3/2} \rho (\beta g \Delta T)^{1/2}}{\mu} = \frac{l^3 \rho^2 (\beta g \Delta T)}{\mu^2}$$

The above step has been worked on the postulate that square of a π-term is also non-dimensional.

Following the same procedure, one would obtain :

$$\pi_2 = \mu^a k^b (\beta g \Delta T)^c l^d c_p ; \pi_2 = \frac{\mu c_p}{k}$$

$$\pi_3 = \mu^a k^b (\beta g \Delta T)^c l^d h ; \pi_3 = \frac{hl}{k}$$

Thus the functional relationship becomes :

$$\phi = \left(\frac{l^3 \rho^2 \beta g \Delta T}{\mu^2}, \frac{\mu c_p}{k}, \frac{hl}{k} \right) = 0$$

or $$\frac{hl}{k} = \phi \left(\frac{l^3 \rho^2 \beta g \Delta T}{\mu^2}, \frac{\mu c_p}{k} \right) = 0$$

$$Nu = \phi \,(Gr, Pr)$$

where $Nu = h\, l / k$, a dimensionless group called Nusselt number

$Gr = (l^3 \rho^2 \beta g \Delta T / \mu^2)$, a dimensionless group called Grashof number

$Pr = \mu\, c_p / k$

It is a usual practice to rewrite the above correlation in the form

$$Nu = C\, (Gr)^a\, (Pr)^b$$

The constant C and the exponents a and b are evaluated through experiments.

EXAMPLE 10.5

Neglecting viscous dissipation, the coefficient of convective heat transfer is anticipated to depend upon the following parameters :

fluid viscosity μ, fluid density ρ

fluid thermal conductivity k

fluid specific heat c_p

flow velocity V and significant length l

Considering mass, length, time and temperature as the significant dimensions, set up a suitable correlation in terms of non-dimensional numbers for the heat flow. Use Buckingham's π-theorem method of dimensional analysis.

Solution : The physical quantities affecting the heat flow, along with their dimensions in M-L-T-θ system of units are :

Variable	Symbol	Dimension
Fluid viscosity	μ	$ML^{-1}T^{-1}$
Fluid density	ρ	ML^{-3}
Fluid thermal conductivity	k	$MLT^{-3}\theta^{-1}$
Fluid specific heat	c_p	$L^2T^{-2}\theta^{-1}$
Flow velocity	V	LT^{-1}
Significant length	l	L
Heat transfer coefficient	h	$MT^{-3}\theta^{-1}$

It can be premised that the fuctional relationship is :

$$f(\mu, \rho, k, c_p, V, l, h) = 0$$

There are 7 physical quantities and 4 fundamental units; hence (7 - 4) or 3 π-terms. We choose fluid viscosity μ, thermal conductivity k, flow velocity V and the characteristic length l as the core group (repeated variables) with unknown exponents and establish the π-terms as follows :

$$\pi_1 = \mu^a k^b V^c l^d \rho \quad \text{...(i)}$$

$$1 = (ML^{-1}T^{-1})^a \times (MLT^{-3}\theta^{-1})^b \times (LT^{-1})^c \times (L)^d \times ML^{-3}$$

Equating the exponent of fundamental dimensions on both sides :

M : $0 = a + b + 1$

$L: \quad 0 = -a + b - c + d - 3$
$T: \quad 0 = -a - 3b - c$
$\theta: \quad 0 = -b$

Solution gives : $a = -1$; $b = 0$; $c = 1$ and $d = 1$

Substituting these values in relation (i)

$$\pi_1 = \mu^{-1} \times k^0 \times V^1 \times l^1 \times \rho = \frac{\rho V l}{\mu}$$

Following the same procedure, one would obtain :

$$\pi_2 = \mu^a k^b V^c l^d c_p ; \pi_2 = \frac{\mu c_p}{k}$$

$$\pi_3 = \mu^a k^b V^c l^d h ; \pi_3 = \frac{hl}{k}$$

Thus the functional relation becomes :

$$\phi\left(\frac{\rho V l}{\mu}, \frac{\mu c_p}{k}, \frac{hl}{k}\right) = 0$$

or $$\frac{hl}{k} = \phi\left(\frac{\rho V l}{\mu}, \frac{\mu c_p}{k}\right)$$

$$Nu = \phi\,(Re, Pr)$$

10.5. MODEL STUDIES AND SIMILITUDE

10.5.1. Prototype and Model

A prototype is the full size structure employed in the actual engineering design. The prototype operates under the actual working conditions. A model is generally a small scale replica of the prototype. Experimental observations made on the model bear a definite relationship to the prototype. Undoubtedly the performance characteristics of the prototype and other similar systems can be well ascertained by having test runs on a model.

10.5.2. Similitude

The term similitude refers to the theory and art of predicting prototype conditions from model observations. Similitude prescribes the relationship between a full scale flow and a flow involving smaller but geometrically similar boundaries. Through principle of similarity, the performance of apparatus and devices can be investigated on small scale models instead of cumbersome and full size equipment. This simplifies considerably the investigations and reduces time and cost of experimental work. However, the results obtained from experiments on models can be applied to the prototype only if a complete similarity exists between the model and prototype and for that the two systems must be (i) geometrically (ii) kinematically (iii) thermally and (iv) dynamically similar.

- *Geometrical similarity* refers to the similarity of shape or form. The geometrically similar systems may differ in size but they are identical in shape. There is point to point correspondence between the two systems. Geometric similarity prescribes that the ratio of the corresponding linear dimensions of the two systems are same.
- *Kinematic similarity* refers to the similarity of motion. Both the systems undergo similar rates of change of motion and evidently there is similarity in the streamline patterns. Kinematic similarity stipulates that (i) paths of the homologous moving particles are geometrically similar (ii) ratios of the kinematic parameters (velocity and acceleration) of the homologous particles are equal. Homologous particles refer to the corresponding fluid particles of the two systems. Velocity distributions within the given fields are similar in magnitude, direction and turbulence pattern when their Reynolds number are same.
- *Thermal similarity* refers to the comparison of two systems made on the basis of their temperature, specific heat and heat flux. A similarity in thermal quantities is achieved when Prandtl number is same for both the fields.
- *Dynamic similarity* refers to the similarity of masses and forces of the corresponding particles of flow. Systems are dynamically similar if the corresponding particles experience similar force, and the ratio of all the forces acting at homologous particles in the model and the prototype remains constant. Both geometric and kinematic similarities are pre-requisite for dynamic similarity.

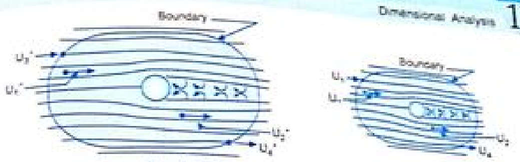

Fig. 10.3. Similarity in forced convection

When Reynolds number and Prandtl number are same, the two systems (fields) are said to be dynamically or physically similar from the point of ratio of forced convection heat transfer. For such dynamically similar systems, the Nusselt number for corresponding surface elements is same for both solid bodies.

10.5.3. Similarity Constants

Consider two cylinders (Fig. 10.3) and the limited region around them in which the fluid flows; these regions are referred to as the fields.

The behaviour of two systems will be similar if the ratio of their physical parameters (linear dimensions, velocity, temperature and force) are same for the homologous (corresponding points of the systems). The ratios are called similarity constants or scale ratios. For instance

$$C_l = \frac{l_1'}{l_1} ; C_v = \frac{V_1'}{V_1} ; C_t = \frac{t_1'}{t_1} ; C_f = \frac{f_1'}{f_1}$$

where length l, velocity V, temperature t and force f are the physical parameters of the two systems. C_l, C_v, C_t and C_f are the similarity constants of the corresponding parameters.

10.6. ADVANTAGES AND LIMITATIONS OF DIMENSIONAL ANALYSIS

Advantages :

(i) Dimensional analysis is a useful tool in the analysis and correlation of experimental data, in the planning of experiments and in the formulation of empirical correlations describing a particular phenomenon.

(ii) The presentation of data in non-dimensional groups allows the application of empirical correlations to a wide range of physical conditions. Further, the non-consistent dimensional data is suitable for use in any consistent system of units.

(iii) With the aid of dimensional analysis, the results of one series of tests can be applied to a large number of other similar problems.

Fig. 10.4 depicts the variation of Nusselt number with Reynolds number for flow of air over a pipe of 25 mm outside diameter. This correlation curve permits the evaluation of Nusselt number for air flow over any size of pipe as long as Reynolds number of that arrangement lies within the range covered by this experiment.

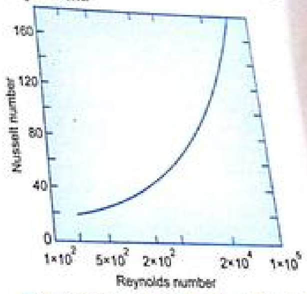

Fig. 10.4. Variation of Nusselt number with Reynolds number

(iv) Through principal of similarity and dimensional analysis, the preformance of apparatus and devices can be investigated on

small scale models instead of cumbersome and full size equipment. This simplifies considerably the investigations and reduces time and cost of experimental work.

(v) The important variables involved in a physical phenomenon are systematically orgainsed into dimensionless groups which are less numerous than the original variables. Few experiments then need to be conducted to cover a wide range of parameters and that results in a considerably saving in time, money and efforts.

Let it be premised that for a particular heat flow situation,

$$h = f(\mu, \rho, k, c_p, V, d)$$

Through dimensional analysis, we would obtain :

$$\frac{hd}{k} = C\left(\frac{Vd\rho}{\mu}, \frac{\mu c_p}{k}\right)$$

If the effect of each original variables upon the convective coefficient is to be investigated individually, then the test runs will have to be made with say 5 different cylinder diameters and then conducting experiments with each of the cylinder with 5 fluids of different thermal conductivities, further vary fluids density 5 times and so on. This implies that $5^6 = 15625$ tests will have to be conducted which is rather cumbersome and time consuming.

With the aid of dimensional analysis, the original variables have been recast into 3 dimensionless groups. Obviously the same information can now be obtained through $5^2 = 25$ experiments.

Limitations:

(i) Information from previous experiments is necessary to know the different variables which are likely to affect the phenomenon being analysed. The success or failure of the method depends upon proper selection of these variables.

(ii) Dimensional analysis only helps to set up an equation consisting of variables involved in the process and some unknown constants. These constants have to be determined through experimental investigations.

(iii) No information is given about the internal mechanism of the physical phenomenon.

10.7. SIGNIFICANCE OF DIMENSIONLESS GROUPS

• **Reynolds Number Re** : Ratio of the interia force to the viscous force :

$$Re = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho V^2 l^2}{\mu V l} = \frac{\rho V l}{\mu}$$

Reynolds number is indicative of the relative importance of inertial and viscous effects in a fluid motion. At low Reynolds number, the viscous effects dominate and the fluid motion is laminar. At high Reynolds number, the inertial effects lead to turbulent flow and the associated turbulence level dominates the momentum and energy flux.

Reynolds number constitutes an important criterion of kinematic and dynamic similarity in forced convection heat transfer. Velocity within the given fields would be similar in magnitude, direction and turbulence pattern when their Reynolds number are same.

• **Grashof Number Gr** indicates the relative strength of the buoyant to viscous forces. From its mathematical formulation,

$$Gr = \frac{l^3 \rho^2 \beta g \Delta T}{\mu^2}$$

$$= \left(\rho l^3 \beta g \Delta T\right)\frac{\rho}{\mu^2}$$

$$= \left(\rho l^3 \beta g \Delta T\right) \times \frac{\rho V^2 l^2}{(\mu V l)^2}$$

$$= \text{buoyant force} \times \frac{\text{inertia force}}{(\text{viscous force})^2}$$

Obviously the Grashof number represents the ratio of the product of buoyant amd inertia forces to the square of the viscous forces. Grashof number has a role in free convection similar to that played by Reynolds number in forced convection. Free convection is usually suppressed at sufficiently small Gr, begins at some critical value of Gr (depending upon

the arrangement) and then becomes more and more effective as Gr increases.

• **Prandtl Number Pr** is indicative of the relative ability of the fluid to diffuse momentum and internal energy by molecular mechanisms. From its mathematical formulation,

$$Pr = \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k} = \frac{\nu}{(k/\rho c_p)}$$

Recalling that the parameter $(k/\rho c_p)$ is thermal diffusivity α of the fluid,

$$Pr = \frac{\nu}{\alpha} = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$$

Apparently Pr is the ratio of the kinematic viscosity to thermal diffusivity of the fluid. The kinematic viscosity indicates the momentum transport by molecular friction and thermal diffusivity represents the heat energy transport through conduction. Obviously Pr provides a measure of the relative effectiveness of momentum and energy transport by diffusion. For highly viscous oils, Pr is quite large (100 to 10,000) and that indicates rapid diffusion of momentum by viscous action compared to the diffusion of energy. Prandtl number for gases is near unity and accordingly the momentum and energy transfer by diffusion are comparable. In contrast, the liquid metals have $Pr = 0.003$ to 0.01 and that indicates more rapid diffusion of energy compared to the momentum diffusion rate.

The Prandtl number is connecting link between the velocity field and the temperature field, and its value strongly influences relative growth of velocity and thermal boundary layers. Mathematically,

$$\frac{\delta}{\delta_t} = (Pr)^n$$

where δ and δ_t are the thickness of velocity and thermal boundary layers respectively, and n is a positive exponent. For oils $\delta_t << \delta$; for gases $\delta_t = \delta$; and for liquid metals $\delta_t >> \delta$.

• **Nusselt Number Nu** establishes the relation between convective film coefficient h, thermal conductivity of the fluid k and a significant length parameter l of the physical system :

$$Nu = (h\,l/k)$$

An energy balance at the surface of a heated plate stipulates that energy transport by conduction must equal the convective heat transfer into the fluid flowing past the plate. Thus

$$Q = -kA\left(\frac{\partial t}{\partial y}\right)_{y=0} = hA(t_s - t_\infty)$$

$$h = \frac{-k(\partial t/\partial y)_{y=0}}{(t_s - t_\infty)}$$

$$\frac{hl}{k} = \frac{-(\partial t/\partial y)_{y=0}}{(t_s - t_\infty)/l}$$

Apparently the Nusselt number $h\,l/k$ may be interpreted as the ratio of temperature gradient at the surface to an overall reference temperature gradient.

The Nusselt number is a convenient measure of the convective heat transfer coefficient. For a given value of the Nusselt number, the convective heat transfer coefficient is directly proportional to thermal conductivity of the fluid and inversely proportional to the signigicant length parameter.

• **Stanton Number St** is the ratio of heat transfer coefficient to the flow of heat per unit temperature rise due to the velocity of the fluid.

$$St = \frac{h}{\rho V c_p}$$

$$= \frac{hl/k}{(\rho V l/\mu) \times (\mu c_p/k)} = \frac{Nu}{Re \times Pr}$$

Thus the Stanton number can be expressed in terms of other dimensionless numbers as :

$$St = \frac{\text{Nusselt number}}{\text{Reynolds number} \times \text{Prandtl number}}$$

It should be noted that Stanton number can be used only in correlating forced convection data. This becomes obvious when we observe the velocity V contained in the expression for Stanton number.

Through dimensional analysis, we have obtained the following possible forms for correlation of convection data :

(a) Forced convection :

$$Nu = f_1\,(Re,\ Pr) \text{ or } St = f_2\,(Re,\ Pr)$$

(b) Free convection :

$$Nu = f_2\,(Gr,\ Pr)$$

• **Peclet Number** Pe is the ratio of heat flow rate by convection to flow rate by conduction under a unit temperature gradient and through thickness l

$$Q_{conv} = mc_p = (\rho AV)c_p\ ;$$

$$Q_{cond} = \frac{kA\Delta t}{l} = \frac{kA}{l}$$

$$\therefore \quad Pe = \frac{Q_{conv}}{Q_{cond}} = \frac{(\rho AV)c_p}{kA/l} = \frac{\rho c_p}{k} lV$$

$$= \frac{lV}{\alpha}$$

The Peclet number can be recast as

$$Pe = \frac{\rho c_p}{k} lV = \frac{\rho lV}{\mu} \times \frac{\mu c_p}{k}$$

$$= Re \times Pr$$

i.e. the Peclet number is a function of Reynolds number and Prandtl number.

• **Graetz Number** (Gr) represents the ratio of heat capacity of fluid flowing through the pipe per unit of length to the conductivity of pipe material.

$$Gr = \frac{mc_p/l}{k} = \frac{mc_p}{lk}$$

$$= \frac{(\rho AV)c_p}{lk} = \frac{\rho \times \left(\frac{\pi}{4} d^2 \times V\right)c_p}{lk}$$

$$= \frac{\pi}{4} \frac{\rho V d}{\mu} \times \frac{\mu c_p}{k} \times \frac{d}{l}$$

$$= \frac{\pi}{4} (Re \times Pr) \times \frac{d}{l}$$

where d and l are the diameter and length of pipe respectively. Graetz number can be recast in another form as

$$Gr = \frac{mc_p}{lk} = \frac{\rho AVc_p}{lk} = \frac{AV}{\alpha l}$$

$$= \frac{\pi}{4} d^2 \frac{V}{\alpha l} = \frac{Vd}{\alpha}\left(\frac{\pi d}{4l}\right)$$

$$= Pe\left(\frac{\pi d}{4l}\right)$$

i.e. the Graetz number is merely a product of a constant and the Peclet number.

SALIENT POINTS

1. *Equations describing a physical phenomenon must be dimensionally homogenous and the units there must be consistent.*
2. *The principle of dimensional homogeneity implies that only the variables with the same dimensions must be added or subtracted.*
3. *Dimensional analysis is the system of forming dimensionless groups from the number of independent parameters influencing a physical phenomenon.*
4. *The non-dimensional groups are formulated by the Rayleigh method and Buckingham's P_i theorem.*
5. *The Buckingham's P_i -theorem states that:*
   "If there are n variables in a dimensionally homogeneous equation and if these variables contain m primary dimensions, then the variables can be grouped into $(n - m)$ non-dimensional groups."

   The dimensions normally used for mechanical systems are : mass (M), length (L), time (T) and temperature θ, and heat H.
6. The different variables representing free convection of fluid flow over a flat plate are:
   Fluid viscosity (μ), fluid density (ρ), fluid thermal conductivity (k), fluid heat capacity (c_p), fluid coefficient of thermal expansion (β), temperature difference Δt, significant length (l) and heat transfer coefficient (h).

   Through dimensional analysis, the functional relation would be

$$Nu = \phi\,(Gr,\ Pr)$$

   where $Nu = \dfrac{hl}{k}$, a dimensionless group called Nusselt number

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2}, \text{ a dimensionless}$$

group called Grashof number

$$Pr = \frac{\mu c_p}{k}, \text{ a dimensionless group}$$

called Prandtl number

7. The different variables representing forced convection of fluid flow over a flat plate are:
   $\mu, \rho, k, c_p, \Delta t, l, h$ and flow velocity V
   Through dimensional analysis, we get the following functional relations

$$Nu = \phi\,(R_e,\ P_r)$$

and

$$St = \phi\,(R_e,\ P_r)$$

   where $R_e = \dfrac{\rho Vl}{\mu}$, a dimensionless group called Reynolds number

$$St = \frac{h}{\rho V c_p}, \text{ a dimensionless group}$$

   called the Stanton number

8. The dimensionless groups which appear so often in heat transfer are listed below along with their significances:

   (i) Reynolds number (R_e)

$$R_e = \frac{\rho Vl}{\mu} = \frac{\text{inertia force}}{\text{viscous force}}$$

   (ii) Grashof number (Gr)

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2}$$

$$= \text{buoyant force} \times \frac{\text{inertia force}}{(\text{viscous force})^2}$$

   (iii) Prandtl number (Pr)

$$Pr = \frac{\mu c_p}{k}$$

$$= \frac{\text{kinematic viscosity } (\nu)}{\text{thermal diffusivity } (\alpha)}$$

   (iv) Nusselt number (Nu)

$$Nu = \frac{hl}{k}$$

$$= \frac{\text{temperature gradient at the surface}}{\text{overall reference temperature gradient}}$$

   (v) Stanton number (St)

$$St = \frac{h}{\rho V c_p}$$

$$= \frac{\text{heat transfer coefficient}}{\text{heat flow per unti temperature rise due to flow velocity}}$$

$$= \frac{N_u}{Rr \times Pr}$$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. What is meant by dimensional homogeneity? Explain some of its applications.
2. State the scope and application of dimensional analysis in heat transfer processes.
3. What are fundamental dimensions? Express thermal resistance, thermal diffusivity and convection heat transfer coefficient in fundamental dimensions.
4. Explain the Rayleigh method and the Buckingham's π-theorem for dimensional analysis. What are repeating variables and how are they selected for dimensional analysis?
5. Using dimensional analysis, demonstrate that the parameters

$$\frac{t - t_\infty}{t_0 - t_\infty}, \frac{x}{l}, \frac{\alpha l}{l^2} \text{ and } \frac{hl}{k}$$

   are possible combinations of the appropriate variables in describing unsteady state conduction in a plane wall.
6. Through dimensional analysis establish the following correlation for the heat transfer rate Q from a rigid body of linear dimeasion l and submerged in a fluid flowing with the undisturbed velocity U_∞.

$$\frac{Q}{hl\,\Delta t} = C\left(\frac{k}{c_p\, l U_\infty}\right)^n$$

   where h is the convective coefficeient, k is the thermal conductivity, Δt is the temperature

difference and c_p is the specific heat. C and m are numerical constants.

7. Under steady state conditions, the rate of conduction heat transfer through a plane wall is known to depend upon the length of the heat flow passage, its cross-sectional area, temperature difference across the faces and thermal conductivity of the wall material. Through dimensional analysis, establish an expression for the heat flow rate in terms of other variables.

8. *By the method of dimensional analysis, determine the following correlation for the film coefficient h for forced convection in laminar flow:*

$$h = C\left[\frac{V\rho c_p k^2}{DL}\right]^m$$

The symbols have their usual meanings and C and m are numerial constants.

9. *The cable of electrical transmission system can be assumed to be a long homogeneous cylinder. Experiments were performed on one such cable to find out an expression for the temperature distribution while it is carrying a current resulting in uniform heat generation. By using dimensional analysis, determine the relation between the steady state temperature at the centre of the cylinder, the cable diameter, the thermal conductivity and the rate of heat generation. The temperature of the cable surface may be taken as the datum.*

10. *The expression hl/k gives the Biot number as well as the Nusselt number. What is the difference between the two ?*

11. *What thermo-physical variables are involved in the equations describing the phenomenon of free convection ?*

Apply dimensional analysis to these variables and develop a generalised correlation between certain non-dimensional parameters. Discuss the physical significance of those parameters.

12. *List the variables that affect the forced heat transfer coefficient.*

Using dimensional analysis, demonstrate that the following dimensionless parameters are possible combinations of the appropriate varibles describing forced convection :

$$\frac{Vx\rho}{\mu}, \frac{\mu c_p}{k}, \frac{hx}{k} \text{ and } \frac{h}{\rho c_p V}$$

Name these groups and discuss their physical significance.

13. (i) What is a dimensionless number ? How and why are they used in heat transfer ?
 (ii) Define and state the physical significance of Prandtl number and Stanton number.

14. How does dimensional analysis help in correlating the experimental data.

15. Differentiate between similitude and similarity. State the utility of similarity principle. Explain the advantages and limitatons of dimensional analysis.

16. Discuss the similarities that need to be ensured between model and prototype while testing a heat transfer equipment.

B. Fill in the blanks with appropriate word/words :

1. A .... is a measure of physical quantity without numerical values.
2. An equation which experiences a physical phenomenon must be ....
3. *If there are n-variables in a dimensionally homogeneous equation and if these contain m-fundamental dimensions, then the variables can be arranged into .... dimensionless groups.*
4. *The .... must not form dimensionless groups among them solves.*
5. *Grashoff number has a role in .... convection similar to that played by Reynolds number in .... convection.*
6. *Prandtl number $P_r = \frac{\mu c_p}{k}$ essentially represents the ratio of .... to ....*
7. *.................... number defined as the ratio of heat transfer coefficient to the flow of heat per unit temperature rise due to the velocity of the fluid, and is used only in correlating .... convection.*
8. *Based on dimensional analysis*

$$Nu = C(Re)^m\,(Pr)^n$$

The valus of constants C, m and n are evaluated ....

9. *In geometrically similar systems, the values of the Nusselt number will be same for given values of .... number and .... number.*

Answers : 1. dimension; 2. dimensionally homogenous; 3. $(n - m)$; 4. repeating variables; 5. free, forced; 6. kinematic viscosity, thermal diffusivity; 7. Stanton, forced; 8. experimentally; 9. Reynolds, Prandtl.

C. Multiple choice questions :

1. Free convection heat flow depends on all of the followings, except
 (a) density
 (b) coefficent of viscosity
 (c) gravitational force
 (d) velocity

2. The free convection heat transfer is significantly affected by
 (a) Reynolds number
 (b) Grashoff number
 (c) Prandtl number
 (d) Stanton number

3. The dimensionless parameter $\frac{l^3\rho^2\beta g\Delta T}{\mu^2}$ is referred to as
 (a) Stanton number (b) Schmidt number
 (c) Grashoff number (d) Peclet number

4. Consider natural convection heat transfer between a vertical tube surface and a fluid surrounding it. For dimensional analysis of the problem, the characteristic length corresponds to
 (a) length of the tube
 (b) diameter of the tube
 (c) perimeter of the tube
 (d) either length or diameter of the tube

5. The Nusselt number in convective heat transfer
 (a) signifies the velocity gradient at the surface
 (b) is the ratio of conduction to convection resistance
 (c) represents the ratio of viscous to inertia force
 (d) is the ratio of molecular momentum diffusivity to thermal diffusivity

6. The Nusselt number in natural transfer is a function of fluid Prandtl number and
 (a) Stanton number (b) Biot number
 (c) Grashoff number (d) Reynolds number

7. The ratio of heat transfer by convection to that by conduction is called
 (a) Stanton number (b) Nusselt number
 (c) Biot number (d) Peclet number

8. Choose the wrong statement with respect to Nusselt number and convective heat transfer coefficient
 (a) Nusselt number represents the ratio of temperature gradient at the surface to an overall or reference temperature gradient
 (b) Nusselt number represents the dimensionless slope of the temperature distribution curve at the surface
 (c) The convective coefficient can be evaluated from a knowledge of fluid temperature distribution in the neighbourhood of the surface
 (d) For a given Nusselt number, the convective coefficient is inversely proportional to thermal conductivity of the fluid

9. Heat is lost from a 100 mm diameter steam pipe placed horizontally in ambient air at 30°C. If the Nusselt number is 25 W/m²K and thermal conductivity of air is 0.03 W/mK, then the heat transfer coefficient will be
 (a) 7.5 W/m²K (b) 16.5 W/m²K
 (c) 25 W/m²K (d) 30 W/m²K

10. Which one of the following numbers represents the ratio of kinematic viscosity to thermal diffusivity?
 (a) Grashoff number (b) Prandtl number
 (c) Mach number (d) Nusselt number

11. Prandtl number is
 (a) a measure of temperature gradient at the surface
 (b) ratio of conduction to convection resistance
 (c) ratio of molecular momentum diffusivity to thermal diffusivity
 (d) mass diffused to momentum diffused

12. The Prandtl number will be lowest for
 (a) water (b) liquid metal
 (c) aqueous solution (d) lube oil

13. The value of Prandtl number for air is about
 (a) 0.1 (b) 0.4
 (c) 0.7 (d) 1.1

14. Stanton number may be interpreted as the ratio of
(a) temperature gradient at the surface to an overall or reference temperature gradient
(b) buoyant to inertia force
(c) heat transfer coefficient to the flow of heat per unit temperature rise due to velocity of fluid
(d) heat conducted to heat capacity

15. The non-dimensional parameter known as Stanton number is used in
(a) forced convection heat transfer in flow over flat plate
(b) condensation heat transfer with laminar film layer
(c) natural convection heat transfer over flat plate
(d) unsteady heat transfer from bodies in which internal temperature gradients cannot be neglected.

16. Peclet number is defined as
(a) $\dfrac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$
(b) $\dfrac{\text{convective heat transfer}}{\text{conduction heat transfer}}$
(c) $\dfrac{\text{buoyancy force} \times \text{inertial force}}{\text{viscous force}}$
(d) $\dfrac{\text{wall heat transfer rate}}{\text{convection heat transfer}}$

17. Which dimensionless number has a significant role in forced convection ?
(a) Prandtl number (b) Reynolds number
(c) Mach number (d) Peclet number

18. For forced convection, Nusselt number is a function of
(a) Prandtl and Grashoff number
(b) Reynolds and Prandtl number
(c) Reynolds and Grashoff number
(d) Reynolds number only

19. Match List-I with List-II and select the correct answer :

List-I	List-II
A. Reynolds number	1. Film coefficient, pipe diameter, thermal conductivity
B. Prandtl number	2. Flow velocity, acoustic velocity
C. Nusselt number	3. Heat capacity, dynamic viscosity, thermal conductivity
D. Mach number	4. Flow velocity, pipe diameter, kinematic viscosity

Codes :	A	B	C	D
(a)	4	1	3	2
(b)	4	3	1	2
(c)	2	3	1	4
(d)	2	1	3	4

20. Which one of the following non-dimensional numbers is used to determine the transition from laminar to turbulent flow in free convection?
(a) Reynolds number (b) Grashoff number
(c) Peclet number (d) Rayleigh number

Answers :

1. (d)	2. (b)	3. (c)	4. (a)	5. (b)
6. (c)	7. (b)	8. (d)	9. (a)	10. (b)
11. (c)	12. (b)	13. (c)	14. (c)	15. (a)
16. (b)	17. (b)	18. (b)	19. (b)	20. (b)

HINTS AND COMMENTS

5(b):

The Nussel number $\dfrac{hl}{k}$ is the ratio of conduction resistance to heat flow $\left(\dfrac{l}{k}\right)$ and the convection resistance to heat flow $\left(\dfrac{1}{h}\right)$.

8(d):

$$N_u = \frac{hl}{k} \quad \text{or} \quad h = N_u \frac{k}{l}$$

Obviously for a given Nusselt number, the convective coefficient is directly proportional to thermal conductivity of the fluid.

9(a):

$$N_u = \frac{hl}{k}$$

$$25 = \frac{h \times 0.1}{0.03};$$

$$h = 7.5\ \text{W/m}^2\text{K}$$

11(c):

From mathematical formulation

$$P_r = \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k}$$

$$= \frac{\nu}{\dfrac{k}{\rho c_p}} = \frac{\nu}{\alpha}$$

Apparently P_r is the ratio of kinematic viscosity to thermal diffusivity of the fluid. The kinematic viscosity indicates the momentum transport by molecular friction and thermal diffusivity represents the heat energy transport through conduction. As such, the Prandtl number represents the ratio of molecular momentum diffusivity to molecular heat diffusivity.

14(c):

From mathematical formulation

$$S_t = \frac{h}{\rho c_p U}$$

The physical significance of Stanton number is

$$S_t = \frac{h\,\Delta t}{\rho c_p U\,\Delta T}$$

$$= \frac{\text{actual heat flux to fluid}}{\text{heat flux capacity of the fluid flow}}$$

Further,

$$S_t = \frac{\dfrac{hl}{k}}{\left(\dfrac{\mu c_p}{k}\right) \times \left(\dfrac{U l \rho}{\mu}\right)}$$

Apparently, the Stanton number represents the Nusselt number divided by the product of Reynolds number and Prandtl number.

Chapter

11 Empirical Correlations for Free and Forced Convection

Learning objectives : The subject matter included in this chapter will enable the readers to

- *define and differentiate between*
  - *bulk temperature and mean film temperature*
  - *local and average convective coefficient*
- *know about the various correlations for free and forced convection depending nature of flow, geometrical configuration and its orientation*
- *understand and apply the simplified free convection relations for air*

Mathematical analysis of convective heat problems is complicated due to the large number of variables involved. Majority of the convective problems are, therefore, analysed through the technique of dimensional analysis supported by experimental investigations. With variables pertinent to the situation known, the dimensional analysis helps to develop certain correlations for the convective coefficient. The constants and exponents apperating in these correlations for a particular situation are worked out through experiments. Attention has been directed in this chapter to present a collection of the existing relations for the more frequently encountered cases of free and forced convection.

11.1. BULK TEMPERATURE AND MEAN FILM TEMPERATURE

The physical properties (μ, k, ρ, c_p) of a fluid are temperature dependent. Undoubtedly the accuracy of the results obtained by using theoretical relations and the dimensionless empirical correlations would depend upon the temperature chosen for the evaluation of these properties. No uniform procedure has been attained in the selection of this reference temperature. However, it is customary to evaluate the fluid properties either on the basis of bulk temperature or the mean film temperature. The ***mean bulk temperature*** t_b denotes the equilibrium temperature that would result if the fluid at a cross-section was thoroughly mixed in an adiabtic container. For turbulent flow of fluids in ducts, this temperature is very nearly equal to the fluid temperature near the duct axis. In heat exchangers, the fluid flowing through the tubes may be heated or cooled during its flow passage. The bulk temperature is then taken to be the arithmetic mean of the temperatures at inlet to and at exit from the heat echanger tube, *i.e.*, $t_b = (t_i + t_e)/2$. The ***mean film temperature*** t_f is the arithmetic mean of the surface temperature t_s of a solid and the undisturbed temperature t_∞ of the fluid which flows past it; $t_f = (t_s + t_\infty)/2$.

11.2. LOCAL AND AVERAGE CONVECTIVE COEFFICIENT

Consider the flow condition depicted in Fig. 11.1 where a fluid with velocity U_∞ and temperature t_∞ flows past a stationary flat plate of length l and width B. If temperature t_s at the plate surface is greater than the free stream temperature t_∞ then convective heat transfer occurs from the plate to the fluid. The flow conditions vary from point to point on the surface, and as such the convective film coefficient and the heat flux would also vary along the surface. However for an elementary strip of length dx located at a distance x form the leading edge, the convective coefficient can be assumed to be practically constant. The local heat flux is then given by

$$Q_x = h_x\, dA\, (t_s - t_\infty) = h_x\, (B\, dx)\, (t_s - t_\infty) \quad ...(11.1)$$

where h_x is the local convection coefficient.

The total heat transfer rate Q is obtained by integrating the local flux over the entire surface.

That is,

$$Q = \int_0^l Q_x\, dx = B\,(t_s - t_\infty) \int_0^l h_x\, dx \quad ...(11.2)$$

Defining an average coefficient $\bar{h}$ for the entire surface, the total heat transfer rate may also be expressed as :

$$Q = \bar{h}\, (Bl)\, (t_s - t_\infty) \quad ...(11.3)$$

From equations 11.2 and 11.3, the local and average convection coefficients are related by an expression of the form,

$$\bar{h} = \frac{1}{l}\int_0^l h_x\, dx \quad ...(11.4)$$

The average value of convection coefficient based upon the total surface area is used in the Newton-Rikhman relation to calculate the total amount of heat that is transferred for the given set of conditions. The local convection coefficient becomes the controlling factor when the local temperature in the boundary layer has a certain temperature limit which is not to be exceeded.

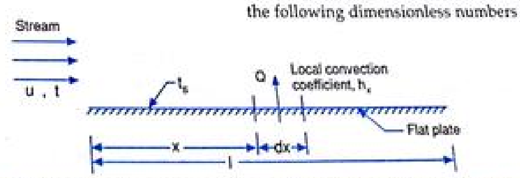

Fig. 11.1. Local and average convection coefficient for flow past a flat plate

EXAMPLE 11.1

Experimental results indicate that the local heat transfer coefficient h_x for flow over a flat plate with an extremely rough surface is approximated by the relation:

$$h_x = a\, x^{-0.12}$$

where a is a constant coefficient and x is distance from the leading edge of the plate. Set up a relation between this local heat transfer coefficient and the average heat transfer coefficient $\bar{h}$ for a plate of length x.

Solution : The local and average convection coefficients are related by an expression of the form :

$$\bar{h} = \frac{1}{x}\int_0^x h_x\, dx = \frac{1}{x}\int_0^x a\, x^{-0.12}\, dx$$

$$= \frac{a}{x}\left[\frac{x^{0.88}}{0.88}\right] = 1.136\, a\, x^{-0.12}$$

$$= 1.136\, h_x$$

11.3. CORRELATIONS FOR FREE CONVECTION

For the usual free convection circumstances the following dimensionless numbers apply

- **Nusselt number**

$$Nu = \frac{hl}{k}$$

- **Grashof number**

$$Gr = \frac{l^3 \rho^2 (\beta g \Delta t)}{\mu^2}$$

- **Prandtl number**

$$Pr = \frac{\mu c_p}{k}$$

The basic equation developed from dimensional analysis for use in determining the value of convection coefficient h is :

$$Nu = f(Gr, Pr)$$
$$= C\,(Gr)^a\,(Pr)^b \qquad ...(11.5)$$

Through rigorous experimental tests of free convection with various fluids (both liquids and gases) flowing past horizontal cylinders and vertical plates, the exponents a and b have been found to be numerically same. This permits the correlation 11.5 to be rewritten in the form

$$Nu = C\,(Gr \times Pr)^m \qquad ...(11.6)$$

The constant C and exponent m depend upon the nature of flow (laminar or turbulent), geometrical configuration (plate or cylinder) and its orientation (vertical or horizontal). For free convection, the fluid properties needed for the determination of Gr and Pr are generally evaluated at the mean film temperature. The product $(Gr \times Pr)$ is often referred to a Rayleigh number, and its value sets the criterion of laminar or turbulent character of flow.

Thus,

$10^4 < Gr\,Pr < 10^9$ .... for laminar flow

$Gr\,Pr > 10^9$ .... for turbulent flow

11.3.1. Horizontal Plates, Cylinders and Wires

(*i*) Plates: heated surface up or cooled surface down

- laminar flow

$$2 \times 10^5 < Gr\,Pr < 2 \times 10^7$$
$$Nu = 0.54\,(Gr\,Pr)^{0.25} \qquad ...(11.7)$$

- turbulent flow

$$2 \times 10^7 < Gr\,Pr < 3 \times 10^{10}$$
$$Nu = 0.14\,(Gr\,Pr)^{0.33} \qquad ...(11.8)$$

(*ii*) Plates : heated surface down or cooled surface up

- laminar flow

$$3 \times 10^5 < Gr\,Pr < 7 \times 10^8$$
$$Nu = 0.27\,(Gr\,Pr)^{0.25} \qquad ...(11.9)$$

- turbulent flow

$$7 \times 10^8 < Gr\,Pr < 11 \times 10^{10}$$
$$Nu = 0.107\,(Gr\,Pr)^{0.33} \qquad ...(11.10)$$

In these correlations, the characteristic length used in computing the Nusselt and Grashof numbers is the length of the side of a square, mean of the two dimensions of a rectangular surface or 0.9 times the diameter of a circular disk.

(*iii*) Long cylinder $L/D > 60$

- laminar flow

$$10^4 < Gr\,Pr < 10^9$$
$$Nu = 0.53\,(Gr\,Pr)^{0.25} \qquad ...(11.11)$$

- turbulent flow

$$10^9 < Gr\,Pr < 10^{12}$$
$$Nu = 0.13\,(Gr\,Pr)^{0.33} \qquad ...(11.12)$$

For cross flow over horizontal pipes,

$$Nu = 0.37\,(Gr\,Pr)^{0.25} \qquad ...(11.13)$$

The characteristic length is the cylinder diameter.

(*iv*) Fine horizontal wires $D < 0.005$ cm

$$Nu = 0.4\,(Gr\,Pr)^{0.0} = 0.4 \qquad ...(11.14)$$

The characteristic length is the wire diameter.

11.3.2. Vertical Plates and Large Cylinders

- laminar flow

$$10^4 < Gr\,Pr < 10^9$$
$$Nu = 0.59\,(Gr\,Pr)^{0.25} \qquad ...(11.15)$$

- turbulent flow

$$10^9 < Gr\,Pr < 10^{12}$$
$$Nu = 0.13\,(Gr\,Pr) \qquad ...(11.16)$$

The characteristic length is the vertical dimension of the plate or cylinder.

11.3.3. Inclined Plates

Multiply Grashof number by $\cos\theta$, where θ is the angle of inclination from the vertical and use vertical plate constants.

11.3.4. Miscellaneous Solids (Spheres, Short Cylinders and Blocks)

The following correlation has been suggested for free convection for a sphere

$$Nu = 2 + 0.43\,(Gr\,Pr)^{0.25} \qquad ...(11.17)$$

For higher values $(3 \times 10^5 < Gr\,Pr < 8 \times 10^{10})$ 0.43 is replaced by 0.50.

For short cylinders $(D = H)$

$$Nu = 0.775\,(Gr\,Pr)^{0.208} \qquad ...(11.18)$$

For other solids

$$Nu = 0.52\,(Gr\,Pr)^{0.25} \qquad ...(11.19)$$

where the characteristic length is the distance travelled by a particle in the boundary layer.

For rectangular solids, the characteristic length is worked out from the relation

$$\frac{1}{l} = \frac{1}{l_h} + \frac{1}{l_v}$$

where l_v and l_h refer to the significant vetical and horizontal dimensions. Further when flow is considered through a non-circular conduit such as square, rectangular duct, annular space between concentric pipes and tubes, the significant length is the equivalent diameter defined as ;

$$d_{eq} = 4 \times \frac{\text{cross-sectional area}}{\text{wetted perimeter}}$$

EXAMPLE 11.2

A horizontal heated plate at 200°C and facing upwards has been placed in still air at 20°C. If the plate measures 1.25 m × 1 m, make calculations for the heat loss by natural convection.

The convective film coefficient for free convection is given by the following empirical relation :

$$h = 0.32\,(\theta)^{0.25}\ \text{W/m}^2\text{K}$$

where θ is the mean film temperature in degrees kelvin.

Solution : Mean film temperature,

$$\theta = 273 + \frac{200 + 20}{2} = 383°C$$

∴ Convective film coefficient,

$$h = 3.02 \times (383)^{0.25}$$
$$= 13.36\ \text{W/m}^2\text{-deg}$$

Rate of heat loss, Q

$$= h\,A\,\Delta t$$
$$= 13.36 \times (1.25 \times 1) \times (200 - 20)$$
$$= 3006\ \text{W}$$

EXAMPLE 11.3

Two horizontal steam mains with diameters 5 cm and 15 cm are so laid in a boiler house that any mutual heat effect is precluded. The mains are at the same surface temperature of 500°C whilst the ambient air is at 50°C. Work out the ratios of the heat transfer coefficients, and of the heat losses from one metre length of the mains.

Solution : The steam mains are located in the boiler house where the ambient air is stationary. The situation then corresponds to that of free convection for which the following correlation applies :

$$Nu = C\,(Gr \times Pr)^{0.25}$$

$$Nu = C\left[\frac{d^3 \beta g \Delta t}{v^2} \times \frac{\mu c_p}{k}\right]^{0.25}$$

Since both the mains are at the same temperature and are exposed to the same ambient conditions, the relevant fluid properties are same for both.

$$\therefore \quad Nu \propto (d^3)^{0.25} \propto (d)^{0.75}$$

$$\frac{hd}{k} \propto (d)^{0.75} \text{ or } h \propto \frac{1}{(d)^{0.25}}$$

$$\therefore \quad \frac{h_1}{h_2} = \left(\frac{d_2}{d_1}\right)^{0.25} = \left(\frac{15}{5}\right)^{0.25} = 1.316$$

Heat loss $Q = h\,A\,\Delta t$

$$\therefore \quad \frac{Q_1}{Q_2} = \frac{h_1 \times \pi d_1 l}{h_2 \times \pi d_2 l}$$
$$= \frac{h_1 d_1}{h_2 d_2} = 1.316 \times \frac{5}{15} = 0.438$$

EXAMPLE 11.4

A spherical heater of 20 cm diameter and at 60°C is immersed in a tank of water at 20°C. Determine the value of convective heat transfer coefficient.

Solution : At the mean film temperature,

$$t_f = \frac{60 + 20}{2} = 40°C$$

the thermo-physical properties of water are :

$$\rho = 992.2 \text{ kg/m}^3$$
$$\nu = 0.659 \times 10^{6} \text{ m}^2/\text{s}$$
$$Pr = 4.34$$
$$k = 0.633 \text{ W/m-deg}$$

and $\beta = 0.41 \times 10^{-3}$ per degree kelvin

$$Gr = \frac{l_c^3 \rho^2 \beta g \Delta t}{\mu^2}$$

$$= \frac{d^3 \beta g \Delta t}{\nu^2} \qquad (l_c = d)$$

$$= \frac{(0.2)^3 \times 0.41 \times 10^{-3} \times 9.81 \times (60 - 20)}{(0.659 \times 10^{-6})^2}$$

$$= 2.964 \times 10^9$$

$$Gr \times Pr = 2.964 \times 10^9 \times 4.34$$
$$= 12.86 \times 10^9$$

For a sphere, the general correlation is

$$Nu = \frac{hd}{k} = 2 + 0.43\,(Gr\ Pr)^{0.25}$$
$$= 2 + 0.43\,(12.86 \times 10^9) = 146.81$$

$$\therefore \quad h = Nu \times \frac{k}{d}$$

$$= 146.81 \times \frac{0.633}{0.2}$$
$$= \textbf{458.32 W/m}^2\textbf{-deg}$$

EXAMPLE 11.5

A steam pipe 50 mm diameter and 2.5 m long has been placed horizontally and exposed to still air at 25°C. If the pipe wall temperature is 295°C, determine the rate of heat loss. At the mean temperature of 160°C, the thermo-physical properties of air are :

$$k = 3.64 \times 10^{-2} \text{ W/m-deg}$$
$$\nu = 30.09 \times 10^{-6} \text{ m}^2/\text{s}$$
$$P_r = 0.682$$

and $\beta = \frac{1}{160 + 273} = 2.31 \times 10^{-3}$ per k

Solution : Grashof number, Gr

$$= \frac{D^3 \rho^2 (\beta g \Delta T)}{\mu^2}$$

$$= \frac{D^3 (\beta g \Delta T)}{\nu^2}$$

$$= \frac{0.05^3 \times 2.31 \times 10^{-3} \times 9.81\,(295 - 25)}{(30.09 \times 10^{-6})^2}$$

$$= 0.8447 \times 10^6$$

$$Gr \times Pr = (0.8447 \times 10^6) \times 0.682$$
$$= 0.576 \times 10^6$$

For laminar flow over horizontal cylinders within the range $10^3 < Gr \times Pr < 10^9$

$$Nu = 0.53\,(Gr\ Pr)^{1/4}$$

$$\frac{hD}{k} = 0.53\,(0.576 \times 10^6)^{1/4} = 14.60$$

This gives the convective coeficient as

$$h = \frac{14.60\,k}{D}$$

$$= \frac{14.60 \times (3.64 \times 10^{-2})}{0.05}$$

$$= 10.63 \text{ W/m}^2\text{-deg}$$

Heat loss Q

$$= h\,A\,\Delta t$$
$$= 10.63 \times (\pi \times 0.05 \times 2.5) \times (295 - 25)$$
$$= 1126.5 \text{ W}$$

EXAMPLE 11.6

A nuclear reactor with its core constructed of parallel vertical plates 2.25 m high and 1.5 wide has been designed on free convection heating of liquid bismuth. Metallurgical considerations limit the maximum surface temperature of the plate to 975°C and the lowest allowable temperature of bismuth is 325°C. Estimate the maximum possible heat dissipation from both sides of each plate.

The appropriate correlation for the convection coefficient is

$$Nu = 0.13\,(Gr\ Pr)^{1/3}$$

where the different parameters are evaluated as the mean film temperature.

Solution : At the mean film temperature,

$$t_f = \frac{975 + 325}{2} = 650°C$$

the thermo-physical properties of bismuth are:

$$\mu = 3.12 \text{ kg/m-hr}$$
$$\rho = 10^4 \text{ kg/m}^3$$
$$c_p = 150.7 \text{ J/kg-deg}$$
$$k = 13.02 \text{ W/m-deg}$$

$$\beta = \frac{1}{T} = \frac{1}{650 + 273}$$
$$= 1.08 \times 10^{-3} \text{ per deg kelvin}$$

$$\therefore \quad Pr = \frac{\mu c_p}{k} = \frac{(3.12/3600) \times 150.7}{13.02}$$
$$= 0.01$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta T}{\mu^2}$$

$$= \frac{(2.25)^3 \times (10^4)^2 \times 1.08 \times 10^{-3} \times 9.81 \times (975 - 325)}{(3.12/3600)^2}$$

$$= 1043 \times 10^{13}$$

$$Gr \times Pr = 1043 \times 10^{13} \times 0.01$$
$$= 104.3 \times 10^{12}$$

Using the given correlation

$$Nu = \frac{hl}{k} = 0.13\,(Gr\ Pr)^{1/3}$$

$$h = 0.13\,(Gr\ Pr)^{1/3} \times \frac{k}{l}$$

$$= 0.13\,(104.3 \times 10^{12})^{1/3} \times \frac{13.02}{2.25}$$
$$= 34866 \text{ W/m}^2\text{-deg}$$

This gives a heat transfer of :

$$Q = 2\,h\,A\,\Delta t$$

The factor 2 accounts for two sides of the plate.

$$Q = 2 \times 34866 \times (2.25 \times 1.5) \times (975 - 325)$$
$$= 1.53 \times 10^8 \text{ watt} = \textbf{153 MW}$$

EXAMPLE 11.7

A thin walled duct of 0.5 m diameter has been laid in an atmosphere of quiescent air at 15°C and conveys a particular gas at 205°C. The boundary layer flow is laminar and the convective coefficient of heat transfer is given by :

$$h = 1.37\left(\frac{\Delta t}{l}\right)^{0.25} \text{ W/m}^2\text{-deg}$$

where l is the length of the duct in metres. How this value of convective coefficient compares with that computed from the following non-dimensional correlation for laminar flow natural convection for a large vetical cylinder,

$$\frac{hl}{k} = 0.57\,(Gr\ Pr)^{0.25}$$

Base your calculations on one metre length of the duct. Also estimate the convective heat loss from the duct.

Solution : Convective coefficient of heat transfer,

$$h = 1.37\left(\frac{\Delta t}{l}\right)^{0.25}$$

$$= 1.37\left(\frac{205 - 15}{1}\right)^{0.25}$$
$$= 5.086 \text{ W/m}^2\text{-deg}$$

(b) At the mean film temperature,

$$t_f = \frac{205 + 15}{2} = 110°C$$

the thermo-physical properties of air are :

$$\nu = 24.10 \times 10^{-6} \text{ m}^2/\text{s};$$
$$k = 31.94 \times 10^{-3} \text{ W/m-deg}$$
$$Pr = 0.704$$

and $\beta = \frac{1}{T} = \frac{1}{110 + 273} = 2.61 \times 10^{-3}$ per deg kelvin

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{1^3 \times 2.61 \times 10^{-3} \times 9.81 \times (205 - 15)}{(24.10 \times 10^{-6})^2}$$

$$= 8.37 \times 10^9$$

Using the given correlation,

$$\frac{hl}{k} = 0.57\,(8.37 \times 10^9 \times 0.704)^{0.25}$$
$$= 157.89$$

This gives the convective coefficient a

$$h = 157.89 \times \frac{k}{l}$$

$$= 157.89 \times \frac{31.94 \times 10^{-3}}{1}$$

$$= 5.043 \text{ W/m}^2\text{-deg}$$

$\therefore$ Heat lost by convection

$$= hA\,\Delta t = h\,(\pi\,d\,l)\,\Delta t$$

$$= 5.043 \times (\pi \times 0.5 \times 1) \times (205 - 15)$$

$$= 1504 \text{ W}$$

EXAMPLE 11.8

Estimate the heat transfer from a 40 W incandescent bulb at 125°C to 25°C in quiescent air. Approximate the bulb as a 50 mm diameter sphere. What percent of the power is lost by free convection ?

The appropriate correlation for the convection coefficient is

$$Nu = 0.60\,(Gr\,Pr)^{0.25}$$

where the different parameters are evaluated at the mean film temperature, and the characteristic length is the diameter of the sphere.

Solution : At the mean film temperature,

$$t_f = \frac{125 + 25}{2} = 75°C$$

the thermo-physical properties for air are :

$$\nu = 20.55 \times 10^{-6} \text{ m}^2/\text{s}$$

$$k = 0.03 \text{W/m-deg}$$

$$Pr = 0.693$$

and $\beta = \dfrac{1}{273 + 75}$

$$= 2.87 \times 10^{-3} \text{ per deg kelvin}$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{(0.05)^3 \times 2.87 \times 10^{-3} \times 9.81 \times (125 - 25)}{(20.55 \times 10^{-6})^2}$$

$$= 8.0 \times 10^5$$

Using the given correlation,

$$Nu = \frac{hl}{k} = 0.60\,(8.0 \times 10^5 \times 0.693)^{0.25}$$

$$h = \frac{0.03}{0.05} \times 0.60\,(8.0 \times 10^5 \times 0.693)^{0.25}$$

$$= 9.823 \text{ W/m}^2\text{K}$$

This gives a heat transfer of

$$Q = h\,A\,\Delta t$$

$$= 9.823 \times \pi \times 0.05^2 \times (125 - 25)$$

$$= 7.71 \text{ W}$$

Therefore the percent of heat loss by free convection is :

$$\frac{7.71}{40} \times 100 = 19.278 \;\%$$

EXAMPLE 11.9

A hot square plate 40 cm × 40 cm at 100°C is exposed to atmospheric air at 20°C. Make calculations for the heat loss from both surfaces of the plate, if (a) the plate is kept vertical (b) the plate is kept horizontal.

The following empirical aorrelations have been suggested :

$$Nu = 0.125\,(Gr\,Pr)^{0.33}$$

for vertical position of plate, and

$$Nu = 0.72\,(Gr\,Pr)^{0.25} \text{ for upper surface}$$

$$= 0.35\,(Gr\,Pr)^{0.25} \text{ for lower surface}$$

where the air properties are evaluated at the mean temperature

Solution : At the mean temperature,

$$t = \frac{100 + 20}{2} = 60°C$$

the thermo-physical proberties of air are

$$\rho = 1.06 \text{ kg/m}^3$$

$$k = 0.028 \text{ W/m-deg}$$

$$c_p = 1.008 \text{ kJ/kg K}$$

and $\nu = 18.97 \times 10^{-6} \text{ m}^2/\text{s}$

$$\beta = \frac{1}{T} = \frac{1}{273 + 60}$$

$$= 0.003 \text{ per degree kelvin}$$

$$\therefore \quad Pr = \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k}$$

$$= \frac{1.06 \times (18.97 \times 10^{-6}) \times (1.008 \times 1000)}{0.028}$$

$$= 0.724$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{0.4^3 \times 0.003 \times 9.81\,(100 - 20)}{(18.97 \times 10^{-6})^2}$$

$$= 4.19 \times 10^8$$

$$Gr \times Pr = (4.19 \times 10^8) \times 0.724$$

$$= 3.033 \times 10^8$$

(*a*) when the plate is oriented vertically,

$$Nu = 0.125 \times (3.033 \times 10^8)^{0.33}$$

$$= 78.69$$

$$h = Nu \times \frac{k}{l}$$

$$= 78.69 \times \frac{0.028}{0.4}$$

$$= 5.508 \text{ W/m}^2\text{ K}$$

This gives a heat tronsfer of :

$$Q = 2\,h\,A\,\Delta t$$

The factor 2 accounts for two sides of the plate

$$Q = 2 \times 5.508 \times (0.4 \times 0.4) \times (100 - 20)$$

$$= 141 \text{ W}$$

(*b*) When the plate is positioned horizontally

(*i*) For upper surface :

$$Nu = 0.72\,(3.003 \times 10^8)^{0.25} = 95$$

$$h = Nu\frac{k}{l}$$

$$= 95 \times \frac{0.028}{0.4} = 6.65 \text{ W/m}^2\text{ K}$$

$$Q_u = h\,A\,\Delta t = 6.65 \times (0.4 \times 0.4) \times (100 - 20)$$

$$= 85.12 \text{ W}$$

(*ii*) For lower surface :

$$Nu = 0.35\,(3.033 \times 10^8)^{0.25} = 46.19$$

$$h = Nu\frac{k}{l}$$

$$= 46.19 \times \frac{0.028}{0.4} = 3.23 \text{ W/m}^2\text{ K}$$

$$Q_l = h\,A\,\Delta t$$

$$= 3.23 \times (0.4 \times 0.4) \times (100 - 20)$$

$$= 41.35 \text{ W}$$

$$\therefore \quad Q = Q_i + Q_l$$

$$= 85.12 + 41.35 = 126.47 \text{ W}$$

Comments : The above calculations show that the plate loses more heat when it is oriented vertically. Obviously natural cooling can be achieved more effectively by keeping the plate in vertical position.

EXAMPLE 11.10

A hot plate of 15 cm² area maintained at 200°C is exposed to still air at 30°C temperature. When the smaller side of the plate is held vertical, convective heat transfer rate is 15 percent higher than when bigger side of the plate is held vertical. Make calculations for the size of the plate and heat transfer rate in both cases.

The appropriate correlation for the convection coefficient is

$$Nu = 0.60\,(Gr\,Pr)^{0.25}$$

Solution : At the mean film temperature,

$$t_f = \frac{200 + 30}{2} = 115°C$$

the thermo-physical properties of air are :

$$\rho = 0.91 \text{ kg/m}^3$$

$$c_p = 1.008 \text{ kJ/kgK}$$

$$\mu = 22.65 \times 10^{-6} \text{ Ns/m}^2$$

and $k = 0.033 \text{ W/mK}$

When the plate is oriented with smaller side vertical

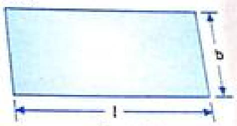

Fig. 11.2.

$$Pr_1 = \frac{\mu c_p}{k}$$

$$Gr_1 = \frac{b^3 \rho^2 \beta g \Delta t}{\mu^2}$$

$$Nu_1 = \frac{h_1 b}{k}$$

$$= 0.60\,(Gr_1\,Pr_1)^{0.25}$$

$$= 0.60 \left[\frac{b^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right.$$

$$\therefore \quad h_1 = Nu_1 \times \frac{k}{b}$$

$$= 0.60\left[\frac{b^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right]^{0.25} \times \frac{k}{b}$$

and $Q_1 = h_1 A \Delta t$

$$= 0.60\left[\frac{b^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right]^{0.25} \times \frac{k}{b} \times (b \times l)\, \Delta t \quad ...(i)$$

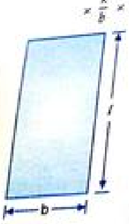

Fig. 11.3.

When the plate is oriented with larger side vertical

$$Pr_2 = \frac{\mu c_p}{k}$$

$$Gr_2 = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2}$$

$$Nu_2 = \frac{h_1 l}{k}$$

$$= 0.60\, (Gr_2\, Pr_2)^{0.25}$$

$$= 0.60\left[\frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right]^{0.25}$$

$$\therefore \quad h_2 = Nu_2 \times \frac{k}{l}$$

$$= 0.60\left[\frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right]^{0.25} \times \frac{k}{l}$$

and $Q_2 = h_2 A \Delta t$

$$= 0.60\left[\frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right]^{0.25} \times \frac{k}{l} \times (b \times l)\, \Delta t \quad ...(ii)$$

From expressions (i) and (ii) and from the given candition

$$\frac{Q_1}{Q_2} = 1.15$$

$$= \left(\frac{b^3}{l^3}\right)^{0.25} \times \frac{l}{b} = \left(\frac{b}{l}\right)^{0.75} \times \frac{l}{b} = \left(\frac{l}{b}\right)^{0.25}$$

$$\therefore \quad \frac{l}{b} = (1.15)^{1/0.25} = 1.749$$

$l \times b = 15\ \text{cm}^2$ (given)

$1.749\, b \times b = 15$

$\therefore \quad$ **$b = 2.928$ cm**

$l = 5.123$ cm

Now, $\beta = \frac{1}{T}$

$$= \frac{1}{115 + 273} = \frac{1}{388} = 0.00258$$

per degree kelvin

$$Gr_1 = \frac{(0.02928)^3 \times (0.91)^2 \times 0.00258 \times 9.81 \times (200 - 30)}{(22.65 \times 10^{-6})^2}$$

$$= 0.174 \times 10^6$$

$$Pr_1 = \frac{\mu c_p}{k}$$

$$= \frac{22.65 \times 10^{-6} \times 1.008 \times 10^3}{0.033}$$

$$= 0.692$$

$$Nu_1 = \frac{h_1 \times b}{k}$$

$$= 0.60\, (0.174 \times 10^6 \times 0.692)^{0.25}$$

$$= 11.177$$

$$\therefore \quad h_1 = 11.177 \times \frac{0.033}{0.02928}$$

$$= 12.597\ \text{W/m}^2\text{-deg}$$

$$Q_1 = h_1 A \Delta t$$

$$= 12.597 \times (15 \times 10^{-4}) \times (200 - 30)$$

$$= \mathbf{3.21\ W}$$

and $Q_2 = \frac{Q}{1.15} = \frac{3.21}{1.15} = $ **2.79 W**

EXAMPLE 11.11

A 3.5 kW plate heater of 15 cm × 30 cm is held vertically with larger side vertical in a water bath at 40°C. Make calculations for the steady state temperature attained by the heater if heat transfer is only due to convection.

Use the following correlation

$$Nu = 0.13\, (Gr\, Pr)^{0.33}$$

and take $\beta = 4.15 \times 10^{-4}$ *per degree centigrade.*

Solution : Let it be preusmed that the plate temperature t_p is 100°C. Then at the mean film temperature

$$t_f = \frac{100 + 40}{2} = 70°\text{C}$$

the thermo-physical properties of water are :

$\rho = 977.8\ \text{kg/m}^3$

$c_p = 4.187\ \text{kJ/kgK}$

$\nu = 0.415 \times 10^6\ \text{m}^2/\text{s}$

and $k = 0.667\ \text{W/mK}$

Then $Pr = \frac{\mu c_p}{k} = \frac{\rho \nu \times c_p}{k}$

$$= \frac{977.8 \times 0.415 \times 10^{-6} \times 4.187 \times 1000}{0.667}$$

$$= 2.547$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2}$$

$$= \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{(0.3)^3 \times 4.15 \times 10^{-4} \times 9.81 \times (t_p - 40)}{(0.415 \times 10^{-6})^2}$$

$$= 6.38 \times 10^8\, (t_p - 40)$$

$$Nu = \frac{hl}{k}$$

$$= 0.13\left[6.38 \times 10^8 (t_p - 40) \times 2.547\right]^{0.33}$$

$$\therefore h = Nu \times \frac{k}{l}$$

$$= 0.13\left[6.38 \times 10^8 (t_p - 40) \times 2.547\right]^{0.33} \times \frac{0.667}{0.3}$$

$$= 316.6\, (t_p - 40)^{0.33}$$

and $Q = h A \Delta t$

$$= 316.6\, (t_p - 40)^{0.33} \times (0.2 \times 0.3) \times (t_p - 40)$$

Under steady state conditions, this heat flow must equal the heat capacity of the heater. That is

$$3500 = 316.6\, (t_p - 40)^{0.33} \times (0.2 \times 0.3) \times (t_p - 40)$$

$$(t_p - 40)^{1.33} = \frac{3500}{316.6 \times (0.2 \times 0.3)} = 184.25$$

$$t_p - 40 = (184.25)^{1/1.33}$$

$$= (184.25)^{0.752} = 50.53$$

$\therefore$ Plate temperature t_p

$$= 50.53 + 40 = \mathbf{90.53°C}$$

Note : Calculations may be repeated by taking the properties of water at (90 + 40)/2 = 65°C. However not much change in the value of plate temperature is expected.

EXAMPLE 11.12

Consider a cubical block 10 cm × 10 cm in size and suspended in still air at 20°C. All surfaces of the block are maintained at 160°C and one of its surface lies in the horizontal position. Determine the total heat loss from the block.

Solution : At the mean film temperature,

$$t_f = \frac{160 + 20}{2} = 90°\text{C}$$

the thermo-physical properties of air are :

$\nu = 22.10 \times 10^6\ \text{m}^2/\text{s}$

$k = 0.03217\ \text{W/m-deg}$

$Pr = 0.690$

and $\beta = \frac{1}{T} = \frac{1}{90 + 273} = \frac{1}{363}$

$= 0.00275$ per degree kelvin

$$Gr = \frac{l_c^3 \rho^2 \beta g \Delta t}{\mu^2}$$

(l_c is the reference length)

$$= l_c^3 \frac{\beta g \Delta t}{\nu^2}$$

$$= l_c^3 \times \frac{0.00275 \times 9.81 \times (160 - 20)}{(22.10 \times 10^{-6})^2}$$

$$= 7.73 \times 10^9 \, l_c^3$$

$$(Gr \times Pr) = 7.73 \times 10^9 l_c^3 \times 0.690$$

$$= 5.33 \times 10^9 \, l_c^3$$

(*i*) For four vertical faces of the cube

$$= l_c = l = 10 \text{ cm} = 0.1 \text{ m}$$

$$Gr\, Pr = 5.33 \times 10^9 \times (0.1)^3$$

$$= 5.33 \times 10^6$$

Obviously the boundary layer is laminar, and accordingly

$$Nu = \frac{hl}{k} = 0.59\,(Gr\, Pr)^{0.25}$$

$$= 0.59\,(5.33 \times 10^6)^{0.25} = 28.348$$

$$\therefore \quad h = 28.348 \times \frac{0.03217}{0.1}$$

$$= 9.12 \text{ W/m}^2\text{-deg}$$

Heat loss, Q_1

$$= h\, A\, \Delta t$$

$$= 9.12 \times (4 \times 0.1 \times 0.1) \times (160 - 20)$$

$$= 51.07 \text{ W}$$

(*ii*) For top face of the cube

$$l_c = \frac{A}{P} = \frac{ab}{2(a + b)}$$

$$= \frac{0.1 \times 0.1}{2(0.1 + 0.1)} = 0.025 \text{ m}$$

$$Gr\, Pr = 5.33 \times 10^9 \times (0.025)^3 = 83281$$

Obviously the boundary layer is laminar for the horizontal top face of the cube, and accordingly

$$Nu = 0.54\,(Gr\, Pr)^{0.25}$$

...(heated surface up)

$$= 0.54\,(83281)^{0.25} = 9.17$$

$$\therefore \quad h = Nu \frac{k}{l_c} = 9.17 \times \frac{0.03217}{0.025}$$

$$= 11.8 \text{ W/m}^2\text{-deg}$$

Heat loss, Q_2

$$= h\, A\, \Delta t$$

$$= 11.8 \times (0.1 \times 0.1) \times (160 - 20)$$

$$= 16.52 \text{ W}$$

(*iii*) For bottom face of the cube

$$l_c = \frac{A}{P} = \frac{0.1 \times 0.1}{2(0.1 + 0.1)} = 0.025$$

$$Gr\, Pr = 5.33 \times 10^9 \times (0.025)^3 = 83281$$

Obviously the boundary layer is laminar for the horizontal bottom face of the cube, and accordingly

$$Nu = 0.27\,(Gr\, Pr)^{0.25}$$

...(heated surface down)

$$= 0.27\,(83281)^{0.25} = 4.587$$

$$\therefore \quad h = Nu \frac{k}{l_c} = 4.587 \times \frac{0.03217}{0.025}$$

$$= 5.9 \text{ W/m}^2\text{-deg}$$

Heat loss, Q_3

$$= h\, A\, \Delta t$$

$$= 5.9 \times (0.1 \times 0.1) \times (160 - 20)$$

$$= 8.26 \text{ W}$$

$\therefore$ Total heat loss from the cubical block

$$= 51.07 + 16.52 + 8.26$$

$$= \mathbf{75.85 \text{ W}}$$

EXAMPLE 11.13

A hot plate 1 m × 0.5 m at 180°C is kept in still air at 20°C with 0.5 m side vertical. The plate has a mass of 20 kg and is made of a material having specific heat 400 J/kg-deg. If convection takes place from both sides of the plate, determine heat transfer coefficient, initial rate of cooling the plate, and time required in cooling the plate from 120°C to 80°C.

Solution : At the mean film temperature,

$$t_f = \frac{180 + 20}{2} = 100°\text{C}$$

the thermo-physical properties of air are :

$$k = 0.03208 \text{ W/m-deg}$$

$$\nu = 23.13 \times 10^{-6} \text{ m}^2/\text{s}$$

$$Pr = 0.688$$

$$\beta = \frac{1}{T} = \frac{1}{100 + 273} = 0.00268$$

per degree kelvin

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{(0.5)^3 \times 0.00268 \times 9.81\,(180 - 20)}{(23.13 \times 10^{-6})^2}$$

$$= 982.84 \times 10^6$$

$$Gr\, Pr = (982.84 \times 10^6) \times 0.688$$

$$= 676.2 \times 10^6$$

Obviously the boundary layer is turbulent, and accordingly

$$Nu = \frac{hl}{k} = 0.14\,(Gr \times Pr)^{0.33}$$

$$= 0.14\,(676.2 \times 10^6)^{0.33} = 114.83$$

$$\therefore \quad h = 114.83 \times \frac{0.03208}{0.5}$$

$$= \mathbf{7.367 \text{ W/m}^2\text{-deg}}$$

(*ii*) The initial rate of cooling can be obtained by energy balance

Rate of decrease of internal energy = Rate of heat convection from the plate

$$m\, c \frac{dt}{d\tau} = h\, A\, \Delta t$$

or $$\frac{dt}{d\tau} = \frac{h\, A\, \Delta t}{m\, c}$$

$$= \frac{7.367 \times (2 \times 1 \times 0.5) \times (180 - 20)}{20 \times 400}$$

$$= \mathbf{0.1473°C/s}$$

(*iii*) The time required to cool the plate from 180°C to 80°C can be worked out by using the relation

$$\frac{t - t_a}{t_i - t_a} = \exp\left(-\frac{h\, A}{\rho\, V\, c}\tau\right)$$

$$= \exp\left(-\frac{h\, A}{m\, c}\tau\right)$$

or $$\tau = -\frac{mc}{hA} \log_e \frac{t - t_a}{t_i - t_a}$$

$$= -\frac{20 \times 400}{7.367 \times (2 \times 1 \times 0.5)} \times \log_e \frac{80 - 20}{180 - 20}$$

$$= -1085.92 \times (-0.9808)$$

$$= \mathbf{1065 \text{ s}}$$

EXAMPLE 11.14

A vertical plate is under free convection with ambient still air at 20°C. If the plate is heated from one side and maintained at 80°C, workout the local heat transfer coefficient at 20 cm from the lower edge. What would be the average value of convective coefficient over the 20 cm length ?

Adopt the following correlation for the local Nusselt number.

$$Nu_x = 0.52\left(\frac{Pr}{0.95 + Pr}\right)^{0.25} (Gr\, Pr)^{0.25}$$

Solution : At the mean film temperature,

$$t_f = \frac{80 + 20}{2} = 50°\text{C}$$

the thermo-physical properties of air are :

$$\rho = 1.093 \text{ kg/m}^3$$

$$Pr = 0.698$$

$$k = 10.17 \times 10^{-2} \text{ kJ/m-hr-K}$$

and $$\nu = 17.95 \times 10^{-6} \text{ m}^2/\text{s}$$

$$\beta = \frac{1}{273 + 50}$$

$$= 3.096 \times 10^{-3} \text{ per degree kelvin}$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta T}{\nu^2}$$

$$= \frac{0.2^3 \times 3.096 \times 10^{-3} \times 9.81 \times (80 - 20)}{(17.95 \times 10^{-6})^2}$$

$$= 45.2 \times 10^6$$

$$\therefore \quad Nu_x = 0.52\left(\frac{0.698}{0.95 + 0.698}\right)^{0.25} \times (45.2 \times 10^6 \times 0.698)^{0.25}$$

$$= 31.439$$

Thus the local heat transfer coefficient,

$$h_x = \frac{21.439 \times k}{l}$$

$$= \frac{21.439 \times (10.17 \times 10^{-2})}{0.20}$$

$$= 15.98 \text{ kJ/m}^2\text{-hr-deg}$$

(ii) The local and average convection coefficient are related by an expression of the form:

$$\bar{h} = \frac{1}{l}\int_0^l h_x \, dx$$

$$= \frac{1}{l}\int_0^l \frac{k}{x} \times 0.52\left(\frac{Pr}{0.95 + Pr}\right)^{0.25} \left(\frac{x^3 \beta g \Delta T}{\nu^2} \times Pr\right)^{0.25} dx$$

$$= 0.52\,\frac{k}{l}\left(\frac{Pr}{0.95 + Pr}\right)^{0.25}\left(\frac{\beta g \Delta T}{\nu^2}\,Pr\right)^{0.25} \int_0^l \frac{\left(x^3\right)^{0.25}}{x}\,dx$$

$$= 0.52\,\frac{k}{l}\left(\frac{Pr}{0.95 + Pr}\right)^{0.25}\left(\frac{\beta g \Delta T}{\nu^2}\,Pr\right)^{0.25} \times \frac{4}{3} l^{0.75}$$

$$= \frac{4}{3}\,\frac{k}{l}\left(\frac{Pr}{0.95 + Pr}\right)^{0.25}\left(\frac{l^3 \beta g \Delta T}{\nu^2}\,Pr\right)^{0.25}$$

$$= \frac{4}{3} \text{ local convective coefficient}$$

$$= \frac{4}{3} \times 15.98$$

$$= 21.31 \text{ kJ/m}^2\text{-hr-deg}$$

EXAMPLE 11.15

Calculate the rate of heat loss from a human body which may be considered as a vertical cylinder 30 cm in diameter and 175 cm high in still air at 15°C. The skin temperature is 35°C and emissivity at the skin surface is 0.4. Neglect sweating and effect of clothing.

Solution : At the mean film temperature,

$$t_f = \frac{35 + 15}{2} = 25°\text{C}$$

the thermo-physical properties of air are:

$$\nu = 15.53 \times 10^{-6} \text{ m}^2/\text{s}$$

$$k = 0.0263 \text{ W/m-deg}$$

$$Pr = 0.7$$

and $\beta = \frac{1}{T} = \frac{1}{273 + 25} = \frac{1}{298}$

$= 0.00335$ per degree kelvin

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{(1.75)^3 \times 0.00335 \times 9.81 \times (35 - 15)}{(15.53 \times 10^{-6})^2}$$

$$= 1.46 \times 10^{10}$$

$$Gr \times Pr = 1.46 \times 10^{10} \times 0.7 = 1.022 \times 10^{10}$$

Since $(Gr \times Pr)$ is more than 10^9, the flow is turbulent, and accordingly

$$Nu = \frac{hl}{k} = 0.13\,(Gr \times Pr)^{0.33}$$

$$= 0.13\,(1.022 \times 10^{10})^{0.33} = 261.08$$

$$h = Nu \times \frac{k}{l} = 261.08 \times \frac{0.0263}{1.75}$$

$$= 3.924 \text{ W/m}^2\text{-deg}$$

Convective heat loss, $Q_c = h\,A\,\Delta t$

$$= h\,\pi\,d\,l\,\Delta t = 3.924 \times (\pi \times 0.3 \times 1.75) \times (35 - 15)$$

$$= 129.37 \text{ W}$$

Radiation heat loss, Q_r

$$= \epsilon\,\sigma_0\,A\,(T_1^4 - T_2^4)$$

$$= 0.4 \times (5.67 \times 10^{-8})\,(\pi \times 0.3 \times 1.75)\,(308^4 - 288^4)$$

$$= 79.24 \text{ W}$$

$\therefore$ Total loss from the human body, Q_t

$$= Q_c + Q_r$$

$$= 129.37 + 79.24 = 208.61 \text{ W}$$

EXAMPLE 11.16

A wood burning stove of 1 m outside diameter utilises a hollow lining with radiation shields to reduce its outside temperature. Determine the number of shields required to keep the outside of stove at 100°C. Assume the emissivity of all surfaces

to be 0.85. The temperature of the still atmospheric air is 20°C and the temperature in the stove is 500°C.

Solution : At the mean film temperature,

$$t_f = \frac{100 + 20}{2} = 60°\text{C}$$

the relevant thermo- physical properties of air are

$$k = 0.0289 \text{ W/mK}$$

$$\nu = 18.87 \times 10^{-6} \text{ m}^2/\text{s}$$

$$Pr = 0.696$$

and $\beta = \frac{1}{60 + 273}$

$= 3 \times 10^{-3}$ per degree kelvin

Grashof number, Gr

$$= \frac{D^3 \rho^2 \beta g \Delta t}{\mu^2}$$

$$= \frac{D^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{1^3 \times 3 \times 10^{-3} \times 9.81 \times (100 - 20)}{\left(18.97 \times 10^{-6}\right)^2}$$

$$= 6.54 \times 10^9$$

$$Gr \times Pr = 6.54 \times 10^9 \times 0.696 = 4.55 \times 10^9$$

A quiescent atmosphere corresponds to natural free convection. Further for such a flow with $10^9 < Gr\,Pr < 10^{12}$;

$$Nu = \frac{hd}{k} = 0.13\,(Gr\,Pr)^{0.33}$$

$$= 0.13\,(4.55 \times 10^9)^{0.33}$$

$$= 214.33$$

Convective coefficient, h

$$= \frac{k\,Nu}{d} = \frac{0.0289 \times 214.33}{1}$$

$$= 6.20 \text{ W/m}^2\text{-deg}$$

$\therefore$ Convective heat loss,

$$\frac{Q_c}{A} = h\,\Delta t = h\,(t_2 - t_3)$$

$$= 6.20\,(100 - 20)$$

$$= 496 \text{ W/m}^2$$

Radiation heat loss Q_r to surroundings,

$$\frac{Q_r}{A} = \left(F_g\right)_{23}\,\sigma_b\left(T_2^4 - T_3^4\right)$$

where factor

$$\left(F_g\right)_{23} = \frac{1}{\frac{1-\epsilon_2}{\epsilon_2} + \frac{1}{F_{23}} + \frac{1-\epsilon_3}{\epsilon_3} \times \frac{A_2}{A_3}}$$

For the given arrangement $A_2 <<< A_3$ and $F_{23} = 1$ and therefore

$$\left(F_g\right)_{23} = \frac{1}{\frac{1-\epsilon_2}{\epsilon_2} + 1} = \epsilon_2 = 0.85$$

$$\therefore \quad \frac{Q_r}{A} = 0.85 \times (5.67 \times 10^{-8}) \times (373^4 - 293^4)$$

$$= 577.7 \text{ W/m}^2$$

$\therefore$ Total heat loss from the surface to stove

$$\frac{Q}{A} = 496 + 577.7 = 1063.7 \text{ W/m}^2$$

With n-radiation shields of equal emissivities, the heat flow from internal surface of the hollow lining to external surface of stove is

$$\frac{Q_{12}}{A} = \frac{1}{n+1}\,\frac{\sigma_b\left(T_1^4 - T_2^4\right)}{\left[\frac{2}{\epsilon} - 1\right]}$$

$$= \frac{1}{n+1}\,\frac{5.67 \times 10^{-8}\left(773^4 - 373^4\right)}{\left[\frac{2}{0.85} - 1\right]}$$

$$= \frac{14150}{n+1} \text{ W/m}^2$$

From energy balalnce,

$$\frac{14150}{n+1} = 1063.7 \;;$$

$$n + 1 = 13.3$$

$\therefore$ Radiation shields required are $n = 13$

EXAMPLE 11.17

A 20 cm diameter pipe has been laid in an atmosphere of quiescent air at 15°C and conveys gas at 370°C. Find the heat lost per metre length of the bare pipe if the convection heat transfer coefficient from a hot cylindrical surface freely exposed to still air is prescribed by the relation

$$h = 1.53\left(\frac{\Delta t}{d}\right)^{0.25} \text{ W/m}^2\text{-deg}$$

where Δt is the temperature difference and d is the diameter of the cylinder in metres.

What percentage reduction in heat loss would occur if the pipe is covered with 5 cm thick layer of material whose thermal conductivity is 0.069 W/m-deg ? Neglect any temperature drop through the metal.

Solution : Convective heat transfer coefficient,

$$h = 1.53\left(\frac{370-15}{0.2}\right)^{0.25}$$
$$= 9.93 \text{ W/m}^2\text{-deg}$$

Convective heat loss from the bare pipe,

$$Q = h\,A\,\Delta t$$
$$= 9.93 \times (\pi \times 0.2 \times 1) \times (370 - 15)$$
$$= \mathbf{2214\ W}$$

(*b*) When the pipe is lagged

$$d_1 = 20 \text{ cm} ;$$
$$d_2 = 20 + (2 \times 5) = 30 \text{ cm}$$

When steady state is attained, the heat conducted through the lagging equals the convective heat loss from the outer surface of the lagging. That is

$$\frac{2\pi k\,l\,(t_1 - t)}{\log_e \frac{r_2}{r_1}} = h \times (\pi\,d_2\,l) \times (t - t_a)$$

where t is the temperature at the outer surface of the lagging

Inserting the appropriate values,

$$\frac{2\pi \times 0.069 \times 1 \times (370 - t)}{\log_e \frac{0.15}{0.1}} =$$
$$1.53\left(\frac{t-15}{0.3}\right)^{0.25} \times (\pi \times 0.3 \times 1) \times (t - 15)$$

or $370 - t = 1.82\,(t - 15)^{1.25}$

By trial and error : $t = 74°C$

∴ *Heat loss from the lagged pipe,*

$$Q = \frac{2\pi k\,l\,(t_1 - t)}{\log_e \frac{r_2}{r_1}}$$

$$= \frac{2\pi \times 0.069 \times 1\,(374 - 74)}{\log_e \frac{0.15}{0.1}}$$
$$= \mathbf{316.7\ W}$$

Reduction in heat loss

$$\frac{2214 - 316.7}{2214} = 0.857 \text{ or } \mathbf{85.7\ \%}$$

EXAMPLE 11.18

A 20 cm diameter pipe carrying gas at 350°C has been laid horizontally and exposed to ambient air at 25°C. Workout the heat loss by convection from the pipe. The convective coefficient is given by :

$$h = 1.82\left(\frac{\Delta t}{d}\right)^{1/4} \text{ kJ/m}^2\text{-hr-deg}$$

where d is the pipe diameter in metres and Δt is the temperature differential.

(b) The pipe is now covered with a layer of insulating material 50 mm thick for which the coefficient of conductivity is 0.398 kJ/m-hr-deg. Make calculations to show that the outer surface of the material will attain a temperature of about 166.5°C. Also determine the heat loss per metre length of the pipe. Neglect temperature drop through the metallic pipe.

Solution : (*a*) When the pipe is bare :

Convective heat coefficient, h

$$= 1.82\left(\frac{350-25}{0.20}\right)^{1/4}$$
$$= 11.55 \text{ kJ/m}^2\text{-hr-deg}$$

Heat loss, Q

$$= h\,A\,\Delta t$$
$$= 11.55 \times (\pi \times 0.2 \times 1)\,(350 - 25)$$
$$= \mathbf{2357.35\ kJ/hr}$$

(*b*) When the pipe is lagged :

$$d_i = 20 \text{ cm} \quad \text{and} \quad d_0 = 30 \text{ cm}$$

Under steady conditions, the heat conducted through the insulated layer will be equal to the heat loss by convection from the outer surface. Thus

$$\frac{2\pi k\,l\,(t_i - t_0)}{\log_e \frac{r_0}{r_1}} = h_0\,\pi\,d_0\,l\,(t_o - t_a)$$

where t_o is the temperature at the outer surface of insulation. Inserting the appropriate values :

$$\frac{2\pi \times 0.398 \times 1\,(350 - t_0)}{\log_e \frac{0.15}{0.1}} =$$
$$1.82\left(\frac{t_0 - 25}{0.30}\right)^{1/4} \pi \times 0.3 \times 1\,(t_0 - 25)$$

or $6.171\,(350 - t_0) = 2.317\,(t_0 - 25)^{1.25}$

or $2.66\,(350 - t_0) = (t_0 - 25)^{1.25}$

Solution, through trial and error, yields

$$t_0 = \mathbf{166.5°C}$$

Heat loss from the lagged pipe,

$$= \frac{2\pi \times 0.398\,(350 - 166.5)}{\log_e \frac{0.15}{0.1}}$$

(for unit length of pipe)

$$= \mathbf{1132.5\ kJ/hr}$$

EXAMPLE 11.19

A thin walled vertical duct of circular cross-section is 0.4 m in diameter. That duct carries a gas at 470 K and the surrounding air may be considered still at 290 K. Determine the heat transfer rate from one metre length of the duct assuming that the boundary layer is laminar.

The general non-dimensional correlation for laminar flow, natural convection from large vertical cylinders is :

$$\frac{hl}{k} = 0.56\left(\frac{l^3\,\rho^2\beta\,g\,\Delta t}{\mu^2}\right)^{0.25}\left(\frac{\mu\,c_p}{k}\right)^{0.25}$$

The fluid properties are to be evaluated at film temperature which is defined as the average of the bulk fluid and wall temperature.

(b) The heat transfer coefficient is to be prescribed by the relation:

$$h = C\left(\frac{\Delta t}{l}\right)^{0.25} \text{ W/m}^2\text{ K}$$

where the length parameter l is in metres. Calculate the value of constant C which would give the same heat transfer rate.

At the film temperature, the thermo-physical properties of the gas are:

$$\rho = 0.9315 \text{ kg/m}^3$$
$$c_p = 1.012 \text{ kJ/kgK}$$
$$\mu = 22.016 \times 10^{-6} \text{ kg/ms}$$
$$k = 3.2215 \times 10^{-2} \text{ W/mK}$$
and
$$\beta = 2.631 \times 10^{-3} \text{ per deg kelvin}$$

Solution :

$$\frac{\mu\,c_p}{k} = \frac{22.016 \times 10^{-6} \times (1.012 \times 10^3)}{3.2215 \times 10^{-2}}$$
$$= 0.6916$$

$$\frac{l^3\rho^2\beta\,g\,\Delta t}{\mu^2} = \frac{1^3 \times 0.9315^2 \times 2.631 \times 10^{-3} \times 9.81 \times (470 - 290)}{(22.016 \times 10^{-6})^2}$$
$$= 8.316 \times 10^9$$

$$\frac{hl}{k} = 0.56\,(8.316 \times 10^9 \times 0.6916)^{0.25}$$
$$= 154.21$$

Thus the convective film coefficient,

$$h = 154.21 \times \frac{k}{l}$$
$$= 154.21 \times \frac{3.2215 \times 10^{-2}}{1}$$
$$= 4.968 \text{ W/m}^2\text{ K}$$

Heat flow rate Q

$$= h\,A\,\Delta t$$
$$= 4.968\,(\pi \times 0.4 \times 1)\,(470 - 290)$$
$$= \mathbf{1123.16\ W}$$

(*b*) Substituting the relevant data in the prescribed correlation,

$$4.968 = C\left(\frac{470-290}{1}\right)^{0.25}$$

$$\text{Constant } C = \frac{4.968}{180^{0.25}} = \mathbf{1.3566}$$

EXAMPLE 11.20

An electrically heated square plate, 50 cm × 50 cm, has one of its surface thermally insulated and the other surface dissipates heat by free convection into atmospheric air at 20°C. The heat flux over the surface of the plate is uniform and results in a mean temperature of 60°C. If the plate is inclined at an angle of 50 degree from the vertical, make calculations for the heat loss from the plate for the heated surface facing up.

Solution : For constant heat flux conditions the thermo-physical properties are evaluated at

$$t_f = t_w - 0.25\,(t_w - t_\infty)$$
$$= 60 - 0.25\,(60 - 20) = 50°C$$
$$k = 0.02824 \text{ W/m-deg}$$
$$\nu = 17.95 \times 10^{-6} \text{ m}^2/\text{s}$$
$$Pr = 0.698$$

β is evaluated at t_∞ :

$$\beta = \frac{1}{273 + 20}$$
$$= 0.0034 \text{ per degree kelvin}$$
$$l_c = \text{length of the side of square} = 0.5 \text{ m}$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$
$$= \frac{(0.5)^3 \times 0.0034 \times 9.81 \times (60 - 20)}{(17.95 \times 10^{-6})^2}$$
$$= 517.6 \times 10^6$$
$$Gr \times \cos\theta = 517.6 \times 10^6 \times \cos 50°$$
$$= 332.70 \times 10^6$$
$$Gr \cos\theta \times Pr = 332.70 \times 10^6 \times 0.698$$
$$= 232.22 \times 10^6$$

Since the heated surface is up and the parameter $(Gr \cos\theta \times P_r)$ lies between

$$2 \times 10^7 \text{ and } 3 \times 10^{10}$$

the boundary layer is turbulent and accordingly

$$Nu = 0.14\,(Gr\,Pr\cos\theta)^{0.33}$$
$$= 0.14\,(232.22 \times 10^6)^{0.33} = 80.70$$
$$h = Nu \times \frac{k}{l_c}$$
$$= 80.70 \times \frac{0.02824}{0.5}$$
$$= 4.56 \text{ W/m}^2\text{-deg}$$

Heat trasfer rate

$$= h\,A\,\Delta t$$
$$= 4.56 \times (0.5 \times 0.5) \times (60 - 20)$$
$$= 45.6 \text{ W}$$

EXAMPLE 11.21

Two horizontal surfaces separated by a distance of 5 cm have air between them at atmospheric pressure. Make calculations for the heat flux if the upper surface is at 50°C and the lower surface is at 26°C.

Solution : At the mean temperature,

$$\frac{50 - 26}{2} = 38°C$$

the thermo-physical properties of air are :

$$\nu = 1.68 \times 10^{-5} \text{ m}^2/\text{s}$$
$$k = 0.028 \text{ W/m-deg}$$
$$\beta = \frac{1}{T_m} = \frac{1}{273 + 38} = \frac{1}{311}$$
$$= 0.00321 \text{ per degree kelvin}$$

The following correlations have been suggested by Jacob for horizontal enclosed air spaces

$$\frac{k_e}{k} = \overline{Nu} = \frac{\bar{h}\,l}{k}$$
$$= 0.195\,(Gr)^{1/4}$$
for $10^4 < Gr < 4 \times 10^5$
$$= 0.068\,(Gr)^{1/3} \text{ for } 4 \times 10^5 < Gr$$

where k_e = effective thermal conductivity,

$$\frac{Q}{A} = k_e\left(\frac{t_1 - t_2}{l}\right)$$

where l is thickness of the air space and Gr is based on thickness of air space.

$$\text{Now, } Gr = \frac{l^3 \beta g \Delta t}{\nu^2}$$
$$= \frac{(0.05)^3 \times 0.00321 \times 9.81 \times (50 - 26)}{(1.68 \times 10^{-5})^2}$$
$$= 3.347 \times 10^5$$
$$\therefore \quad \overline{Nu} = 0.195 \times (3.347 \times 10^5)^{1/4} = 4.69$$
$$\bar{h} = 4.69 \times \frac{0.028}{0.05} = 2.63 \text{ W/m}^2\text{-deg}$$
$$\frac{Q}{A} = \bar{h}\,(t_h - t_c)$$
$$= 2.63 \times (50 - 26) = 63.12 \text{ W/m}^2$$

EXAMPLE 11.22

Two parallel walls, each 1.25 m high, form a 7.5 cm thick vertical slot containing air at atmospheric pressure. Make calculations for the effective thermal conductivity and heat flux if the hotter and cooler walls are at 77°C and 27°C temperature respectively.

Solution : At the mean temperature,

$$\frac{77 + 27}{2} = 52°C$$

the thermo-physical properties of air are :

$$\nu = 1.824 \times 10^{-5} \text{ m}^2/\text{s}$$
$$k = 0.0281 \text{ W/m-deg}$$
$$\beta = \frac{1}{T_m} = \frac{1}{273 + 52} = \frac{1}{325}$$
$$= 0.003077 \text{ per degree kelvin}$$

The following correlations have been suggested by Jacob for vertical enclosed spaces:

$$\frac{k_e}{k} = \overline{Nu} = \frac{\bar{h}\,l}{k} = 0.18\,(Gr)^{1/4}\,(h/l)^{-1/9}$$
for $2000 < Gr < 2 \times 10^4$
$$= 0.065\,(Gr)^{1/3}\,(h/l)^{-1/9}$$
for $2 \times 10^4 < Gr$

where k_e is effective thermal conductivity

Here h and l are height and thickness of air space and Gr is based on thickness of air space

$$\text{Now, } Gr = \frac{l^3 \beta g \Delta t}{\nu^2}$$
$$= \frac{(0.075)^3 \times 0.003077 \times 9.81 \times (77 - 27)}{(1.824 \times 10^{-5})^2}$$
$$= 1.913 \times 10^6$$
$$\therefore \quad \overline{Nu} = 0.065\,(1.913 \times 10^6)^{1/3} \times \left(\frac{1.25}{0.075}\right)^{-1/9}$$
$$= 5.876$$
$$\bar{h} = \bar{N}_u \times \frac{k}{l} = 5.876 \times \frac{0.0281}{0.075}$$
$$= 2.20 \text{ W/m}^2\text{-deg}$$

$$\frac{Q}{A} = \bar{h}\,(t_1 - t_2) = 2.20 \times (77 - 27)$$
$$= 110 \text{ W/m}^2$$

Effective thermal conductivity k_e

$$= k\,\overline{Nu} = 0.0281 \times 5.876$$
$$= 0.165 \text{ W/m-deg}$$

Calculations for the heat flux may then be made by using the relation

$$\frac{Q}{A} = \frac{k_e\,(t_1 - t_2)}{l} = \frac{0.165\,(77 - 27)}{0.075}$$
$$= 110 \text{ W/m}^2$$

11.3.5. Simplified Free Convection Relations for Air

Recall the following correlation for free convection,

$$Nu = C\,(Gr\,Pr)^m$$
$$\frac{hl}{k} = C\left(\frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} \times \frac{\mu c_p}{k}\right)^m$$
$$\text{or} \quad h = C\,k\left(\frac{\rho^2 \beta g c_p}{\mu k}\right)^m \frac{\Delta t^m}{l^{1-3m}}$$

The group $(\rho^2 \beta g c_p)/(\mu k)$ contains only fluid properties and is called the free convection modulus. For the purpose of rapid and approximating estimate of free convection coefficient for air at normal atmosphere conditions, a mean value of

$$k\,[(\rho^2 \beta g c_p)/(\mu k)]^m$$

is chosen and taken as constant. With that stipulation, the free convection coefficient may be computed from the correlation,

$$h = C_1 \frac{\Delta t^m}{l^{1-3m}}$$

The exponent m is usually 0.5 for laminar flow and 0.33 for turbulent flow. That gives :

$$h = C_1\left(\frac{\Delta t}{l}\right)^{0.25} \quad \text{laminar flow}$$
$$\text{and} \quad h = C_1\,(\Delta t)^{0.33} \quad \text{turbulent flow}$$

In general, the simplified relation is

$$h = C\left(\frac{\Delta t}{l}\right)^m \qquad ...(11.20)$$

Table 11.1. Simplified free convection equations for air

Surface and its Orientation	Laminar flow $10^4 < Gr\,Pr \le 10^9$	Turbulent flow $Gr\,Pr > 10^9$
(i) Vertical planes or cylinders	$1.42\left(\frac{\Delta t}{l}\right)^{1/4}$	$1.31\ (\Delta t)^{1/3}$
(ii) Horizontal cylinders	$1.32\left(\frac{\Delta t}{d}\right)^{1/4}$	$1.24\ (\Delta t)^{1/3}$
(iii) Horizontal plates : Heated plates facing upward or cooled plates facing downward	$1.32\left(\frac{\Delta t}{l}\right)^{1/4}$	$1.52\ (\Delta t)^{1/3}$
(iv) Horizontal plates : Heated plates facing downward or cooled plates facing upward.	$0.59\left(\frac{\Delta t}{l}\right)^{1/4}$	... ...

where C and m are constants depending on geometry and flow conditions; the significant length l is also a function of geometry and flow. Table 11.1 lists the values suggested by Mc Adams for various geometries, orientations and flow conditions indicated by the magnitude of product $(Gr\,Pr)$.

The terms in these correlations are dimensional: h is measured in W/m^{2}K, Δt in degrees kelvin, and d (or l) in metres

EXAMPLE 11.23

A steam pipe 6 cm in diameter is covered with 2 cm thick layer of insulation which has a surface emissivity of 0.92. The insulation surface temperature is 75°C and the pipe is placed in atmospheric air at 25°C. Considering heat loss both by radiation and natural convection, estimate the heat loss from 5 m length of the pipe. Also calculate the overall heat transfer coefficient and the heat transfer coefficient due to radiation alone.

Solution : At the mean film temperature,

$$\frac{75+25}{2} = 50°C$$

the thermo-physical properties of air are :

$\rho = 1.092$ kg/m^3

$c_p = 1007$ J/kg-deg

$\mu = 19.57 \times 10^{-6}$ kg/ms

and $k = 27.81 \times 10^{-3}$ W/m-deg

Corresponding to these properties, the relevant parameters are

$$\beta = \frac{1}{T} = \frac{1}{273+50}$$

$$= 3.096 \times 10^{-3} \text{ per degree kelvin}$$

$$Pr = \frac{\mu\, c_p}{k} = \frac{19.57 \times 10^{-6} \times 1007}{27.81 \times 10^{-3}}$$

$$= 0.708$$

$$Gr = \frac{d^3 \rho^2 \beta\, g\, \Delta t}{\mu^2};$$

$$d = 6 + 2 \times 2 = 10 \text{ cm} = 0.1 \text{ m}$$

$$= \frac{0.1^3 \times (1.092)^2 \times 3.096 \times 10^{-3} \times 9.81 \times (75-25)}{\left(19.57 \times 10^{-6}\right)^2}$$

$$= 4.73 \times 10^6$$

A quiescent atmosphere corresponds to natural free convection for which we can use the following correlation for finding the convective heat transfer coefficient

$$Nu = \frac{h\,d}{k} = 0.53\,(Gr\,Pr)^{0.25}$$

$$\therefore \quad h = \frac{k}{d} \times 0.53\,(Gr\,Pr)^{0.25}$$

$$= \frac{27.81 \times 10^{-3}}{0.1} \times 0.53 \times \left(4.73 \times 10^6 \times 0.708\right)^{0.25}$$

$$= 6.30 \text{ W/m}^2\text{-deg}$$

Heat lost by convection, Q_c

$$= h_c A\,\Delta t = h_c\,\pi\, d\, l\,\Delta t$$

$$= 6.30 \times (\pi \times 0.1 \times 5) \times (75 - 25)$$

$$= 494.55 \text{ W}$$

Heat lost by radiation, Q_r

$$= \epsilon\,\sigma_b A\left(T_1^4 - T_2^4\right)$$

$$= 0.92 \times (5.67 \times 10^{-8}) \times (\pi \times 0.1 \times 5) \times (348^4 - 298^4)$$

$$= 555.28 \text{ W}$$

Total heat loss, Q

$$= 494.55 + 555.28 = 1049.83 \text{ W}$$

In terms of total (over all) heat transfer coefficient

$$Q = h_t A\,\Delta t$$

$$\therefore \quad h_t = \frac{Q}{A\,\Delta t}$$

$$= \frac{1049.83}{(\pi \times 0.1 \times 0.5) \times (75-25)}$$

$$= \mathbf{13.37\ W/m^2\text{-deg}}$$

Heat transfer coefficient due to radiation

$$h_r = h_t - h_c$$

$$= 13.37 - 6.30 = \mathbf{7.07\ W/m^2\text{-deg}}$$

EXAMPLE 11.24

A transformer is immersed in an oil bath and the combination housed in a cylindrical container 0.75 m in a diameter and 1.2 m high. Neglecting the energy transfer from the bottom of the tank, estimate the surface temperature of the tank if the electrical loss is 1.5 kW. The entire loss is assumed to be by natural convection to the surrounding air at 20°C.

The boundary layer is laminar in character and the convective coefficient is prescribed by the relations:

$$h = 1.32\left(\frac{\Delta t}{l}\right)^{0.25} \quad \textit{for a cylindrical plane}$$

$$h = 1.42\left(\frac{\Delta t}{l}\right)^{0.25} \quad \textit{for a vertical plane}$$

where l is significant length in metres, Δt is the temperature difference, and the convective coefficient h is in W/m^2-deg

Solution : The areas to be considered are :

$$A_{top} = \frac{\pi}{4}(0.75)^2 = 0.4416 \text{ m}^2$$

$$A_{side} = \pi \times 0.75 \times 1.2 = 2.826 \text{ m}^2$$

The heat transfer is sum of the contribution from each surface. Individually the heat transfer rates, $Q = h\,A\,\Delta t$, are :

$$Q_{top} = h_{top} \times 0.4416\,(t - 20)$$

and $$Q_{side} = h_{side} \times 2.826\,(t - 20)$$

Evaluating h_{top} and h_{side} from the given correlations; we obtain

$$Q_{top} = 1.32\left(\frac{t-20}{0.9 \times 0.75}\right)^{0.25} \times 0.4416\,(t-20)$$

$$= 0.643\,(t - 20)^{1.25}$$

and $$Q_{side} = 1.42\left(\frac{t-20}{1.2}\right)^{0.25} \times 2.826\,(t-20)$$

$$= 3.834\,(t - 20)^{1.25}$$

For the cylindrcial top the significant length has been taken as 90% of the diameter.

Setting the sum of Q's equal to 1.5 kW, we obtain

$$1500 = 0.643\,(t - 20)^{1.25} + 3.834\,(t - 20)^{1.25}$$

$$t - 20 = \left(\frac{1500}{0.643 + 3.834}\right)^{1/1.25} = 104.73$$

The surface temperature then works and to be :

$$t = 104.73 + 20 = \mathbf{124.73°C}$$

11.4. CORRELATIONS FOR FORCED CONVECTION

For the usual forced circumstances, the following dimensionless numbers apply :

(ii) Fluid properties are evaluated at the bulk temperature

(iii) Re and Nu are calculated on the basis of pipe diameter as length parameter

(iv) Prandtl number lies in the range 0.5–100

(v) Constant wall temperature and fully turbulent flow.

2. $$Nu = \frac{hd}{k}$$

$$= 1.86\left(\frac{d}{l}\,Pr\,Re\right)^{0.33}\left(\frac{\mu}{\mu_s}\right)^{0.14}$$

...(11.25)

which is essentially valid for short tubes. The fluid properties are evaluated at the bulk temperature except μ_s which is calculated at the mean surface temperature. The pipe diameter is the significant length parameter. Further Re, l/d and μ should have values within the limits:

$$Re < 2100\ ;\ \frac{l}{d} > 2 \text{ and } \mu > 1$$

B. TURBULENT FLOW

11.4.3. Turbulent Flow Over Flat Plate

The general equations giving the local heat transfer coefficient for *turbulent flow* ($Re_x > 5 \times 10^5$) past flat plate are:

$$Nu_x = \frac{0.0292\,(Re_x)^{0.8}\,Pr}{1 - 2.12\,(Re_x)^{-0.1}\,(Pr)^{-1}}$$

and $Nu_x = 0.0292\ (Re_x)^{0.8}\ (Pr)^{0.33}$

...(11.26)

where the properties are evaluated at the mean film temperature

For a plate of length l, average Nusselt number would be given by:

$$\overline{Nu} = 0.036\ (Re_l)^{0.8}\ (Pr)^{0.33}$$

...(11.27)

When the flow lies in the transition range,

$$\overline{Nu} = 0.036\ (Pr)^{0.33}\ [(Re_l)^{0.8} - A]$$

...(11.28)

The constant A takes the value 18700 at transitional Reynolds number 4×10^5 and 23100 at transitional Reynolds number 5×10^5.

11.4.4. Turbulent Flow in Tubes

1. Mc Adam has suggested the following general correlation for heating and cooling of fluids in turbulent flow through long pipes.

$$Nu = 0.023\ (Re)^{0.8}\ (Pr)^n$$

...(11.29)

where

(i) $n = 0.4$ if the fluid is being heated
$= 0.3$ if the fluid is being cooled

(ii) Fluid properties are evaluated at the mean bulk temperature

(iii) $1 \times 10^4 < Re < 12 \times 10^4$

(iv) $0.7 < Pr < 120$ and

(v) $l/d > 60$

For gases such as air, Pr is essentially constant and can be dropped by changing the constant to 0.02.

2. Colburn proposed a correlation in terms of Stanton number.

$$St = 0.023\ (Re)^{-0.2}\ (Pr)^{-0.67}$$

...(11.30)

where

(i) St is evaluated at the mean bulk temperature

(ii) Re and Pr are calculated at the average film temperature

(iii) Re, Pr and l/d should have the values within the limits

$Re > 10^4\ ;\ 0.7 < Pr < 160\ ;\ l/d \geq 60$

3. The Mc Adam's and Colburn correlations are fairly accurate for temperature difference of 5°C in case of liquids and 50°C for gases. For greater temperature differences, Sieder and Tate proposed the following correlations

$$St = 0.023\,Re^{-0.2}\,Pr^{-0.67}\left(\frac{\mu}{\mu_w}\right)^{-0.14}$$

...(11.31)

where

(i) all fluid properties except μ_w are calculated at the bulk temperature; μ_w is evaluated at the wall temperature

(ii) $Re > 10^4$; $0.7 < Pr < 17000$; $l/d < 60$

For flow in short passages, these correlations need to be modified to account for the variable velocity and temperature profile along the axis of flow. The following equations have been suggested:

$$\frac{h}{h_\infty} = 1 + \left(\frac{d}{l}\right)^{0.7} \quad \text{for} \quad 2 < \frac{l}{d} < 20$$

and $$\frac{h}{h_\infty} = 1 + 6\frac{d}{l} \quad \text{for} \quad 20 < \frac{l}{d} < 60 \qquad ...(11.32)$$

4. For very large temperature difference $(t_s - t_b)$ with air, Desman and Sams have suggested the following correlation

$$Nu = 0.026\,(Re)^{0.8}\,(Pr)^{0.4} \qquad ...(11.33)$$

where

(i) t_s/t_b upto 3.55

(ii) $Re \geq 10^4$

(iii) Nu and Pr are evaluated at mean bulk temperature

(iv) Re is evaluated at mean film temperature

11.4.5. Turbulent Flow Over Cylinders

$$Nu = \frac{h\,d}{k} = C\,(Re)^n\,(Pr)^{1/3} \qquad ...(11.34)$$

The values of constants C and n depend on the flow Reynolds number. Further, all thermo- physical properties of fluids are evaluated at the film temperature.

11.4.6. Turbulent Flow Over Spheres

1. For flow of gases over spheres

$$Nu = 0.37\,(Re)^{0.6} \qquad ...(11.35)$$

for $25 < Re < 10^5$

Fluid properties are to be evaluated at the film temperature.

2. For flow of liquids past spheres

$$Nu = [0.97 + 0.68\,(Re)^{0.5}]\,Pr^{0.3} \qquad ...(11.36)$$

for $1 < Re < 2000$

Fluid properties are to be evaluated at the film temperature.

EXAMPLE 11.25

Air flows at 1.8 m/s past a 15 cm diameter pipe in a direction normal to the axis. The pipe surface is at 30°C and it receives 4.95 kW/m² of heat from the air at 250°C temperature. The convective heat transfer coefficient is prescribed by the relation

$$\left[\frac{hd}{k}\right] = \left[\frac{V\,d\,c_p}{k}\right]^{0.7}$$

Under similar operating conditions, make calculations for the heat loss from carbon dioxide at 200°C when it flows at 1.5 m/s normal to a 25 cm diameter pipe at 25°C.

The relevant thermo-physical properties for air and carbon dioxide are :

For air at 250°C :

$c_p = 1.038$ kJ/kgK

$k = 4.270 \times 10^{-2}$ W/mK

For CO_2 at 200°C :

$c_p = 0.99$ kJ/kgK

$k = 3.12 \times 10^{-2}$ W/mK

Solution : Let suffix 1 denote air and suffix 2 denote carbon dioxide. Then,

$$\frac{h_1 d_1}{k_1} = C\left[\frac{V_1 d_1 c_{p1}}{k_1}\right]^{0.7}$$

$$\frac{h_2 d_2}{k_2} = C\left[\frac{V_2 d_2 c_{p2}}{k_1}\right]^{0.7}$$

$$\therefore \frac{h_1}{h_2} \times \frac{d_1}{d_2} \times \frac{k_2}{k_1} = \left[\frac{V_1}{V_2} \times \frac{d_1}{d_2} \times \frac{c_{p1}}{c_{p2}} \times \frac{k_2}{k_1}\right]^{0.7}$$

$$\frac{h_1}{h_2} \times \frac{0.15}{0.25} \times \frac{3.12 \times 10^{-2}}{4.27 \times 10^{-2}} =$$

$$\left[\frac{1.8}{1.6} \times \frac{0.15}{0.25} \times \frac{1.038}{0.99} \times \frac{3.12 \times 10^{-2}}{4.27 \times 10^{-2}}\right]^{0.7}$$

$$0.438\frac{h_1}{h_2} = 0.659 \text{ or } \frac{h_1}{h_2} = \frac{0.659}{0.438} = 1.505$$

From the convective heat equation, $Q = h\,A\,\Delta t$

Therefore for the same heat flow area,

$$\frac{Q_1}{Q_2} = \frac{h_1\,\Delta t}{h_2\,\Delta t_2} = 1.505 \times \frac{(250 - 30)}{(200 - 25)} = 1.892$$

$\therefore$ Heat loss from carbon dioxide,

$$Q_2 = \frac{4.95}{1.892} = \mathbf{2616\ kW/m^2}$$

EXAMPLE 11.26

Investigate the effect of following conditions on the average value of heat transfer coefficient in flow through a tube :

(i) Two- fold increase in flow velocity by varying mass flow rate;

(ii) Two-fold increase in the diameter of tube, the flow velocity is maintained constant by a change in the rate of liquid flow.

It may be presumed that there is no change in the temperatures of the liquid and the tube wall, and that the flow through the tube is turbulent in character.

Solution : In case of turbulent flow through a tube, the average value of heat transfer coefficient is prescribed by the relation

$$Nu = \frac{h\,d}{k} = 0.023\,(Re)^{0.8}\,(Pr)^{0.33}$$

or $$h = \frac{k}{d} \times 0.023\left(\frac{\rho V d}{\mu}\right)^{0.8}\left(\frac{\mu c_p}{k}\right)^{0.33}$$

$$= 0.023\,k\left(\frac{\rho}{\mu}\right)^{0.8}\left(\frac{\mu c_p}{k}\right)^{0.33}\frac{(V)^{0.8}}{(d)^{0.2}}$$

Further as there is no change in the temperatures, the fluid properties remain constant.

(i) When the tube diameter and the fluid properties remain constant

$$h \propto (V)^{0.8}$$

$$\frac{h_2}{h_1} = \left(\frac{V_2}{V_1}\right)^{0.8} = (2)^{0.8} = 1.741$$

That is, a two-fold increase in the flow velocity results into 74.1% increase in heat transfer.

(ii) When the flow velocity and the fluid properties remain unchanged

$$h \propto \frac{1}{(d)^{0.2}}$$

$$\frac{h_2}{h_1} = \left(\frac{d_1}{d_2}\right)^{0.2} = \left(\frac{1}{2}\right)^{0.2} = 0.87$$

That is, with two-fold increase in tube diameter, the heat transfer decreases by 13%.

EXAMPLE 11.27

Calculate the rate of heat loss from a human body which may be considered as a vertical cylinder 30 cm in diameter, and 175 cm high while standing in a 30 km/hr wind at 15°C. The surface temperature of the human is 35°C.

Solution : At the mean film temperature,

$$t_f = \frac{35 + 15}{2} = 25°C$$

the thermo-physical properties of air are :

$\nu = 15.33 \times 10^{-6}$ m²/s

$k = 0.0263$ W/m-deg

$Pr = 0.7$

Wind velocity = 30 km/hr

$$= \frac{30 \times 1000}{3600} = 8.33 \text{ m/s}$$

$$Re = \frac{V\,d\,\rho}{\mu} = \frac{V\,d}{\nu} = \frac{8.33 \times 0.3}{15.33 \times 10^{-6}} = 163014 \text{ (laminar)}$$

$$Nu = \frac{h\,d}{k} = 0.664\,(Re)^{0.5} \times (Pr)^{0.33}$$

$$= 0.664\,(163014)^{0.5} \times (0.7)^{0.33} = 238.33$$

$$h = Nu \times \frac{k}{d} = 238.33 \times \frac{0.0263}{0.3} = 20.89 \text{ W/m}^2\text{-deg}$$

Convective heat loss,

$$Q = h\,A\,\Delta t$$

$$= h\,(\pi\,d\,l)\,\Delta t$$

$$= 20.89 \times (\pi \times 0.3 \times 1.75) \times (35 - 15)$$

$$= 688.86 \text{ W}$$

EXAMPLE 11.28

Air moving at 0.3 m/s flows over the top of a chest type freezer. The top of the freezer measures 0.9 m by 1.5 m and is poorly insulated so that the surface remains at 10°C. If the temperature of air is 30°C, make calculations for the maximum heat transfer by forced convection from the top of the freezer.

Solution : At the mean film temperature,

$$t_f = \frac{10 + 30}{2} = 20°C$$

the thermo-physical properties of air are :

$$\nu = 15.06 \times 10^{-6}\ m^2/s$$
$$k = 0.0259\ W/m\text{-deg}$$
$$Pr = 0.703$$

The maximum average heat transfer occurs when the air flow is in the direction of the shorter dimension.

$$Re = \frac{\rho V l}{\mu} = \frac{V l}{\nu} = \frac{0.3 \times 0.9}{15.06 \times 10^{-6}}$$
$$= 17928 \text{ (laminar)}$$
$$Nu = 0.664\ (Re)^{0.5}\ (Pr)^{0.33}$$
$$= 0.664\ (17928)^{0.5} \times (0.703)^{0.33}$$
$$= 79.14$$
$$h = Nu \times \frac{k}{l} = 79.14 \times \frac{0.0259}{0.9}$$
$$= \mathbf{2.28\ W/m^2\text{-deg}}$$

$\therefore$ Convective heat flow, Q

$$= h\,A\,\Delta t$$
$$= 2.28 \times (0.9 \times 1.5) \times (10 - 30)$$
$$= \mathbf{-61.56\ W}$$

The negative sign indicates that heat transfer is towards the freezer.

EXAMPLE 11.29

The oil pan of an 1.6 engine approximates a flat plate 0.3 m wide by 0.45 m long and protrudes below the framework of the automobile. The engine oil is at 95°C and the ambient air temperature is 35°C. If the automobile runs at 36 km/hr, make calculations for the rate of heat transfer from the oil pan surface. Assume negligible resistance to conduction through the oil pan.

Solution : At the mean film temperature,

$$t_f = \frac{95 + 35}{2} = 65°C$$

the thermo-physical properties of air are :

$$\nu = 18.46 \times 10^{-6}\ m^2/s$$
$$k = 0.0293\ W/m\text{-deg}$$
$$Pr = 0.695$$

Velocity of automobile V

$$= 36\ km/hr$$
$$= \frac{36 \times 1000}{3600} = 10\ m/s$$
$$Re = \frac{\rho V l}{\mu} = \frac{V l}{\nu} = \frac{10 \times 0.45}{18.46 \times 10^{-6}}$$
$$= 2.438 \times 10^5 \text{ (laminar)}$$
$$Nu = 0.664\ (Re)^{0.5}\ (Pr)^{0.33}$$
$$= 0.664\ (2.438 \times 10^5)^{0.5} \times (0.695)^{0.33}$$
$$= 290.81$$
$$h = Nu \times \frac{k}{l}$$
$$= 290.81 \times \frac{0.0293}{0.45}$$
$$= 18.93\ W/m\text{-deg}$$

Convective heat transfer, Q

$$= h\,A\,\Delta t$$
$$= 18.93 \times (0.3 \times 0.45) \times (95 - 35)$$
$$= \mathbf{170.37\ W}$$

EXAMPLE 11.30

A heat-treat steel plate measures 3 m × 1 m and is initially at 30°C. It is cooled by blowing air parallel to 1 m edge at 9 km/hr. If the air is at 10°C, calculate the convective heat transfer from both sides of the plate.

Solution : At the mean film temperature,

$$t_f = \frac{30 + 10}{2} = 20°C$$

the thermo-physical properties of air are :

$$\nu = 15.06 \times 10^{-6}\ m^2/s$$
$$k = 0.0259\ W/m\text{-deg}$$
$$Pr = 0.703$$

Velocity of air $V = 9$ km/hr

$$= \frac{9 \times 1000}{3600} = 2.5\ m/s$$

$$Re = \frac{\rho V l}{\mu} = \frac{V l}{\nu} = \frac{2.5 \times 1}{15.06 \times 10^{-6}}$$
$$= 166003 \text{ (laminar)}$$
$$Nu = \frac{h l}{k} = 0.664\ (Re)^{0.5}\ (Pr)^{0.33}$$
$$= 0.664\ (166003)^{0.5} \times (0.703)^{0.33}$$
$$= 240.83$$
$$h = Nu \times \frac{k}{l} = 240.83 \times \frac{0.0259}{1}$$
$$= 6.237\ W/m\text{-deg}$$

Convective heat transfer from both sides of the plate,

$$Q = h\,(2\,A)\,\Delta t$$
$$= 6.23 \times (2 \times 3 \times 1) \times (30 - 10)$$
$$= \mathbf{747.6\ W}$$

EXAMPLE 11.31

A metallic cylinder of 12.5 mm diameter and 95 mm length was heated internally by an electrical heater, and was subjected to cross flow of air in a low speed wind tunnel. Under a specific set of operating conditions, the following data were recorded :

Velocity and temperature of free stream air = 10 m/s and 25.5°C respectively

Average temperature of cylinder surface = 128.5°C

Power disspation by heater = 45 W

If 15% of the power dissipation is lost through the insulated end pieces of the cylinder, determine the experimental value of the convective heat transfer coefficient. How this value compares with the covection coefficient obtained by using the correlation :

$$Nu = 0.26\ (Re)^{0.6}\ (Pr)^{0.36} \left(\frac{Pr}{Pr_s}\right)^{0.25}$$

where all properties, except Pr_s are evaluated at the mean bulk (free stream) temperature of air.

$$k = 0.0264\ W/mK$$
$$\nu = 15.85 \times 10^{-6}\ m^2/s$$
$$\text{and} \quad Pr = 0.706$$

Pr_s is the Prandtl number of air evaluated at the average temperature of cylinder surface;

$$Pr_s = 0.691$$

Solution : Heat flow from the heater to the air flowing past it is given by :

$$Q = h\,A\,\Delta t$$
$$0.85 \times 45 = h\,(\pi \times 0.0125 \times 0.095) \times (128.5 - 25.5)$$

$\therefore$ Convective film coefficient,

$$h = \frac{0.85 \times 45}{(\pi \times 0.0125 \times 0.095) \times 103}$$
$$= \mathbf{99.5\ W/m^2K}$$

(b) $$Re = \frac{V d}{\nu} = \frac{10 \times 0.0125}{15.85 \times 10^{-6}} = 7886$$

$$Nu = \frac{h d}{k}$$
$$= 0.26\ (7886)^{0.6}\ (0.706)^{0.36} \left(\frac{0.706}{0.691}\right)^{0.25}$$
$$= 50.23$$

$\therefore$ Convective film coefficient,

$$h = Nu \times \frac{k}{d}$$
$$= 50.23 \times \frac{0.0264}{0.0125} = \mathbf{106\ W/m^2K}$$

EXAMPLE 11.32

Air flows through a 10 cm internal diameter tube at the rate of 75 kg/hr. Measurements indicate that at a particular point in the tube, the pressure and temperature of air are 1.5 bar and 325 K respectively whilst the tube wall temperature is 375 K. Make calculations for the heat transfer rate from one metre length in the region of this point.

The general non-dimensional correlations for turbulent flow in the tube is :

$$Nu = 0.023\ Re^{0.8}\ Pr^{0.4}$$

where the fluid properties are evaluated at the bulk temperature.

Solution : For air at $t_f = 325$ K and $\rho = 1$ atm, the thermo-physical properties of air are :

$$\mu = 1.967 \times 10^{-5}\ kg/ms$$
$$k = 0.02792\ W/mK$$
$$\text{and} \quad Pr = 0.713$$

These properties may be assumed to be independent of pressure to an excellent approximation.

$$Re = \frac{\rho V d}{\mu} = \frac{m d}{A \mu} = \frac{4m}{\pi d \mu}$$

$$= \frac{4 \times (75/3600)}{\pi \times 0.1 \times 1.967 \times 10^{-5}} = 13492$$

This is well in excess of the critical Reynolds number for flow in tubes (2500); the flow is turbulent and the given correlation applies

$$Nu = 0.023\ (13492)^{0.8}\ (0.713)^{0.4}$$
$$= 40.46$$

Convective film coefficient,

$$h = \frac{Nu \times k}{d} = \frac{40.46 \times 0.02792}{0.1}$$
$$= 11.296\ \text{W/m}^2\text{K}$$

Heat flow rate, Q

$$= h\ A\ \Delta t$$
$$= 11.296 \times (\pi \times 0.1 \times 1) \times (375 - 325)$$
$$= 177.35\ \text{W}$$

EXAMPLE 11.33

A square channel with a side 10 mm and length 1.5 m carries water with a velocity of 5 m/s. Measurements indicate that lengthwise mean temperature of water is 30°C whilst the inner surface of channel is at 80°C. Calculate the convective coefficient of heat trnasfer from the channel wall to the water. Use the correlation :

$$Nu = 0.021\ Re^{0.8}\ Pr^{0.43} \left(\frac{Pr}{Pr_w}\right)^{0.25}$$

where the thermo-physical properties pertain to those at the mean bulk temperature of water. Pr_w corresponds to the value of Prandtl number at the channel surface temperature and equivalent diameter is the reference dimension.

The physical properties of water at 30°C are :

$$\rho = 995.67\ kg/m^3$$
$$c_p = 4174\ J/kgK$$
$$k = 0.6172\ W/mK$$
$$\mu = 2.88\ kg/m\text{-}hr$$
$$\nu = 0.805 \times 10^{-6}\ m^2/s$$

$$P_r = 5.42$$

At wall : temperature t_w
$= 80°C$ and $Pr_w = 2.21$

Solution : Equivalent diameter of the channel

$$d_{eq} = 4\frac{A}{P} = 4\ \frac{0.01 \times 0.01}{4 \times 0.01} = 0.01\ m$$

where A is cross-sectional area of the channel and P is perimeter of the channel.

$$Re = \frac{V d_{eq}}{\nu} = \frac{5 \times 0.01}{0.805 \times 10^{-6}} = 62100$$

$$Nu = 0.021\ (62100)^{0.8}$$

$$(5.42)^{0.43}\left(\frac{5.42}{2.21}\right)^{0.25} = 371.4$$

$$\therefore \quad h = \frac{Nu \times k}{d_{eq}} = \frac{371.4 \times 0.6172}{0.01}$$
$$= 22923\ \text{W/m}^2\text{K}$$

EXAMPLE 11.34

Air at atmospheric pressure and 20°C flows with 6 m/s velocity through main trunck duct of air conditioning system. The duct is rectangular in cross-section and measures 40 cm × 80 cm. Determine heat loss per metre length of duct corresponding to unit temperature difference. The relevant thermo-physical properties of air are :

$$\nu = 15 \times 10^{-6}\ m^2/s$$
$$\alpha = 7.7 \times 10^{-2}\ m^2/hr$$
and $$k = 0.026\ W/m\text{-}deg$$

Solution : For the rectangular duct, characteristic length is

$$l = \frac{2ab}{a+b} = \frac{2 \times 80 \times 40}{80 + 40} = 53.3\ \text{cm}$$

$$Pr = \frac{\mu c_p}{k} = \frac{\nu}{\alpha} = \frac{15 \times 10^{-6}}{(7.7 \times 10^{-2})/3600}$$
$$= 0.701$$

$$Re = \frac{\rho V l}{\mu} = \frac{V l}{\nu} = \frac{6 \times 0.533}{15 \times 10^{-6}}$$
$$= 0.213 \times 10^6$$

Obviously the flow is turbulent and accordingly

$$Nu = \frac{h l}{k} = 0.023\ (Re)^{0.8} \times (Pr)^{0.4}$$
$$= 0.023\ (0.213 \times 10^6)^{0.8} \times (0.701)^{0.4}$$
$$= 365.34$$

$\therefore$ Convective coefficient, h

$$= 365.34 \times \frac{k}{l}$$
$$= 365.34 \times \frac{0.026}{0.533}$$
$$= 17.82\ \text{W/m}^2\text{-deg}$$

Area of duct per metre length

$$= 2\ (a + b) \times 1$$
$$= 2\ (0.8 + 0.4) \times 1 = 2.4\ m^2$$

$\therefore$ Convective heat loss, Q

$$= h\ A\ \Delta t$$
$$= 17.82 \times 2.4 \times 1 = 42.768\ \text{W}$$

EXAMPLE 11.35

What factors affect the value of convection coefficient for water flowing inside a circular tube?

Within a condenser shell, water flows through one hundred thin-walled circular tubes (diameter = 22.5 mm and length 5 m) which have been arranged in parallel. The mass flow rate of water is 65 kg/s. and its inlet and outlet temperatures are known to be 22°C and 28°C respectively. Predict the average convection coefficient associated with water flow.

Solution : A the mean bulk temperature,

$$t_b = \frac{22 + 28}{2} = 25°C$$

the thermo-physical properties of water are :

$$\rho = 996.65\ kg/m^3$$
$$\mu = 903.01 \times 10^{-6}\ kg/ms$$
$$c_p = 4.1776\ kJ/kgK$$
$$k = 2.1893\ kJ/m^2hrK$$

$$Pr = \frac{\mu c_p}{k}$$
$$= \frac{(903.1 \times 10^{-6} \times 3600) \times 4.1776}{2.1893}$$
$$= 6.2$$

Mass flow of water through each tube,

$$m = \frac{65}{100} = 0.65\ kg/s$$

$$Re = \frac{V d \rho}{\mu} = \frac{m d}{A \mu} = \frac{4m}{\pi d \mu}$$
$$= \frac{4 \times 0.65}{\pi \times 0.0225 \times 903.06 \times 10^{-6}}$$
$$= 40749.78$$

This is well in excess of the critical Reynolds number for flow through tubes (2500); the flow is turbulent and the following correlation applies :

$$Nu = 0.023\ (Re)^{0.8}\ (Pr)^{0.4}$$
$$= 0.023\ (40749.78)^{0.8} \times (6.2)^{0.4}$$
$$= 232.69$$

$$\therefore \quad h = \frac{Nu \times k}{d}$$
$$= \frac{232.69 \times 2.1893}{0.0225}$$
$$= 22641\ \text{kJ/m}^2\text{-hr-deg}$$

EXAMPLE 11.36

A motor cycle cylinder consists of 10 fins; each 15 cm outside diameter and 7.5 cm inside diameter. Calculate the rate of heat dissipation from the cylinder fins when (i) motor cycle is stationary and (ii) motor cycle is running at 60 km/hr.

The atmospheric air is at 20°C and the average fin temperature is 480°C. The relevant thermo-physical properties at the average temperature of 250°C are :

$$\rho = 0.674\ kg/m^3\ ;\ c_p = 1038\ J/kgK$$
$$k = 0.427\ W/mK\ ;\ P_r = 0.677$$
$$\nu = 40.61 \times 10^{-6}\ m^2/s\ \text{and}$$

$$\beta = \frac{1}{250 + 273}$$
$$= 1.912 \times 10^{-3}\ \text{per degree kelvin}$$

The approximate value of heat transfer coefficient may be evaluated by idealing the fins as a single horizontal flat plate of the same area.

Solution : (a) Motor cycle stationary (free convection)

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

where the significant length $l = 0.9\ d$

$$Gr = \frac{(0.9 \times 0.15)^3 \times 1.912 \times 10^{-3} \times 9.81 \times (480 - 20)}{(40.61 \times 10^{-6})^2}$$

$$= 1.287 \times 10^7$$

$$Gr \times Pr = 1.287 \times 10^7 \times 0.677$$

$$= 0.8715 \times 10^7$$

For laminar flow within the range

$$10^5 < Gr\,Pr < 2 \times 10^7,$$

the following correlation applies;

$$Nu = 0.54\,(Gr\,Pr)^{1/4}$$

$$= 0.54\,(0.8715 \times 10^7)^{1/4} = 29.34$$

This gives the convective coefficient as

$$h = \frac{Nu \times k}{l}$$

$$= \frac{29.34 \times 0.0427}{0.9 \times 0.15}$$

$$= 9.28\ \text{W/m}^2\ \text{K}$$

Since both sides of each fin are exposed to the air, the surface area for convective heat transfer is

$$= 2 \times 10 \times \frac{\pi}{4}\left(0.15^2 - 0.075^2\right)$$

$$= 0.265\ \text{m}^2$$

Heat dissipation, Q

$$= h\,A\,\Delta t$$

$$= 9.28 \times 0.265\,(480 - 20)$$

$$= 1131.2\ \text{W}$$

(b) Motor cycle is running (forced convection)

Speed of motor cycle

$$= 60\ \text{km/hr}$$

$$= \frac{60 \times 1000}{3600} = 16.67\ \text{m/s}$$

$$Re = \frac{V\,d\,l}{\mu} = \frac{V\,l}{\nu}$$

$$= \frac{16.67 \times (0.9 \times 0.15)}{40.61 \times 10^{-6}}$$

$$= 55416$$

which indicates a turbulent flow for which the following correlation applies

$$Nu = 0.036\,(Re)^{0.8}\,(Pr)^{0.33}$$

$$= 0.036\,(55416)^{0.8}\,(0.677)^{0.33}$$

$$= 197.11$$

$$h = \frac{Nu \times k}{l} = \frac{197.11 \times 0.423}{0.9 \times 0.15}$$

$$= 61.76\ \text{W/m}^2\ \text{K}$$

Heat dissipation Q

$$= h\,A\,\Delta t$$

$$= 61.76 \times 0.265\,(480 - 20)$$

$$= 7528.6\ \text{W}$$

EXAMPLE 11.37

In a certain glass making process, a square plate of glass 0.8 m² area and 3 mm thick is heated uniformly to 90°C. Subsequently it is cooled by 20°C temperature air that flows at 2 m/s over both sides parallel to the plate. Neglecting temperature gradient in the glass plate and considering only forced convection, estimate the initial cooling rate of the plate. How would the cooling rate be affected if the velocity is increased to 2.5 m/s.

For glass take

$$\rho = 2500\ \text{kg/m}^3$$

and $c_p = 0.65$ *kJ/kg-deg*

Solution : At the mean air temperature,

$$t = \frac{90 + 20}{2} = 55°\text{C}$$

the thermo-physical properties of air are.

$$\rho = 1.076\ \text{kg/m}^3$$

$$c_p = 1008\ \text{J/kgK}$$

$$k = 0.0286\ \text{W/mK}$$

and $\mu = 19.8 \times 10^{-6}\ \text{Ns/m}^2$

$$Re = \frac{\rho\,l\,V}{\mu} = \frac{1.076 \times 0.8 \times 2}{19.8 \times 10^{-6}} = 86950$$

$$Pr = \frac{\mu\,c_p}{k} = \frac{19.8 \times 10^{-6} \times 1008}{0.0286}$$

$$= 0.698$$

$$Nu = \frac{h\,l}{k}$$

$$= 0.664\,(Re)^{0.5} \times (Pr)^{0.33}$$

$$= 0.664\,(86950)^{0.5} \times (0.698)^{0.33}$$

$$= 173.87$$

$$\therefore \quad h = 173.87 \times \frac{k}{l}$$

$$= 173.87 \times \frac{0.0286}{\sqrt{0.8}}$$

$$= 5.562\ \text{W/m}^2\text{-deg}$$

Heat flow from both sides of the plate is given by

$$Q = h\,(2A)\,\Delta t$$

$$= 5.562 \times (2 \times 0.8) \times (90 - 20)$$

$$= 623\ \text{W}$$

The instantaneous heat loss from the plate is also given by

$$Q = m\,c_p\,\Delta t = (\rho\,A\,\delta)\,c_p\,\Delta\tau$$

$$= (2500 \times 0.8 \times 0.003) \times (0.65 \times 1000)\,\Delta t$$

$$= 3900\,\Delta\tau$$

$\therefore$ Rate of cooling, $\Delta\tau$

$$= \frac{623}{3900} = 0.1597°\text{C/sec}$$

(ii)
$$\frac{(\Delta t)_2}{(\Delta t)_1} = \frac{h_2}{h_1} = \left(\frac{Re_2}{Re_1}\right)^{0.5}$$

$$= \left(\frac{V_2}{V_1}\right)^{0.5} = \left(\frac{2.5}{2}\right)^{0.5} = 1.118$$

$\therefore$ Percentage increase in cooling rate

$$= 11.8\%$$

EXAMPLE 11.38

A copper bus bar of round cross-section with 15 mm diameter is cooled with a cross flow of dry air. The air is at 20°C and it flows past the bus bar with a velocity of 1.5 m/s. Make calculations for the coefficient of heat transfer from the surface of bus bar to the cooling air. What would be the permissible current intensity for the bus bar if its surface temperature is not to exceed 75°C ? Resistivity of copper = 0.0175 × 10⁻⁶ ohm-m²/m.

For a single cylinder placed in cross flow, the following empirical correlations have been suggested:

$$Nu = 0.44\ Re^{0.5} \qquad 10 < Re < 10^3$$

and $Nu = 0.22\ Re^{0.6} \qquad 10^3 < Re < 2 \times 10^5$

The reference dimension is the diameter of the cylinder and the relevant physical properties at the temperature of free stream air flow (t = 20°C) are :

$$\nu = 15.06 \times 10^{-6}\ \text{m}^2\text{/s}$$

$$k = 0.0259\ \text{W/m-deg}$$

Solution : Reynolds number :

$$Re = \frac{Vd}{\nu} = \frac{1.5 \times 0.015}{15.06 \times 10^{-6}}$$

$$= 1.494 \times 10^3$$

Apparently, for the given conditions, the second correlation applies

$$Nu = 0.22 \times (1.494 \times 10^3)^{0.6} = 17.66$$

$$h = \frac{Nu \times k}{d}$$

$$= \frac{17.66 \times 0.0259}{0.015}$$

$$= 30.49\ \text{W/m}^2\text{-deg}$$

With the limiting surface temperature of 75°C, the heat dissipation to air is :

$$Q = h\,(\pi\,d\,l)\,\Delta t$$

$$= 30.49\,(\pi \times 0.015 \times 1)\,(75 - 20)$$

$$= 78.985\ \text{W/m length}$$

If I is the current intensity, then heat generated in the bus bar

$$= I^2R = I^2\frac{\rho\,l}{A}$$

$$= \frac{I^2 \times (0.0175 \times 10^{-6}) \times 1}{\frac{\pi}{4}(0.015)^2}$$

$$= 99.08 \times 10^{-6}\,I^2$$

Then from energy balance,

$$99.08 \times 10^{-6}\,I^2 = 78.985$$

$$I = \sqrt{\frac{78.985}{99.08 \times 10^{-6}}}$$

$$= 892.85\ \text{amperes}$$

EXAMPLE 11.39

A copper bus bar of 1.5 cm diameter is required to operate at a temperature not exceeding 75°C. The task is accomplished by cooling it with air moving with a velocity of 1.2 m/s. If the air is at 20°C, estimate:

(a) the heat transfer coefficient from the bus bar to the cooling air, (b) the maximum allowable current capacity of the bus bar. How these parameters will change if the air velocity is reduced

to one-half ? All other conditions remain unchanged? Take resistivity of copper

$$\rho = 0.0175\ 10^{-6}\ \Omega\ m$$

and use the following correlation for finding the average heat transfer coefficient :

$$Nu = \frac{hd}{k} = 0.42\sqrt{Re}$$

The properties of air are to be evaluated at its bulk temperature at 20°C.

Solution : At 20°C, the properties of air are :

$$\rho = 1.204\ kg/m^3$$
$$\mu = 18.17 \times 10^{-6}\ kg/ms$$
$$k = 25.64 \times 10^{-3}\ W/m\text{-}deg$$

$$\therefore \quad Re = \frac{Vd\rho}{\mu}$$
$$= \frac{1.2 \times 0.015 \times 1.204}{18.17 \times 10^{-6}} = 1192$$

and $$h = 0.42\sqrt{Re} \times \frac{k}{d} \quad ...(i)$$
$$= 0.42 \times \sqrt{1192} \times \frac{25.64 \times 10^{-3}}{0.015}$$
$$= 24.78\ W/m^2\text{-}deg$$

From energy balance:

$$I^2 R = h(\pi d l)\Delta t \quad ...(ii)$$

where resistance of the bus bar,

$$R = \frac{\rho l}{A} = \frac{\rho l}{(\pi/4)d^2} = \frac{4\rho l}{\pi d^2}$$

$$\therefore \quad I^2 \times \frac{4\rho}{\pi d^2} = h\pi d\Delta t$$

or $$I^2 \times \frac{4 \times 0.0175 \times 10^{-6}}{\pi \times (0.015)^2} = 24.78 \times \pi \times 0.015 \times (75 - 20)$$

$\therefore$ Maximum allowable current, I

$$= 805\ amp$$

An insight into relations (i) and (ii) would reveal that

$$h \propto \sqrt{R_e} \propto \sqrt{V} \quad \text{and} \quad I \propto \sqrt{h} \propto \sqrt{V}$$

Accordingly if the velocity is reduced to one-half, the revised values of convective coefficient and maximum allowable current would become :

$$h_r = \frac{24.78}{\sqrt{2}} = 17.52\ W/m^2\text{-}deg$$

and $$I_r = \frac{805}{\sqrt{2}} = 677\ amperes$$

EXAMPLE 11.40

Liquid mercury flows through a copper tube of 2 cm inner diameter at the rate of 1.25 kg/s. The mercury enters at 15°C and is heated to 25°C as it passes through the tube. Determine the tube length which would satisfy the condition of a constant heat flux at the wall which is at an average temperature of 40°C.

For liquid metals, the following correlation is presumed to agree well with experimental results

$$Nu = 7 + 0.025\,(Pe)^{0.8}$$

where Pe is the Peclet number : $Pe = Pr \times Re$

Solution : At the mean bulk temperature,

$$t_b = \frac{15 + 25}{2} = 20°C$$

the thermo-physical properties of the liquid mercury are :

$$\rho = 13580\ kg/m^3$$
$$k = 8.685\ W/mK$$
$$c_p = 139.35\ J/kgK$$
$$v = 1.145 \times 10^{-7}\ m^2/s$$
and $$Pr = 0.0249$$

Heat gained by mercury, Q

$$= m\,c_p\,\Delta t$$
$$= 1.25 \times 139.35 \times (25 - 15)$$
$$= 1741.8\ W$$

$$Re = \frac{Vd\rho}{\mu} = \frac{md}{A\mu} = \frac{4m}{\pi d\mu} = \frac{4m}{\pi d\rho v}$$
$$= \frac{4 \times 1.25}{\pi \times 0.02 \times 13580 \times 1.145 \times 10^{-7}}$$
$$= 51115$$

Then from the given correlation :

$$Nu = 7 + 0.025\,(Pr \times Re)^{0.8}$$
$$= 7 + 0.025\,(51115 \times 0.0249)^{0.8}$$
$$= 14.616$$

$$\therefore \quad h = \frac{Nu \times k}{d} = \frac{14.616 \times 8.685}{0.02}$$
$$= 6347\ W/m^2K$$

The heat gained by the fluid results from the convective process. Thus

$$Q = h\,A\,\Delta T$$
$$1741.87 = 6347\,(\pi \times 0.02 \times l)\,(25 - 15)$$

Therefore the required length of the pipe is

$$l = \frac{1741.87}{6347 \times \pi \times 0.02 \times 10}$$
$$= 1370\ m$$

EXAMPLE 11.41

In a heat exchanger, water flows through a 2 cm inner diameter copper tube at a velocity of 1.5 m/s. The water entering the tube at 15°C is heated by steam condensing at 100°C on the outside surface of the tube. What would be heat transfer coefficient for water if it is to leave the pipe at 45°C ? Use may be made of the Mc Adam's relation and the Colburn equation.

The physical properties of water at the bulk temperature

$$t_b = \frac{15 + 45}{2} = 30°C$$

and at the mean film temperature

$$t_f = \frac{30 + 100}{2} = 65°C$$

are given below :

Temperature	ρ, kg/m³	C_p J/kg K	k, W/mK	ν, m²/s	Pr
30°C	995.7	4174	0.6172	0.805×10^{-6}	5.42
65°C	980.5	4182	0.663	0.446×10^{-6}	2.76

Solution : (a) Mc Adam's correlation :

$$Nu = 0.023\,(Re)^{0.8}\,(Pr)^{0.4}$$

Here all the fluid properties are evaluated at the mean bulk temperature of the fluid.

$$Re = \frac{Vd}{v} = \frac{1.5 \times 0.02}{0.805 \times 10^{-6}} = 37267$$
$$Nu = 0.023\,(37267)^{0.8} \times (5.42)^{0.4}$$
$$= 205.27$$
$$\therefore \quad h = \frac{Nu \times k}{d}$$
$$= \frac{205.27 \times 0.6172}{0.02}$$
$$= 6335.53\ W/m^2K$$

(b) Colburn correlation : St

$$= 0.023\,(Re)^{-0.2} \times (Pr)^{-0.667}$$

Here all the fluid properties, except c_p in the computation of Stanton number, are evaluated at the film temperature. In the Stanton number, the value of c_p at fluid bulk temperature is used

$$Re = \frac{Vd}{v} = \frac{1.5 \times 0.02}{0.446 \times 10^{-6}} = 67264$$
$$St = 0.023\,(67264)^{-0.2} \times (2.76)^{-0.667}$$

$$\therefore \quad \frac{h}{\rho c_p V} = 1.265 \times 10^{-3}$$

That gives :

$$h = (1.265 \times 10^{-3}) \times 980.5 \times 4174 \times 1.5$$
$$= 7765.62\ W/m^2K$$

EXAMPLE 11.42

Air at a temperature of 25°C is blown across a flat plate at a mean velocity of 7.5 m/s. If the plate surface temperature is 575°C, make calculations for the heat transferred per metre width from both sides of the plate over distance of 20 cm from the leading edge.

For heat transfer from a plate with large temperature between the plate and the fluid, the local Nusselt number is given by :

$$Nu_x = 0.332\,(Pr)^{1/3}\,(Re)^{1/2}\left(\frac{T_a}{T_s}\right)^{0.117}$$

where all the properties are at the mean film temperature, T_s and T_a are the absolute temperature of the plate surface and the free stream of air respectively. The characteristic linear dimension is the distance from the leading edge.

Solution : At the mean film temperature

$$t_f = \frac{575 + 25}{2} = 300°C$$

the thermo-physical properties of air are :

$$\rho = 0.615 \text{ kg/m}^3$$
$$c_p = 1.0465 \text{ kJ/kgK}$$
$$k = 0.1659 \text{ kJ/mhrK}$$

and $\mu = 29.724 \times 10^{-6}$ kg/ms

$$Re = \frac{\rho V l}{\mu} = \frac{0.615 \times 7.5 \times 0.2}{29.724 \times 10^{-6}} = 31035$$

$$Pr = \frac{\mu c_p}{k} = \frac{(29.724 \times 10^{-6} \times 3600) \times 1.0465}{0.1659} = 0.675$$

$$Nu_x = 0.332\,(0.675)^{1/3}\,(31035)^{1/2} \times \left(\frac{273 + 575}{273 + 25}\right)^{0.117} = 58$$

$\therefore$ Local heat transfer coefficient,

$$h_x = \frac{58 \times 0.1659}{0.2} = 48.11 \text{ kJ/m}^2\text{hrK}$$

The local and average convective coefficients are related by an expression of the form

$$\bar{h} = \frac{1}{l}\int_0^l h_x\,dx$$
$$= \frac{1}{l}\int_0^l \frac{k}{x} \times 0.332\,(Pr)^{1/3} \left(\frac{T_s}{T_a}\right)^{0.117} \left(\frac{\rho V}{\mu}\right)^{1/2} x^{1/2}\,dx$$
$$= 0.332\,\frac{k}{l}\,(Pr)^{1/3}\left(\frac{T_s}{T_a}\right)^{0.117} \times \left(\frac{\rho V}{\mu}\right)^{1/2} \int_0^l \frac{dx}{\sqrt{x}}$$
$$= 0.332\,\frac{k}{l}\,(Pr)^{1/3}\left(\frac{T_s}{T_a}\right)^{0.117} \times \left(\frac{\rho V}{\mu}\right)^{1/2} 2l^{1/2}$$
$$= 2 \times 0.332\,\frac{k}{l}\,(Pr)^{1/3}\left(\frac{\rho V l}{\mu}\right)^{1/2} \times \left(\frac{T_s}{T_a}\right)^{0.117}$$

= 2 × local heat transfer coefficient

$= 2 \times 48.11 = 96.22$ kJ/m^{2}hrK

Heat loss from both sides of plate

$= 2\,(h\,A\,\Delta t)$

$= 2\,[96.22\,(0.2 \times 1)\,(575 - 25)]$

$=$ **21168.4 kJ/hr**

EXAMPLE 11.43

1000 kg/hr of cream cheese at 15°C is pumped through 1.5 m length of 8 cm inner diameter tube which is maintained at 95°C. Estimate the temperature of cheese leaving the heated section and the rate of heat transfer from the tube to the cheese. The relevant thermo-physical properties of cheese are :

$$\rho = 1150 \text{ kg/m}^3$$
$$\mu = 22.5 \text{ kg/ms}$$
$$c_p = 2750 \text{ J/kg-deg}$$

and $k = 0.42$ *W/m-deg*

Use the following correlation for laminar flow inisde a tube :

$$Nu = \frac{h\,d}{k} = 3.65 + \frac{0.67\,\frac{d}{l}\,Re\,Pr}{1 + 0.04\left(\frac{d}{l}\,Re\,Pr\right)^{0.67}}$$

Solution : Velocity of cheese flowing through the tube,

$$V = \frac{m}{\rho A} = \frac{(1000/3600)}{1150 \times (\pi/4)\,(0.08)^2} = 0.048 \text{ m/s}$$

Reynolds number, Re

$$= \frac{V d \rho}{\mu} = \frac{0.048 \times 0.08 \times 1150}{22.5} = 0.196$$

Apparently the flow is laminar in character and the given correlation is valid for the given flow situation.

Prandtl number, Pr

$$= \frac{\mu c_p}{k} = \frac{22.5 \times 2750}{0.42} = 147321$$

$$\frac{d}{l}\,Re\,Pr = \frac{0.08}{1.5} \times 0.196 \times 147321 = 1540$$

$$\therefore \quad Nu = \frac{hd}{k} = 3.65 + \frac{0.067 \times 1540}{1 + 0.04\,(1540)^{2/3}} = 19.9$$

and $h = 19.9\,\frac{k}{d} = 19.9 \times \frac{0.42}{0.08} = 104.47$ W/m^2-deg

Let t_1 and t_2 denote the temperature of cheese at entrance and exit of the heated section. Then, mean bulk temperature of cheese is,

$$t_b = \frac{t_2 + t_1}{2} = \frac{t_2 + 15}{2}$$

The heat gained by cheese equals the convective heat flow from the tube to cheese. That is:

$$m\,c_p(t_2 - t_1) = h\,A\,(t_s - t_b)$$

or $\frac{1000}{3600} \times 2750\,(t_2 - 15) = 104.47 \times (\pi \times 0.08 \times 1.5) \times \left(95 - \frac{t_2 + 15}{2}\right)$

or $38.81\,(t_2 - 15) = (175 - t_2)$

Therefore temperature of cheese leaving the heated section

$$t_2 = \frac{38.81 \times 15 + 175}{38.81 + 1} = \mathbf{19.02°C}$$

Rise in temperature of cheese

$= 19.02 - 15 =$ **4.02°C**

Heat transfer from tube to cheese

$$= h\,A\,(t_s - t_b)$$
$$= m\,c_p\,(t_2 - t_1)$$
$$= \frac{1000}{3600} \times 2750 \times (19.02 - 15)$$
$$= \mathbf{3071\ W}$$

EXAMPLE 11.44

The wall of a tube 4 m long and 20 mm diameter is held at contant temperature by providing a steam jacket. A viscous fluid enters the tube at 30°C and leaves at 40°C at the rate of 180 kg/hr. Determine the average heat transfer coefficient and the wall temperature.

Use the following correlation

$$Nu = 3.65 + \frac{0.67\,\frac{d}{l}\,Re\,Pr}{1 + 0.04\left(\frac{d}{l}\,Re\,Pr\right)^{0.67}}$$

and take the following thermo-physical properties:-

$$\rho = 850 \text{ kg/m}^3$$
$$k = 0.1396 \text{ W/m-deg}$$
$$c_p = 2000 \text{ J/kg K}$$

and $\nu = 5.1 \times 10^{-6}$ m^2/s

Solution : Velocity of fluid flowing through the tube,

$$V = \frac{m}{\rho A} = \frac{(180/3600)}{850 \times (\pi/4) \times (0.02)^2} = 0.187 \text{ m/s}$$

$$Re = \frac{V d \rho}{\mu} = \frac{V d}{\nu} = \frac{0.187 \times 0.02}{5.1 \times 10^{-6}} = 733$$

$$Pr = \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k} = \frac{850 \times 5.1 \times 10^{-6} \times 2000}{0.1396} = 62.1$$

$$\frac{d}{l}\,Re\,Pr = \frac{0.02}{4} \times 733 \times 62.1 = 227.6$$

Substituting the relevant data in the give[n] expression, we get

$$Nu = \frac{hd}{k}$$

$$= 3.65 + \frac{0.67 \times 227.6}{1 + 0.04 \times (227.6)^{2/3}}$$

$$= 3.65 + \frac{152.49}{2.518} = 64.21$$

$$\therefore \quad h = Nu \times \frac{k}{d}$$

$$= 64.21 \times \frac{0.1396}{0.02}$$

$$= 448.18 \text{ W/m-deg}$$

Heat gained by viscous fluid

$$= m\, c_p\, \Delta t$$

$$= \frac{180}{3600} \times 2000 \times (40 - 30)$$

$$= 1000 W$$

Mean temperature of viscous fluid,

$$t_m = \frac{40 + 30}{2} = 35°C$$

Convective heat flow from tube wall to viscous fluid,

$$= h\,(\pi\, dl) \times \Delta t$$

$$= 448.18 \times (\pi \times 0.02 \times 4) \times (t_w - 35)$$

Under steady state conditions, the heat gained by the viscous fluid equals the convective heat flow from wall to the fluid. That is

$$1000 = 448.18 \times (\pi \times 0.02 \times 4) \times (t_w - 35)$$

$\therefore$ Temperature of the wall,

$$t_w = \frac{1000}{448.18 \times (\pi \times 0.02 \times 4)} + 35$$

$$= 43.88°C$$

EXAMPLE 11.45

Air at 2 bar and 40°C is heated as it flows through a 30 mm diameter tube at a velocity of 10 m/s. If the wall temperature is maintained at 100°C all along the length of tube, make calculations for the heat transfer per unit length of the tube. Proceed to calculate the increase in bulk temperature over one metre length of the tube.

Use the following correlation

$$Nu = 0.023\, Re^{0.8}\, Pr^{0.4}$$

and take the following thermo-physical properties of air at the average film temperature of 70°C.

$$\mu = 20.6 \times 10^{-6} \text{ Ns/m}^2$$

$$c_p = 1.009 \text{ kJ/kg-deg}$$

$$k = 0.0297 \text{ W/m-deg}$$

and $\quad Pr = 0.694$

Solution : $Re = \frac{\rho V d}{\mu}$. The density of air at the given pressure and temperature can be worked out from the characteristic gas equation : $p = \rho RT$

$$\rho = \frac{p}{RT} = \frac{2 \times 10^5}{287 \times (273 + 40)}$$

$$= 2.226 \text{ kg/m}^3$$

$$Re = \frac{2.226 \times 10 \times 0.03}{20.6 \times 10^{-6}} = 32417$$

Then from the given correlation,

$$Nu = \frac{hd}{k}$$

$$= 0.023 \times (32417)^{0.8} \times (0.694)^{0.4}$$

$$= 80.697$$

$$\therefore \quad h = Nu \times \frac{k}{d}$$

$$= 80.697 \times \frac{0.0297}{0.03}$$

$$= 79.89 \text{W/m}^2\text{K}$$

Let t_1 and t_2 denote the temperature of air at entrance and at exit of the heated section. This mean bulk temperature of air is

$$t_b = \frac{t_1 + t_2}{2} = \frac{40 + t_2}{2}$$

Heat given by air

$$= m_a\, c_p\, \Delta t$$

$$= \rho \frac{\pi}{4} d^2 V \times c_p\, \Delta t$$

$$= 2.226 \times \frac{\pi}{4} (0.03)^2 \times 10 \times 1009 \times (t_2 - 40)$$

$$= 15.868\, (t_2 - 40) \quad ...(i)$$

Convective heat flow from tube wall to air

$$= h\,(\pi\, d\, l) \times \Delta t$$

$$= 79.89 \times (\pi \times 0.03 \times 1) \times \left(100 - \frac{40 + t_2}{2}\right)$$

$$= 7.526 \left(100 - \frac{40 + t_2}{2}\right) \quad ...(ii)$$

Under steady state conditions, the heat gained by air equals the convective heat flow from tube wall to air. That is

$$15.868\,(t_2 - 40)$$

$$= 7.526 \left(100 - \frac{40 + t_2}{2}\right)$$

$$= \frac{7.526}{2} (160 - t_2)$$

$$\text{or } t_2 - 40 = \frac{7.526}{2 \times 15.868} (160 - t_2)$$

$$= 0.237\,(160 - t_2)$$

$$= 37.92 - 0.237\, t_2$$

$\therefore$ Temperature of air leaving the heated section,

$$t_2 = \frac{37.92 + 40}{1.237} = 63°C$$

Using expression (*i*), the heat transfer from tube wall to air is

$$Q = 15.868\,(63 - 40)$$

$$= 364.96 \text{ W/m}$$

Rise in bulk temperature of air

$$= 63 - 40 = 23°C/m$$

EXAMPLE 11.46

Air at 2 bar pressure and 200°C temperature gets heated as it flows through 2.5 cm diameter tube with a velocity of 10 m/s. A constant heat flux condition is maintained at the wall and wall temperature is 20°C above the air temperature all along the length of the tube. Make calculations for the heat transfer per unit length of the tube. Also determine the increase in bulk temperature over a 3 metre length of the tube.

The appropriate correlation for the convection coefficient is

$$Nu = 0.023\,(Re)^{0.8}\,(Pr)^{0.4}$$

where the different thermo-physical properties of air are:

$$\mu = 2.57 \times 10^{-5} \text{ Ns/m}^2$$

$$k = 0.0385 \text{ W/m-deg}$$

and $\quad c_p = 1025$ J/kgK

Solution : The density of air at 2 bar pressure and 200°C is worked out from the characteristic gas equation :

$$\rho = \frac{p}{RT} = \frac{2 \times 10^5}{287 \times (273 + 200)}$$

$$= 1.473 \text{ kg/m}^3$$

$$Re = \frac{\rho V d}{\mu} = \frac{1.473 \times 10 \times 0.025}{2.57 \times 10^{-5}}$$

$$= 14329$$

$$Pr = \frac{\mu c_p}{k} = \frac{2.57 \times 10^{-5} \times 1025}{0.0385}$$

$$= 0.684$$

Using the given correlation,

$$Nu = \frac{h d}{k}$$

$$= 0.023\,(14329)^{0.8} \times (0.684)^{0.4}$$

$$= 41.75$$

$\therefore$ Convection coefficient,

$$h = 41.75 \times \frac{k}{d}$$

$$= \frac{41.75 \times 0.0385}{0.025}$$

$$= 64.3 \text{ W/m}^2\text{-deg}$$

Heat transfer, Q

$$= h\, A\, \Delta t = h\, \pi\, d\, l\, \Delta t$$

$$= 64.3 \times (\pi \times 0.025 \times 1) \times 20$$

$$= 100.95 \text{ W per metre length}$$

For 3-metre length of tube, Q

$$= 3 \times 100.95 = 302.85 \text{ W}$$

Also $\quad Q = m\, c_p\, \Delta t = \rho\, A\, V\, c_p\, \Delta t$

$$\therefore 302.85 = 1.49 \times \frac{\pi}{4} (0.025)^2 \times 10 \times 1025 \times \Delta t$$

Increase in bulk temperature $\Delta t = 40.42°C$

EXAMPLE 11.47

3000 kg of water is heated per hour from 30 to 70°C by pumping it through a certain heated section of a 25 mm diameter tube. If the surface of the heated section is maintained at 110°C, estimate length of the heated section and the rate of heat transfer from the tube to water.

(b) Outline the calculation procedure to be used if the pipe length had been specified rather

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than outlet temperature of the fluid; it then being desired to estimate the fluid temperature at the exit section.

Solution : Mean bulk temperature of water,

$$t_b = (30 + 70)/2 = 50°C$$

At the mean film temperature,

$$t_f = (110 + 50)/2 = 80°C,$$

the thermo-physical properties of water are :

$$\rho = 971.6 \text{ kg/m}^3;$$
$$\mu = 0.355 \times 10^{-3} \text{ kg/m-s}$$
$$k = 0.667 \text{ W/m-deg}$$

and $c_p = 4195$ J/kg-deg

Using these properties, the pertinent parameters are :

$$Pr = \frac{\mu c_p}{k} = \frac{0.355 \times 10^{-3} \times 4195}{0.667} = 2.23$$

Flow velocity of water,

$$V = \frac{m}{\rho A} = \frac{3000/3600}{971.6 \times \pi/4 \times (0.025)^2} = 1.748 \text{ m/s}$$

Flow Reynolds number,

$$Re = \frac{V d \rho}{\mu} = \frac{1.748 \times 0.025 \times 971.6}{0.355 \times 10^{-3}} = 1.196 \times 10^5$$

which indicates turbulent flow for which the following correlation applies

$$Nu = 0.023 \, (Re)^{0.8} \, (Pr)^{0.4}$$
$$= 0.023 \times (1.196 \times 10^5)^{0.8} \times (2.23)^{0.4}$$
$$= 366$$

$$\therefore \quad h = \frac{Nu \, k}{d} = \frac{366 \times 0.667}{0.025} = 9765 \text{ W/m}^2\text{-deg}$$

The heat gained by water equals the convective heat flow from the tube surface to water. That is

$$m \, c_p \, (t_2 - t_1) = h \, A \, (t_s - T_b)$$

$$\frac{3000}{3600} \times 4195 \times (70 - 30) = 9765 \times (\pi \times 0.025 \times l) \times (110 - 50)$$

$\therefore$ Length of heated section $l = 304$ m

Heat flow rate, Q

$$= h \, A (t_s - b)$$
$$= 9765 \times (\pi \times 0.025 \times 3.04) \times (110 - 50)$$
$$= \mathbf{1.398 \times 10^5 \text{ W}}$$

(b) If the pipe length had been specified rather than the fluid temperature at the exit section, then the solution is not as straight forward. The correlations require that the thermo-physical properties be evaluated at the mean bulk temperature, $t_b = (t_1 + t_2)/2$, or at an average film temperature, $t_f = (t_s + t_b)/2$. In either case the outlet temperature is unknown, requiring a trial and error solution to the heat balance equation.

A suitable value of t_2 is assumed and an average value of h is calculated for the condition at $t_b = (t_1 + t_2)/2$. The heat balance equation

$$m \, c_p \, (t_2 - t_1) = h \, A \, (t_s - t_b)$$
$$= h \, \pi \, d \, l \, (t_s - t_b)$$

is used with the given length to calculate t_2. This calculated value is compared with the assumed value, and the calculations repeated until a satisfactory agreement is obtained.

EXAMPLE 11.48

0.05 kg/s of hot air flows through an uninsulated sheet metal duct of 15 cm diameter. The air enters the duct at a temperature of 105°C and after a distance of 5 m gets cooled to a temperature of 80°C. Make calculations for

(a) the heat loss from the duct over its 5 m length.

(b) the heat flux and the duct surface temperature at x = 5 m

The heat transfer coefficient between the duct outer surface and the cold ambient air at 5°C is anticipated to have a constant value of 6 W/m²K.

Solution : The arrangement of the system, and the electrical net work for heat flow from the hot air inside the duct to the ambient air outside has been indicated in Fig. 11.4.

At the mean bulk temperature,

$$t_b = \frac{105 + 80}{2} = 92.5°C$$

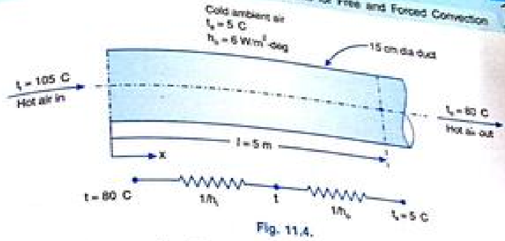

Fig. 11.4.

the thermo-physical properties of air are :

$$c_p = 1009 \text{ J/kgK}$$
$$\mu = 22.5 \times 10^{-6} \text{ kg/ms}$$
$$Pr = 0.691$$
$$k = 0.0315 \text{ W/mK}$$

(a) Heat loss from the duct over its 5 m length is worked out from the relation for energy balance for the entire duct,

$$Q = m \, c_p \, \Delta t$$
$$= 0.05 \times 1009 \times (105 - 80)$$
$$= \mathbf{1261.25 \text{ W}}$$

(b) The heat flux at x = 5 m is inferred from the relation:

$$\frac{Q}{A} = \frac{\Delta t}{\sum R_t} = \frac{\Delta t}{\frac{1}{h_i} + \frac{1}{h_0}}$$

The outside convection coefficient

$$h_o = 6 \text{ W/m}^2 \qquad \text{(given)}$$

and the inside convection coefficient h_i can be estimated from the correlation:

$$Nu = 0.023 \, Re^{0.8} \, Pr^{0.3}$$

where all the thermo-physical properties are evaluated at the mean bulk temperature of the fluid.

$$Re = \frac{V d \rho}{\mu} = \frac{md}{A\mu}$$
$$= \frac{4m}{\pi d \mu} = \frac{4 \times 0.05}{\pi \times 0.15 \times 22.25 \times 10^{-6}}$$
$$= 18872$$

$$Nu = 0.023 \, (18872)^{0.8} \times (0.691)^{0.3}$$
$$= 54.23$$

Inside convection coefficient,

$$h_i = \frac{Nu \times k}{d} = \frac{54.23 \times 0.0315}{0.15} = 11.388 \text{ W/m}^2\text{K}$$

$\therefore$ Thermal resistance, R_t

$$= \frac{1}{11.388} + \frac{1}{6.0} = 0.2545$$

Heat flux, $\frac{Q}{A} = \frac{80 - 5}{0.2545} = \mathbf{294.69 \text{ W/m}^2}$

From the resistance network, $\frac{Q}{A} = \frac{80 - t}{h_i}$

The duct surface temperature then works out to be :

$$t = 80 - \frac{294.69}{11.388} = \mathbf{5412°C}$$

EXAMPLE 11.49

A horizontal steam pipe of 25 mm diameter has a surface temperature of 250°C and a surface emissivity of 0.95. Determine the radiation and convective heat loss from the pipe if

(a) the pipe passes through still air at temperature of 30°C.

(b) instead of still air, air flow is forced at right angles to the pipe with a velocity of 20 m/s.

Solution : The radiation heat loss from the outside surface of the pipe to the surrounding air is given by

$$Q = F_{1-2}\,\sigma A_1 (T_1^4 - T_2^4)$$

where the factor F_{1-2} is equal to

$$\frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{F_{12}} + \frac{1-\epsilon_2}{\epsilon_2}\frac{A_1}{A_2}}$$

The subscripts 1 and 2 refer to pipe surface and surroundings, respectively.

The arrangement (location of pipe in the atmosphere) corresponds to the situation where

$$F_{12} = 1 \text{ and } A_1 <<< A_2$$

$$F_{1-2} = \frac{1}{\frac{1-\epsilon_1}{\epsilon_1} + 1} = \epsilon_1 = 0.85$$

$$\therefore \frac{Q_r}{A_1} = 0.85 \times 5.67 \times 10^{-8} \times (523^4 - 303^4)$$

$$= 3576 \text{ W/m}^2$$

At the mean temperature of

$$\frac{250 + 30}{2} = 140°\text{C}$$

the relevant thermo-physical properties of air:

$$k = 0.03487 \text{ W/mK}$$

$$\nu = 27.8 \times 10^{-6} \text{ m}^2\text{/s and}$$

$$Pr = 0.684$$

$$\beta = \frac{1}{140 + 273}$$

$$= 2.421 \times 10^{-3} \text{ per degree kelvin}$$

Grashof number, Gr

$$= \frac{d^3 \rho^2 \beta g \Delta T}{\mu^2}$$

$$= \frac{d^3 \beta g \Delta T}{\nu^2}$$

$$= \frac{0.025^3 \times 2.421 \times 10^{-3} \times 9.81 \times (250 - 30)}{(27.8 \times 10^{-6})^2}$$

$$= 8.34 \times 10^4$$

$$Gr \times Pr = 8.34 \times 10^4 \times 0.684$$

$$= 5.70 \times 10^4$$

A quiescent atmosphere corresponds to natural free convection. Further for such a flow over horizontal cylinders within the range $10^4 < Gr\,Pr < 10^9$

$$Nu = \frac{hd}{k} = 0.53\,(Gr\,Pr)^{0.25}$$

$\therefore$ Convective coefficient

$$h = \frac{0.03487 \times 0.53 \times (5.70 \times 10^4)^{0.25}}{0.025}$$

$$= 11.42 \text{ W/m}^2\text{-deg}$$

Convective heat loss, $\frac{Q_c}{A}$

$$= h\,\Delta T = 11.42\,(250 - 30)$$

$$= 2512.4 \text{ W/m}^2$$

Total heat loss

$$= Q_r + Q_c$$

$$= 3576 + 2512.4 = 6088.4 \text{ W/m}^2$$

Apparently radiation and free convection are of comparable importance in determining the heat loss from a hot pipe.

(b) At the bulk temperature $t_b = 30°C$, the relevant thermo physical properties of air are:

$$k = 0.02673 \text{ W/mK};$$

$$\nu = 16 \times 10^{-6} \text{ m}^2\text{/s}$$

and $Pr = 0.701$

Reynolds number, Re

$$= \frac{V d \rho}{\mu} = \frac{V d}{\nu}$$

$$= \frac{20 \times 0.025}{16 \times 10^{-6}} = 31250$$

This is well in excess of the critical Reynolds number; the flow is turbulent and the following correlation applies

$$Nu = \frac{hd}{k} = 0.023\,Re^{0.8}\,Pr^{0.4}$$

$$= 0.023\,(31250)^{0.8}\,(0.701)^{0.4}$$

$$= 78.66$$

Convective coefficient, h

$$= \frac{78.66 \times 0.02673}{0.025}$$

$$= 84.13 \text{ W/m}^2\text{-deg}$$

Convective heat loss, $\frac{Q}{A}$

$$= h\,\Delta T = 84.13 \times (250 - 30)$$

$$= 18508.6 \text{ W/m}^2$$

Total heat loss

$$= 3756 + 18508.6$$

$$= 22084.6 \text{ W/m}^2$$

Evidently for forced convection and radiation from a hot pipe, the radiation accounts for a smaller portion of heat loss.

Note : Some more problems on forced convection have been worked out in the text chapter on hydrodynamic and thermal boundary layers.

SALIENT POINTS

1. The heat transfer coefficient for free convection from vertical planes and cylinders, and horizontal cylinders are estimated from the relation
$$N_u = C\,(G_r\,P_r)^m = C\,(R_a)^m$$
where C and m are constants, $R_a = G_r\,P_r$ is the Rayleigh number.
2. For the forced convection the generalised basic equations for use in determining the value of convection coefficient are:
$$N_u = f_1\,(R_e, P_r) = C_1\,(R_e)^m\,(P_r)^n$$
and $$S_t = f_2\,(R_e, P_r) = C_2\,(R_e)^a\,(P_r)^b$$
3. The numerical values of the constants and the exponents are determined by obtaining the best fit to the experimental data.
4. The fluid properties (μ, k, ρ, c_p) needed for calculating the values of dimensionless groups are evaluated either at mean bulk temperature or at mean film temperature as may be stated.
5. The mean bulk temperature t_b denotes the equilibrium temperature that would result if the fluid at a cross-section was throughly mixed in a adiabatic container.
   (i) For turbulent flow, this temperature is nearly equal to the fluid temperature near the duct axis.
   (ii) For heat exchangers, it is taken to be the arithmetic mean of the temperature at inlet to and at exit from the heat exchanger tube.
$$t_b = \frac{t_i + t_e}{2}$$
6. The mean film temperature t_f is the arithmetic mean of the surface temperature t_s of a solid and the undisturbed temperature t_∞ of the fluid which flows past it
$$t_f = \frac{t_s + t_\infty}{2}$$
7. The simplified free convection relations for air are
$$h = C_1\left(\frac{\Delta t}{l}\right)^n \text{ for laminer flow}$$
$$= C_2\,(\Delta t)^{n_1} \text{ for turbulent flow}$$
The constants depend on geometry and flow conditions.
8. For turbulent flow over flat plates
$$N_{ux} = 0.0292\,(R_{ex})^{0.8}\,(P_r)^{0.33}$$
and $$\bar{N}_u = 0.036\,(R_{el})^{0.8}\,(P_r)^{0.33}$$
9. For turbulent flow in tubes, Mc Adam's general correlation for heating and cooling of fluids is
$$N_u = 0.023\,(R_e)^{0.8}\,(P_r)^n$$
where $n = 0.4$ if the fluid is being heated
$= 0.3$ if the fluid is being cooled
10. For turbulent flow over cylinders
$$N_u = \frac{hd}{k} = C(R_e)^n\,(P_r)^{0.33}$$
and for flow of gases over spheres
$$N_u = 0.37\,(R_e)^{0.6}$$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. What is film temperature? How does it differ from bulk temperature?
2. Set up the relationship between local heat transfer coefficient and average heat transfer coefficient for flow past a stationary flat plate.

3. What is Rayleigh number? Give its value that sets the criterion of laminar or turbulent character of flow.

4. Enumerate some of the empirical relations which are used to compute the convective coefficient for free convection.

5. In the product of Gr and Pr, what group of properties is concerned only with the properties and gravitational field ? What is this group of properties called ? What are the dimensional units of this group of properties.

6. Write the correlation that has been suggested for natural convection over a vertical plate or cylinder in turbulent flow region.

7. *The coefficient of free convection at the surface of horizontal pipe may be computed from the relation*

$$Nu = \frac{hd}{k}$$
$$= 0.053\ (Pr)^{0.5} \times (Pr + 0.955)^{-0.25} \times (Gr)^{0.25}$$

where all the properties are evaluated at the surface temperature and coefficient of cubical expansion $\beta = (1/T)$, T being the absolute air temperature.

Use this relation to calculate the heat loss by natrual convection per metre length from a horizontal pipe of 15 cm diameter. The surface temperature of the pipe is 275°C and the surroundings are at 17°C.

At the surface temperature of 275°C, the thermo-physical properties of air are:

$Pr = 0.675$

$\rho = 0.6445$ kg/m³

$k = 0.16$ kJ/m-hr-deg

and $\mu = 28.55 \times 10^{-6}$ Ns/m²

(Ans. 3077 kJ/hr)

8. *A horizontal cylinder 25 mm diameter and 500 mm long is suspended in water at 20°C. Calculate the rate of heat transfer if the cylinder surface is at 60°C. Use the following correlation :*

$Nu = 0.53\ (Gr.\ Pr)^{0.25}$

The relevant physical properties of water at the mean film temperature (40°C) are :

$\rho = 992$ kg/m³

$\mu = 2.35$ kg/hr-m

$k = 0.63$ W/mK and $Pr = 4.3$

(Ans. 2446 W)

9. A restaurant grill 1.0 m × 0.8 m is maintained at 135°C whilst the room temperature is 25°C. Calculate the heat load generated by the grill. The approximate correlation is :

$$Nu = \frac{hl}{k} = 0.14\ (Gr \times Pr)^{1/3}$$

where the significant length is the average of the two edges, and the relevant physical properties of air at the mean film temperature (80°C) are :

$k = 0.0304$ W/mK

$\nu = 21.09 \times 10^{-6}$ m²/s

and $Pr = 0.692$

(Ans. 630.9 W)

10. A horizontal cylindrical heat exchanger of shell diameter 40 cm and surface temperature 200°C is to be cooled by the ambient air at 30°C. Workout the convective coefficient and the rate of heat loss from unit surface area of the heat exchanger.

Use the correlation.

$$Nu = \frac{hd}{k} = 0.5\ (Gr \times Pr)^{0.25}$$

where the physical properties of the fluid and its bulk temperature are :

$\nu = 16 \times 10^{-6}$ m²/s

$k = 2.67 \times 10^{-2}$ W/m-deg

and $Pr = 0.701$

(Ans. 5.9 W/m²-deg, 1003 W/m²)

11. A steel pipe, 20 cm diameter and 15 m long, carrying hot gas at 300°C is placed in still air at 25°C. Determine the heat loss by natural convection if the convective heat transfer coefficient is approximated by the relation

$$h = 1.32\left(\frac{\Delta t}{d}\right)^{1/4} \text{ W/m}^2\text{-deg}$$

where d is in meters and Δt is in degrees kelvin.

12. Calculate the heat generated in the body of a man if for comfortable living, the body is to be at 35°C whilst the environmental conditions are at 15°C. The body of the man may be idealised as a cylinder of 30 cm diameter and 160 cm height.

Use the correlation,

$Nu = 0.12\ (Gr\ Pr)^{1/3}$

13. A long horizontal pipe 15 cm outside diameter and with an oxidised surface passes through a large room. The surface temperature of the pipe is 95°C and the surrounding air is at 25°C. Workout the convective coefficient for free convection. Use the correlation

$$Nu = \frac{hd}{k} = 0.53\ (Gr \times Pr)^{0.25}$$

and take the air properties at the mean film temperature (60°C) as :

$c_p = 1.0$ kJ/kg K

$k = 0.104$ kJ/m-hr-deg

$\nu = 18.97 \times 10^{-6}$ m²/s

and $Pr = 0.696$

(Ans. 22.27 kJ/m²-hr-deg)

14. A 3 mm diameter electrical wire covered with 0.75 mm thick insulation has been placed horizontally and exposed to ambient air at 30°C. The wire dissipates electrical energy into heat due to its resistance and this heat is subsequently convected to the wire. Make calculations for its surface temperature. Assume free convection boundary layer to be laminar and use the correlation :

$h = 1.32\ (\Delta T/d)^{0.25}$

where h is measured in W/m²K, Δt in degree kelvin, and d in metres.

(Ans. 81.35°C)

15. 72 kg/hr of air at a bulk temperature of 325 K flows through 10 cm inside diameter tube. Predict the coefficient of heat transfer by convection between the air and tubes. For the turbulent flow, use the correlation :

$Nu = 0.23\ (Re)^{0.8} \times (Pr)^{0.4}$

and take the following properties at the bulk temperatue (325 K) :

$\mu = 1.96 \times 10^{-6}$ kg/m-s

$k = 0.028$ W/m-deg

and $Pr = 0.70$

If the wall temperature is 375 K, what would be the heat transfer rate for one metre length of the pipe.

(Ans. 10.9 W/m²-deg ; 171 W)

16. Air at 25°C flows normal to 30 mm outer diameter water pipe with a velocity of 1.0 m/s. Estimate the heat transfer per unit length if the surface temperature of the pipe is 75°C. Use the correlation,

$Nu = 0.683\ (Pr)^{0.333}\ (Re)^{0.466}$

The relevant physical properties of air evaluate at the film temperature (50°C) are :

$\nu = 1.795 \times 10^{-5}$ m²/s

$k = 0.0283$ W/mK

and $Pr = 0.698$

(Ans. 85.19 W/m length)

17. Water at 25°C flows through a 5 mm diameter tube with a velocity of 1 m/s. If the tube wall temperature is 75°C, make calculations for the heat transfer coefficient.

Use the correlation,

$St = 0.0232\ Re^{-0.2}\ Pr^{-0.667}$ [illegible]

and take the following thermo-physical properties of air :

$\mu = 1.977$ kg/m-hr

$k = 0.647$ W/mK

$c_p = 4.178$ kJ/kgK

and $\rho = 1000$ kg/m³

(Ans. 6520 W/m²-deg)

18. Estimate the heat transfer from a 40-Watt incandescent bulb at 125°C to 25°C air stream moving at 0.3 m/s. The bulb may be approximated as a 50 mm diameter sphere. Also calculate the percent of power lost by convection.

The average heat transfer coefficient is given as :

$$h = 0.37\frac{k}{d}(Re)^{0.6}$$

The required properties evaluated at the mean film temperature are :

$\nu = 2.055 \times 10^{-5}$ m²/s

and $k = 0.030$ W/mK

(Ans. 2.27 W, 5.67%)

19. For a certain forced convection process, the following correlation applies :

$Nu = 0.031\ (Re)^{0.75}\ (Pr)^{0.30}$

Workout the percentage change in the rate of heat flow per degree temperature difference when the orignal coolant is replaced by another fluid having viscosity equal to two-third that of the original coolant. Assume that other fluid variables and configuration remain the same.

(Ans. 16.7%)

B. Fill in the blanks with appropriate word/words :

1. The basic equation developed from dimensional analysis for use in determining the value of convection coefficient for free convection is : N_u = ...

2. The values of Grashoff number G_r and Prandtl number P_r in the relation $N_u = C\,(G_r\,P_r)^m$ are evaluated at .... temperature.
3. For free convection over inclined plates, Grashoff number is multiplied by .... where θ is the angle of inclination from the vertical and use vertical plate constants.
4. The product $(G_r \times P_r)$ is called .... number and its value sets the exterion of .... character of flow.
5. The simplified free convection relation for air is : $h = C\left(\frac{\Delta t}{l}\right)^m$ where C and m are constants depending upon .... and ....
6. The fluid properties needed for calculating the values of dimensionless groups pertaining to forced convection are generally evaluated at .... temperature.
7. The conventional generalised basic equation $N_u = C\,(R_e)^m\,(P_r)^n$ is used for determining the value of convection coefficient for .... convection.
8. Mc Adam has suggested the following general correlation for heating and cooling of fluids in turbulant flow through long pipes

$$N_u = 0.023\,(R_e)^{0.8}\,(P_r)^n$$

where n = ... if the fluid is being heated and n = ... if the fluid is being cooled.

Answers : 1. $N_u = C\,(G_r)^a\,(P_r)^b$; 2. film; 3. cos θ; 4. Rayleigh number; 5. geometry, flow conditions; 6. bulk; 7. forced; 8. $n = 0.4$ and $n = 0.3$.

C. Multiple choice questions :

1. In respect of free convection over a vertical flat plate, the Nusselt number for laminar and turbulent flows varies respectively with Grashoff number G_r as
   (*a*) G_r and $G_r^{1/4}$ (*b*) $G_r^{1/2}$ and $G_r^{1/3}$
   (*c*) $G_r^{1/4}$ and $G_r^{1/3}$ (*d*) $G_r^{1/3}$ and $G_r^{1/4}$
2. The Nusselt number is related to Reynolds number in laminar and turbulent flows respectively as
   (*a*) $R_e^{-1/2}$ and $R_e^{0.8}$ (*b*) $R_e^{1/2}$ and $R_e^{0.8}$
   (*c*) $R_e^{-1/2}$ and $R_e^{-0.8}$ (*d*) $R_e^{1/2}$ and $R_e^{-0.8}$
3. Consider fully-developed laminar flow and heat transfer in a uniformly heated long circular tube. If the flow velocity is doubled and the tube diameter is halved, the heat transfer coefficient will become/remain .... the original value
   (*a*) four times (*b*) double
   (*c*) half (*d*) same
4. The Nusselt number of convective heat transfer between a horizontal tube and water surrounding it is prescribed by the relation

$$Nu = 0.52\,(Gr\,Pr)^{0.25}$$

For a 4 cm diameter tube, the heat transfer co-efficient is stated to be 5910 kJ/m²-hr-deg. Subsequently the tube is replaced by one with 16 cm diameter tube. If temperature and surface of the fluid remain same, the heat transfer coefficient will change to
   (*a*) 2955
   (*b*) 4185
   (*c*) 11820
   (*d*) 23645 kJ/m²-hr-deg
5. The convective heat transfer coefficient from a hot cylindrical surface exposed to still air varies in accordance with
   (*a*) $(\Delta T)^{0.25}$ (*b*) $(\Delta T)^{0.5}$
   (*c*) $(\Delta T)^{0.75}$ (*d*) $(\Delta T)^{1.25}$
   where ΔT is the temperature difference between the hot surface and the surrounding ambient air
6. For convective heat transfer to fluid flowing past a heated plate, identify the correct relation between the average heat transfer coefficient $\bar{h}$ and the local heat transfer coefficient h_l at the end of plate
   (*a*) $\bar{h} = 0.5\,h_l$ (*b*) $\bar{h} = 0.75\,h_l$
   (*c*) $\bar{h} = 1.25\,h_l$ (*d*) $\bar{h} = 2\,h_l$
   The plate is heated over its entire length
7. Experimental results indicate that the local heat transfer coefficient h_x for flow over a plate with an extremely rough surface is approximated by the relation $h_x = ax^{-0.12}$ where a is a constant coefficient and x is the distance from the leading edge of the plate. Identify the relation between this local heat transfer coefficient and the average heat transfer coefficient $\bar{h}$ for a plate of length x

   (*a*) $\bar{h} = 0.568\,h_x$ (*b*) $\bar{h} = 0.852\,h_x$
   (*c*) $\bar{h} = 1.136\,h_x$ (*d*) $\bar{h} = 1.988\,h_x$
8. For laminar flow over a flat plate, the local heat transfer coefficient h_x varies is $x^{-1/2}$ where x is the distance from the leading edge ($x = 0$) of the plate. The ratio of the average coefficient h_{av} between the leading edge and some location A at $x = x$ on the plate to the local heat transfer coefficient is
   (*a*) 1 (*b*) 2
   (*c*) 4 (*d*) 8
9. The convective heat transfer coefficient in laminar flow over a flat plate
   (*a*) increases with distance
   (*b*) increases if a higher viscosity fluid is used
   (*c*) increases if a denser fluid is used
   (*d*) decreases with increase in free stream velocity
10. For laminar flow over a flat plate, the average value of Nusselt number $\bar{N}_u$ is prescribed by the relation

$$\bar{N}_u = 0.664\,R_e^{0.5}\,P_r^{0.33}$$

Which of the followings is then a false statement? To double the heat transfer coefficient,
   (*a*) density has to be increased four times
   (*b*) plate length has to be decreased four times
   (*c*) specific heat has to be increased eight times
   (*d*) dynamic viscosity has to be decreased sixteen times
11. For turbulent flow over a flat plate, the average value of Nusselt number is prescribed by the relation

$$\bar{N}_u = 0.37\,R_e^{0.8}\,P_r^{0.33}$$

Which of followings is then a false statement ? The average heat transfer coefficient increase as
   (*a*) 2/3 power of thermal conductivity
   (*b*) 1/3 power of specific heat
   (*c*) 4/5 power of free stream velocity
   (*d*) 1/5 power of plate length

Answers :

1. (*c*)	2. (*b*)	3. (*b*)	4. (*b*)	5. (*a*)
6. (*d*)	7. (*c*)	8. (*b*)	9. (*c*)	10. (*d*)
11. (*d*)				

HINTS AND COMMENTS

4(b):

The prescribed relation can be rewritten as

$$\frac{hd}{k} = 0.52\left[\frac{\beta\,g\,\rho^2\,d^3\,\Delta t}{\mu^2}\right]^{0.25}$$

Using identical operating conditions

$$\frac{h_1}{h_2} = \left(\frac{d_2}{d_1}\right)^{0.25} = \left(\frac{16}{4}\right)^{0.25} = \sqrt{2}$$

$$h_2 = \frac{h_1}{\sqrt{2}} = \frac{5190}{\sqrt{2}} = 4180$$

5(a):

For free convection on vertical planes or cylinders, the convection heat transfer coefficient is empirally given by

$$h = 1.42\left(\frac{\Delta t}{d}\right)^{0.25} \quad \text{for laminar flow}$$

$$= 1.31\,(\Delta t)^{0.33} \quad \text{for turbulent flow}$$

The terms in these correlations are dimensional : h is measured in W/m² K, Δt in degrees kelvin and d in metres.

6(d):

For thermal boundary layer on a flat plate, local value of convective heat transfer coefficient is prescribed by the relation

$$Nu_x = \frac{x\,h_x}{k} = 0.332\,(Re_x)^{0.5}\,(P_r)^{0.33}$$

Taking averages over the interval $0 < x < l$, we can workout the average heat transfer coefficient and Nusselt number to be

$$\bar{h} = \frac{1}{l}\int_0^l h_x\,dx = 2 \times 0.332\,\frac{k}{l}\,(Re_l)^{0.5}\,(P_r)^{0.33}$$

$= 2 \times$ local film coefficient for a plate of length l and

$$\overline{Nu} = \frac{\bar{h}\, l}{k}$$

$$= 0.664\, (R_e)^{0.5}\, (P_r)^{0.33}$$

11(c):

The local and average convection coefficients are related by an expression for the form

$$\bar{h} = \frac{1}{x}\int_0^x h_x\, dx$$

$$= \frac{1}{x}\int_0^x a\, x^{-0.12}\, dx$$

$$= 1.136\, a\, x^{-0.12}$$

$$= 1.136\, h_x$$

11(d):

Invoking the relation

$$\overline{Nu} = 0.664\, (R_e)^{0.5}\, (P_r)^{0.33}$$

Substituting for the dimensionless numbers

$$\frac{\bar{h}\, l}{k} = 0.664 \left(\frac{\rho U l}{\mu}\right)^{\frac{1}{2}} \left(\frac{\mu c_p}{k}\right)^{\frac{1}{3}}$$

or $\quad \bar{h} = 0.664\, U^{1/2} \rho^{1/2} k^{2/3} c_p^{1/3}\, l^{-1/2} \mu^{-1/6}$

$$\frac{h_2}{h_1} = \left(\frac{U_2}{U_1}\right)^{\frac{1}{2}} \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}} \left(\frac{k_2}{k_1}\right)^{\frac{2}{3}}$$

$$\times \left(\frac{cp_2}{cp_1}\right)^{\frac{1}{3}} \left(\frac{l_1}{l_2}\right)^{\frac{1}{2}} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{6}}$$

Obviously increased density will increase the convection coefficient.

The dynamic viscosity has an inverse relation to $\frac{1}{6}$ power. To double the convective heat transfer coefficient, th dynamic viscosity has to be decreased 64 times.

11(f):

Substituting the values of physical parameters for the dimensionless numbers in the given identity the following relation can be worked out :

$$\frac{h_2}{h_1} = \left(\frac{U_2}{U_1}\right)^{\frac{4}{5}} \left(\frac{\rho_2}{\rho_1}\right)^{\frac{4}{5}} \left(\frac{k_2}{k_1}\right)^{\frac{2}{3}}$$

$$\times \left(\frac{cp_2}{cp_1}\right)^{\frac{1}{3}} \left(\frac{l_1}{l_2}\right)^{\frac{1}{5}} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{7}{15}}$$

The average heat transfer coefficient reduces with length; as $\frac{1}{5}$th power of the length.

❑❑❑

Chapter 12

Hydrodynamic and Thermal Boundary Layers

Learning objectives : This chapter deals with certain aspects which will enable the readers to

- *understand the concept of hydrodynamic and thermal boundary layers*
- *establish the relationship between thermal and hydrodynamic boundary layers*
- *define the local friction coefficient and average friction coefficient for hydrodynamic boundary layer*
- *workout the correlations for local and average values of heat transfer coefficients for thermal boundary layer*
- *appreciate the Reynolds-Colburn analogy between fluid friction and heat transfer*

When a fluid flows around an object or when the object moves through a body of fluid, there exists a thin layer of fluid close to the solid surface within which shear stresses significantly influence the velocity distribution. The fluid velocity varies from zero at the solid surface to the velocity of free stream flow at a certain distance away from the solid surface. This thin layer of changing velocity has been called the **hydrodynamic boundary layer**; a concept first suggested by Ludwig Prandtl in the year 1904. Heat transfer occurs due to heat conduction and energy transport by moving fluid within this thin layer. Hence, the value of convection coefficient and heat transfer is highly dependent upon the thickness and characteristics of the boundary layer. For better understanding of the combined fluid-dynamic and heat transfer phenomenon, recourse has to be made to the realms of fluid mechanics with particular emphasis on the development and growth of boundary layer. Towards that end, an attempt has been made in this chapter to discuss the concept of boundary layer, derive the relevant governing equations and explain the methods for their solution.

12.1. HYDRODYNAMIC BOUNDARY LAYER : FLAT PLATE

Consider a continuous flow of fluid along the surface of a thin plate with its sharp leading edge set parallel to flow direction (Fig. 12.1). The salient features of the flow situation are :

(*i*) The free stream undisturbed flow has a uniform velocity U_∞ in the *x*-direction. Particles of fluid adhere to the plate surface as they approach it and the fluid is slowed down considerably. The fluid becomes stagnant or virtually so in the immediate vicinity of the plate surface. Generally it is presumed that there is no slip between the fluid and the solid boundary. Thus, there exists a region where the flow velocity changes from that of solid boundary to that of mainstream fluid, and in this region the velocity gradients exist in the fluid. Consequently the flow is

rotational and shear stresses are present. This thin layer of changing velocity has been called the *hydrodynamic boundary layer*.

(*ii*) The condition $\partial u/\partial y \neq 0$ is true for the zone within the boundary layer, whilst the conditions for flow beyond the boundary layer and its outer edge are :

$$\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad u = U_\infty$$

Thus all the variation in fluid velocity is concentrated in a comparatively thin layer in immediate vicinity of the plate surface.

(*iii*) The concepts of boundary layer thickness and outer edge of the boundary layer are quite fictitious as there is no abrupt transition from the boundary layer to the flow beyond or outside it. Velocity within the boundary layer approaches the free stream velocity asymptotically. Usually the boundary layer thickness δ is taken to be the distance from the plate surface to a point at which the velocity is within 1 percent of the asymptotic limit, *i.e.*, $u = 0.99\ U_\infty$. The parameter δ then becomes a nominal measure of the thickness of boundary layer, *i.e.*, of the region in which the major portion of velocity deformation takes place. The thickness is measured normal to the plate thickness. The boundary layer is normally very thin in comparison with the dimensions of the body immersed in the flow.

(*iv*) The thickness of the boundary layer is variable along the flow direction ; it is zero at leading edge of the plate and increases as the distance x from the leading edge is increased. This aspect may be attributed to the viscous forces which dissipate more and more energy of the fluid stream as the flow proceeds. Consequently, a large group of the fluid particles is slowed down.

The boundary layer growth is also governed by other parameters such as the magnitude of the incoming velocity and the kinematic viscosity of the flowing fluid. For higher incoming velocities, there would be less time for viscous forces to act and accordingly there would be less quantum of boundary layer thickness at a particular distance from the leading edge. Further, the boundary layer thickness is greater for the fluids with greater kinematic viscosity.

(*v*) For some distance from the leading edge, the boundary layer is laminar and the velocity profile is parabolic in character. Flow within the laminar boundary layer is smooth and the streamlines are essentially parallel to the plate. Subsequently the laminar boundary layer becomes unstable and the laminar flow undergoes a change in its flow structure at a certain point, called *transition point*, in the flow field. Within a transititon zone, the flow is unstable and is referrred to as transition flow. After going through a transititon zone of finite length, the boundary layer entirely changes to turbulent boundary layer.

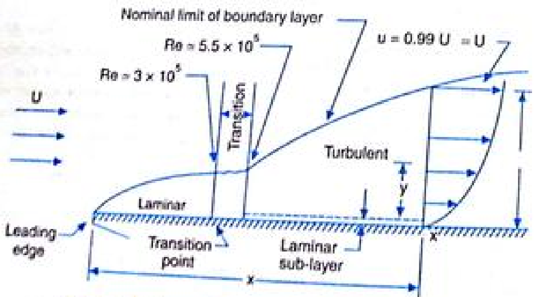

Fig. 12.1. Development of boundary layer on a flat plate

(*vi*) The turbulent boundary layer does not extend to the solid surface. Underlying it, an extremely thin layer, called *laminar sublayer*, is formed wherein the flow is essentially of laminar character. Outside the boundary layer, the main fluid may be either laminar or turbulent.

(*vii*) The pattern of flow in the boundary layer is judged by the Reynolds number $Re = U_\infty x/\nu$ where x is distance along the plate and measured from its leading edge. The transition from laminar to turbulent pattern of flow occurs at values of Reynolds number between 3×10^5 to 5×10^5. Besides this critical Reynolds number, the co-ordinate points at which deterioration of the laminar layer begins and stabilized turbulent flow sets in is dependent on the surface roughness, plate curvature and the pressure gradient, and the intensity of turbulence of the free stream flow.

(*viii*) In a laminar boundary layer, the velocity gradient becomes less steep as one proceeds along the flow. It is because now the change in velocity from no slip at the plate surface to free stream value in the potential core occurs over a greater transverse distance. Nevertheless in a turbulent boundary layer, there occurs an interchange of momentum and energy amongst the individual layers comprising the boundary layer. Consequently, a turbulent boundary layer has a fuller velocity profile and a much steeper velocity gradient at the plate surface when compared to those for a laminar boundary layer (Fig. 12.2).

(*ix*) Velocity gradient and hence the shear stress has a higher value at the plate surface. For a laminar boundary layer the velocity gradient becomes smaller along the flow direction and so does the shear stress. However for a turbulent boundary layer the shear stress at the plate surface again takes up a high value consistent with the steeper velocity gradient.

(*x*) Development of boundary layer for pipe flow proceeds in a fashion similar to that for flow along a flat plate. However, thickness of the boundary layer is limited to the pipe radius because of the flow being within a confined passage. Boundary layers from the pipe walls meet at the center of the pipe and the entire flow acquires the characteristics of a boundary layer. Beyond this point, the velocity profile does not change and it is said to constitute a *fully-developed flow*. Further, the velocity gradient and the wall shear stresses are greatest at the pipe entrance and drop to a steady value at and beyond the region of fully-developed flow. The characteristic velocity distribution of fully developed laminar and turbulent boundary layer flow inside a pipe are depicted in Fig. 12.3.

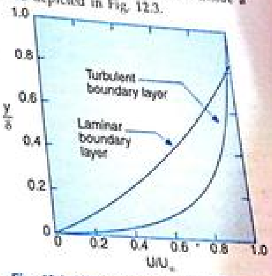

Fig. 12.2. Velocity distribution in laminar and turbulent boundary layer on a flat plate

The entrance length required for the flow to become fully-developed turbulent flow is dependent on the surface finish, initial level of turbulence, downstream conditions, fluid properties and is generally estimated to be 50-80 times the pipe diameter.

12.2. DIFFERENTIAL EQUATIONS FOR THE HYDRODYNAMIC BOUNDARY LAYER

Consider an infinitesimal, two-dimensional control volume ($dx \times dy \times$ unit depth) within the boundary layer region and assume that

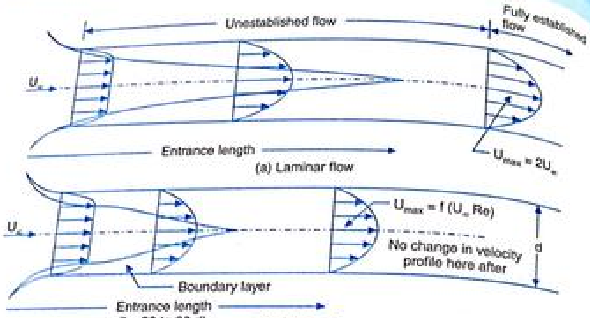

Fig. 12.3. Boundary layer growth in a pipe

(*i*) Flow is steady and the fluid is incompressible

(*ii*) Fluid viscosity is constant

(*iii*) Shear in y-direction is negligible

(*iv*) No pressure variations in the flow field, *i.e.*, stream wise and vertical pressure gradients are negligible

(*v*) Fluid is continuous both in space (*i.e.* no voids occur in the fluid) and time (*i.e.*, the mass is neither created nor destroyed)

The differential equations of motion for the boundary layer are derived by making mass, force and momentum balance on the elemental control volume.

12.2.1. Conservation of Mass: The Continuity Equation

Let u represent the velocity of fluid flow at the left hand face AD. Since the area of this face can be made arbitrarily small, the velocity u may be assumed uniform over the entire surface. The flow velocity changes in the direction of x-axis and the rate of change is given by $\partial u/\partial x$. The change in velocity during distance dx will be $(\partial u/\partial x)dx$. Thus the velocity of fluid motion at surface BC will be $[u + (\partial u/\partial x)dx]$. Likewise the fluid velocity at the bottom face AB and at the top face CD are v and $[v + (\partial v/\partial y)dy]$ respectively.

The mass flow entering the left face of the control volume during time interval $d\tau$ is given by :

$$\text{fluid influx} = \text{density} \times (\text{velocity} \times \text{area}) \times \text{time}$$

$$= \rho\, u dy\, d\tau \qquad ...(12.1)$$

During the same time interval, mass of fluid flowing out from the right face is :

$$\text{fluid efflux} = \rho\left(u + \frac{\partial u}{\partial x}dx\right)dy\, d\tau \qquad ...(12.2)$$

Likewise the mass flow entering the bottom face is $\rho\, v dx\, d\tau$ and the mass leaving the top face is $\rho\, [v + (\partial v/\partial y)\, dy]\, dx\, d\tau$

A mass balance on the element yields :

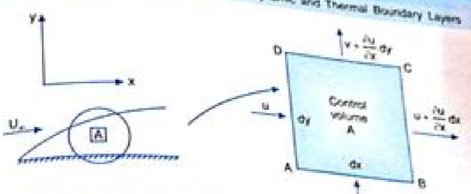

Fig. 12.4. Elemental control volume for mass balance-continuity equation

$$\rho\, u\, dy\, d\tau + \rho\, v\, dx\, d\tau$$

$$= \rho\left(u + \frac{\partial u}{\partial x}dx\right)dy\, d\tau + \rho\left(v + \frac{\partial v}{\partial y}dy\right)dx\, d\tau$$

Simplification gives :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad ...(12.3)$$

This is the mass continuity equation for two-dimensional steady flow of an incompressible fluid. The continuity equation is a mathematical expression of the fact that flow is continuous; it has no breaks in it.

12.2.2. Force or Momentum Equation

For a two-dimensional infinitesimal control volume ($dx \times dy \times$ unit depth) within the boundary layer region, the viscous forces acting along with the momentum of fluid entering and leaving the elementary volume have been indicated in Fig. 12.5.

Momentum change : The momentum flux in x-direction is the product of mass flowing in x-direction and the x-component u of velocity. A fluid mass enters the left face at the rate $\rho\, u\, dy$ producing an x-momentum influx

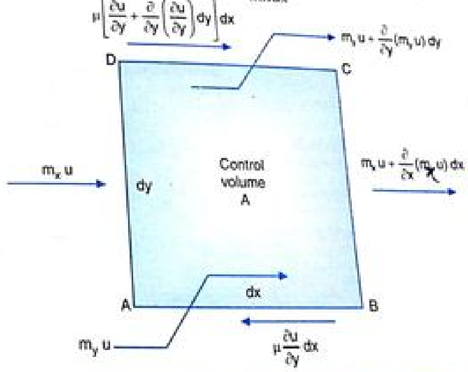

Fig. 12.5. Elemental control volume and force balance

$$m_x u = (\rho\, u\, dy)\, u = \rho\, u^2\, dy \qquad ...(12.4)$$

The momentum efflux through the right face is

$$= m_x u + \frac{\partial}{\partial x}(m_x u)\, dx$$

$$= (\rho\, u\, dy)\, u + \frac{\partial}{\partial x}(\rho\, u\, dy\, u)\, dx$$

$$= \rho\, u^2\, dy + 2\rho u\, \frac{\partial u}{\partial x}\, dx\, dy \qquad ...(12.5)$$

Since we are concerned only with momentum in x-direction, the momentum of the fluid moving in y-direction is obtained by multiplying the mass moving in y-direction also with the x-component u of the velocity. Therefore the momentum influx from the bottom face is

$$m_y u = (\rho\, v\, dx)\, u = \rho\, u\, v\, dx \qquad ...(12.6)$$

and the momentum efflux from the top face is

$$= m_y u + \frac{\partial}{\partial y}\left(m_y u\right) dy$$

$$= (\rho\, v\, dx) u + \frac{\partial}{\partial y}(\rho\, v\, dx\, u)\, dy$$

$$= \rho\, u\, v\, dx + \rho\, u\, \frac{\partial v}{\partial y}\, dx\, dy + \rho\, v\, \frac{\partial u}{\partial y}\, dx\, dy \qquad ...(12.7)$$

The resultant momentum change in x-direction is,

= momentum efflux from the right and top faces – momentum influx from the left and bottom faces

$$= \left(\rho\, u^2\, dy + 2\rho u\, \frac{\partial u}{\partial x}\, dx\, dy\right) + \left(\rho\, u\, v\, dx + \rho u\, \frac{\partial v}{\partial y}\, dx\, dy + \rho v\, \frac{\partial u}{\partial y}\, dx\, dy\right) - \rho u^2\, dy - \rho\, u\, v\, dx$$

$$= 2\rho\, u\, \frac{\partial u}{\partial x}\, dx\, dy + \rho\, u\, \frac{\partial v}{\partial y}\, dx\, dy + \rho\, v\, \frac{\partial u}{\partial y}\, dx\, dy$$

$$= \rho\left(u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y}\right) dx\, dy + \rho\, u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy$$

From the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and therefore the net momentum transfer in x-direction becomes :

$$= \rho\left(u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y}\right) dx\, dy \qquad ...(12.8)$$

Viscous forces : The shearing stress due to fluid viscosity is proportional to the velocity gradient and is given by Newtons' law of viscosity.

$$\text{Shear stress, } \tau = \mu\, \frac{\partial u}{\partial y}$$

where μ is the dynamic viscosity of the fluid

The shearing stress at the lower face of the control volume is $\tau = \mu\, (\partial u/\partial y)$ and the coresponding shearing force for the area $(dx \times 1)$ is $\mu\, (\partial u/\partial y)\, dx$. The shearing stress due to viscosity at the upper face of the control volume is $[\tau + (\partial \tau/\partial y)\, dy]$ and the corresponding shearing force for the area $(dx \times 1)$ is

$$\left(\tau + \frac{\partial \tau}{\partial y}\, dy\right) dx = \left[\mu\, \frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\mu\, \frac{\partial u}{\partial y}\right) dy\right] dx = \left(\mu\, \frac{\partial u}{\partial y} + \mu\, \frac{\partial^2 u}{\partial y^2}\, dy\right) dx$$

Since the main stream flows in x-direction, the shearing force in y-direction can be

neglected. Therefore the net viscous force in the direction of motion is :

$$\left(\mu\, \frac{\partial u}{\partial y} + \mu\, \frac{\partial^2 u}{\partial y^2}\, dy\right) dx - \mu\, \frac{\partial u}{\partial y}\, dx = \mu\, \frac{\partial^2 u}{\partial y^2}\, dx\, dy \qquad ...(12.9)$$

For a fixed control volume and steady flow, the Newton's second law of motion stipulates that the resultant applied x-force equals the net rate of x-momentum transfer out of the volume, *i.e.*,

$$\Sigma F_x = (x\text{-momentum efflux}) - (x\text{-momentum influx})$$

Therefore, in the absence of any pressure and gravitational forces

$$\mu\, \frac{\partial^2 u}{\partial y^2}\, dx\, dy = \rho\left(u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y}\right) dx\, dy$$

or

$$\mu\, \frac{\partial^2 u}{\partial y^2} = \rho\left(u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y}\right)$$

By substituting $\mu/\rho = \nu$, (the kinematic viscosity), we obtain

$$u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y} = \nu\, \frac{\partial^2 u}{\partial y^2} \qquad ...(12.10)$$

This is the force or momentum equation of the boundary layer with constant properties.

12.2.3. Functional Relationship for Boundary Layer Thickness

With stipulations of constant fluid properties and zero pressure gradient, the governing equations of motion for the boundary layer region are :

$$\text{continutity : } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$x\text{-momentum : } u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y} = \nu\, \frac{\partial^2 u}{\partial y^2}$$

Except very close to the surface of the plate, the velocity u within the boundary layer is of the order of undisurbed free stream velocity U_∞, *i.e.*, $u \sim U_\infty$. Likewise, the y-dimension (transverse distance from the plate) is of the order of boundary layer thickness, $y \sim \delta$. The continuity equation may then be approximated as :

$$\frac{U_\infty}{x} + \frac{v}{\delta} = 0$$

This gives : $v \sim \dfrac{U_\infty \delta}{x}$

Inserting the estimates of u, v and y in the x-momentum equation, we obtain :

$$U_\infty \frac{U_\infty}{x} + \frac{U_\infty \delta}{x} \frac{U_\infty}{\delta} = \nu \frac{U_\infty}{\delta^2}$$

or

$$\frac{U^2_\infty}{x} = \nu \frac{U_\infty}{\delta^2}$$

or

$$\delta^2 = \frac{\nu x}{U_\infty}$$

This equation can be rendered dimensionless by dividing it by x^2.

$$\frac{\delta}{x} = \sqrt{\frac{\nu}{U_\infty x}} = \frac{1}{\sqrt{Re_x}} \qquad ...(12.11)$$

where $Re_x = U_\infty x/\nu$ is the Reynolds number based on distance x from the leading edge of the plate.

12.3. BLASIUS SOLUTION FOR LAMINAR BOUNDARY LAYER FLOWS

The Blasius technique for an exact solution of the hydrodynamic boundary layer lies in the conversion of the following partial differential equations :

$$\text{continutity : } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$x\text{-momentum : } u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y} = \nu\, \frac{\partial^2 u}{\partial y^2}$$

into a single full differential equation. Towards that end, the following two parameters are introduced :

Stretching factor : Experimental evidence indicates that velocity profiles at different locations along the plate are geometrically similar, *i.e.*, they differ only by a stretching factor in the y-direction. This implies that the

dimensionless velocity u/U_∞ can be expressed at any location x as a function of the dimensionless distance from the wall, y/δ.

$$\frac{u}{U_\infty} = \phi\left(\frac{y}{\delta}\right) \quad ...(12.12)$$

Invoking the order of magnitude relationship,

$$\frac{\delta}{x} = \sqrt{\frac{\nu}{U_\infty x}}$$

we get :

$$\frac{u}{U_\infty} = g(\eta) \quad ...(12.13)$$

where $\eta = y\sqrt{U_\infty/\nu x}$ denotes the stretching factor.

Stream function : The continuous stream function ψ is the mathematical postulation such that its partial differential with respect to x gives the velocity in the y-direction (generally taken as negative) and its partial differential with respect to y gives the velocity in the x-derection:

$$u = \frac{\partial \psi}{\partial y}\ ;\ v = -\frac{\partial \psi}{\partial x} \quad ...(12.14)$$

If x is kept constant, the integration yields

$$\psi = \int u\,dy + C(x) \quad ...(12.15)$$

The following transformations result from equation 12.13

$$u\,dy = U_\infty\, g(\eta)\, dy$$

$$= U_\infty g(\eta) \frac{dy}{d\eta} d\eta$$

and $\eta = y\sqrt{\dfrac{U_\infty}{\nu x}}$ (Stretching factor)

$$\frac{dy}{d\eta} = \sqrt{\frac{\nu x}{U_\infty}}$$

$$\therefore \quad \psi = U_\infty \int g(\eta)\frac{dy}{d\eta} d\eta + C(x)$$

$$= U_\infty \int g(\eta)\,d\eta \sqrt{\frac{\nu x}{U_\infty}} + C(x)$$

$$= \sqrt{U_\infty \nu x}\int g(\eta)\,d\eta + C(x)$$

$$= \sqrt{U_\infty \nu x}\, f(\eta) \quad ...(12.16)$$

The constant of integration has been dropped on dimensional grounds; ψ and $\sqrt{U_\infty \nu x}$ have the same units, e.g. m²/s.

The velocity components and their derivatives can now be determined in terms of η and $f(\eta)$.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \eta}[\sqrt{U_\infty \nu x}\, f(\eta)] \times \frac{\partial}{\partial y}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right)$$

$$= \sqrt{U_\infty \nu x}\,\frac{df(\eta)}{d\eta}\sqrt{\frac{U_\infty}{\nu x}}$$

$$= U_\infty \frac{df(\eta)}{d\eta} = U_\infty \frac{df}{d\eta} \quad ...(12.17)$$

where f is only a function on η.

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{U_\infty \nu x}\, f(\eta)\right]$$

$$= -\sqrt{\nu U_\infty}\,\frac{\partial}{\partial x}\left[\sqrt{x}\, f(\eta)\right]$$

$$= -\sqrt{\nu U_\infty}\left[\sqrt{x}\,\frac{\partial (f)}{\partial x} + f\frac{\partial}{\partial x}(\sqrt{x})\right]$$

$$= -\sqrt{\nu U_\infty}\left[\sqrt{x}\,\frac{\partial f}{\partial x}\frac{\partial \eta}{\partial x} + \frac{f}{2\sqrt{x}}\right]$$

$$= -\sqrt{\nu U_\infty}\left[\sqrt{x}\,\frac{df}{d\eta}\frac{\partial}{\partial x}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) + \frac{f}{2\sqrt{x}}\right]$$

$$= -\sqrt{\nu U_\infty}\left[\sqrt{x}\,\frac{df}{d\eta}\times y\sqrt{\frac{U_\infty}{\nu x}}\left(-\frac{1}{2}\right)(x)^{-3/2} + \frac{f}{2\sqrt{x}}\right]$$

$$= -\sqrt{\nu U_\infty} \times \frac{1}{2\sqrt{x}}\left[-\frac{df}{d\eta}\, y\sqrt{\frac{U_\infty}{\nu x}} + f\right]$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta\frac{df}{d\eta} - f\right] \quad ...(12.18)$$

and
$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[U_\infty \frac{df}{d\eta}\right]$$

$$= U_\infty \frac{\partial}{\partial \eta}\left(\frac{df}{d\eta}\right)\frac{\partial \eta}{\partial x}$$

$$= U_\infty \frac{d^2 f}{d\eta^2}\frac{\partial}{\partial x}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right)$$

$$= U_\infty \frac{d^2 f}{d\eta^2}\, y\sqrt{\frac{U_\infty}{\nu x}}\left(-\frac{1}{2}\right)(x)^{-3/2}$$

$$= -\frac{U_\infty}{2x}\, y\sqrt{\frac{U_\infty}{\nu x}}\,\frac{d^2 f}{d\eta^2} \quad ...(12.19)$$

$$\therefore \quad u\frac{\partial u}{\partial x} = U_\infty \frac{df}{d\eta}\left[-\frac{U_\infty}{2x}\, y\sqrt{\frac{U_\infty}{\nu x}}\,\frac{d^2 f}{d\eta^2}\right]$$

$$= -\frac{U_\infty^2}{2x}\, y\sqrt{\frac{U_\infty}{\nu x}}\,\frac{d^3 f}{d\eta^3}$$

$$= -\frac{U_\infty^2}{2x}\,\eta\,\frac{d^3 f}{d\eta^3} \quad ...(12.20)$$

Further,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[U_\infty \frac{df}{d\eta}\right]$$

$$= U_\infty \frac{\partial}{\partial \eta}\left(\frac{df}{d\eta}\right)\frac{\partial \eta}{\partial y}$$

$$= U_\infty \frac{d^2 f}{d\eta^2}\frac{\partial}{\partial y}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right)$$

$$= U_\infty \frac{d^2 f}{d\eta^2}\sqrt{\frac{U_\infty}{\nu x}} \quad ...(12.21)$$

$$\therefore \quad v\frac{\partial u}{\partial y} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta\frac{df}{d\eta} - f\right] \times U_\infty \frac{d^2 f}{d\eta^2}\sqrt{\frac{U_\infty}{\nu x}}$$

$$= \frac{1}{2x}\, U_\infty^2 \frac{d^2 f}{d\eta^2}\left[\eta - \frac{df}{d\eta} - 1\right] \quad ...(12.22)$$

Further,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[U_\infty \frac{d^2 f}{d\eta^2}\sqrt{\frac{U_\infty}{\nu x}}\right]$$

$$= U_\infty \sqrt{\frac{U_\infty}{\nu x}}\,\frac{\partial}{\partial \eta}\left(\frac{d^2 f}{d\eta^2}\right)\frac{\partial \eta}{\partial y}$$

$$= U_\infty \sqrt{\frac{U_\infty}{\nu x}}\,\frac{d^3 f}{d\eta^3}\frac{\partial}{\partial \eta}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right)$$

$$= \frac{U_\infty^2}{\nu x}\frac{d^3 f}{d\eta^3} \quad ...(12.23)$$

Substituting the values of relevant parameters into the x-momentum equation, we obtain :

$$-\frac{U_\infty^2}{2x}\,\eta\,\frac{d^3 f}{d\eta^3} + \frac{1}{2x}U_\infty^2 \frac{d^2 f}{d\eta^2}\left(\eta\frac{df}{d\eta} - f\right)$$

$$= \frac{U_\infty^2}{x}\frac{d^3 f}{d\eta^3}$$

$$\text{or } 2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

$$\text{or } 2f''' + ff'' = 0 \quad ...(12.24)$$

which is an ordinary (but non linear) differential equation for f. The number of primes on f denotes the number of successive derivatives of $f(\eta)$ with respect to y. The relevant boundary conditions are :

Physical system	Transformed system
(i) $u = 0$ at $y = 0$	$\frac{df}{d\eta} = 0$ at $\eta = 0$
(ii) $v = 0$ at $y = 0$	$f = 0$ at $\eta = 0$
(iii) $\frac{\partial u}{\partial y} \to 0$ as $y \to \infty$	$\frac{df}{d\eta} \to 1.0$ as $\eta \to \infty$

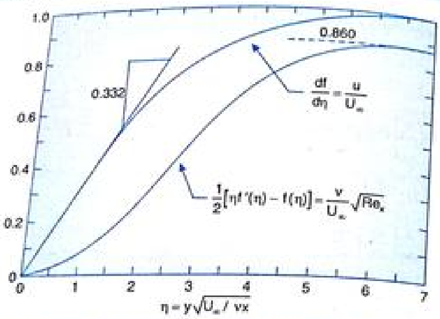

Fig. 12.6. Dimensionless velocity components in a laminar boundary layer

Table 12.1. Laminar boundary layer solution for a flat plate

η	f	$f' = \frac{u}{U_\infty}$	f''	$\frac{1}{2}(\eta f' - f)$
0	0	0	0.3321	0
0.2	0.0664	0.0664	0.3320	0.0033
0.4	0.0266	0.1328	0.3315	0.0133
0.8	0.1061	0.2647	0.3274	0.0528
1.2	0.2379	0.3938	0.3166	0.1173
1.6	0.4203	0.5169	0.2967	0.2032
2.0	0.6500	0.6298	0.2667	0.3047
2.4	0.9223	0.7290	0.2281	0.4136
2.8	1.2310	0.8115	0.1840	0.5206
3.2	1.5691	0.8761	0.1391	0.6172
3.6	1.9294	0.9233	0.0981	0.6972
4.0	2.3058	0.9555	0.0642	0.7582
4.4	2.6924	0.9759	0.0390	0.8007
4.8	3.0853	0.9878	0.0219	0.8280
5.0	3.2833	0.9915	0.0159	0.8372
5.2	3.4820	0.9942	0.0113	0.8441
5.6	3.8803	0.9975	0.0054	0.8528
6.0	4.2796	0.9990	0.0024	0.8571
6.4	4.6794	0.9996	0.0010	0.8591
6.8	5.0793	0.9999	0.0003	0.8599
7.2	5.4792	1.0000	0.0001	0.8602
7.6	5.8792	1.0000	0.0000	0.8603

The Blasius' numerical solution of equation 12.24 with the corresponding values of u and v are plotted in Fig. 12.6; and the results have been given in Table 12.1.

The following results are of particular interest :

(*i*) The boundary layer thickness δ is taken to be the distance from the plate surface to a point at which the velocity is within 1% of the asymptotic limit, *i.e.*, $u = 0.99\ U_\infty$. An accepted value of η at this condition is 5.0 (Fig. 12.6.)

$$\delta = y|_{\eta=5.0} = 5.0\sqrt{\frac{vx}{U_\infty}}$$

or

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{(xU_\infty)/v}} = \frac{5.0}{\sqrt{Re_x}} \quad ...(12.25)$$

where $Re_x = (x\ U_\infty)/v$ is the local Reynolds number based on distance x from the leading edge of the plate. This factor varies from zero at the leading edge to plate Reynolds number $[Re = (lU_\infty)/v]$ at the trailing edge.

(*ii*) The local skin friction coefficient C_{fx} refers to the ratio of the local wall shear stress τ_w to the dynamic pressure of the uniform flow stream.

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}(\rho U_\infty^2)} = \frac{\mu\,(\partial u/\partial y)_{y=0}}{\frac{1}{2}(\rho U_\infty^2)} \quad ...(12.26)$$

The velocity gradient at the surface, $(\partial u/\partial y)_{y=0}$ can be determined from the Blasius solution as follows :

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\partial}{\partial y}\left(U_\infty \frac{df}{d\eta}\right)_{y=0}$$

$$= U_\infty\sqrt{\frac{U_\infty}{vx}}\left(\frac{d^2 f}{d\eta^2}\right)_{\eta=0}$$

$$= U_\infty\sqrt{\frac{U_\infty}{vx}}\, f''(0)$$

The quantity $f''(0)$ represents the solpe of f' curve at $\eta = 0$, and by measurements from Fig. 12.6, $f''(0) = 0.332$. Therefore

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.332\, U_\infty\sqrt{\frac{U_\infty}{vx}}$$

Making the appropriate sustitution into equation 12.26 ;

$$C_{fx} = \mu \times \frac{0.332 U_\infty \sqrt{U_\infty / vx}}{\frac{1}{2}(\rho U_\infty^2)}$$

$$= 0.664\sqrt{\frac{v}{xU_\infty}}$$

$$= \frac{0.664}{\sqrt{Re_x}} \quad ...(12.27)$$

An estimate of the average value of skin friction coefficient $\overline{C}_f$ can be made by integrating the local skin friction coefficient C_f from $x = 0$ to $x = l$ (where l is the length of plate) and then dividing the integrated result by the plate length,

$$\overline{C}_f = \frac{1}{l}\int_0^l C_{fx}\, dx$$

$$= \frac{1}{l}\int_0^l \frac{0.664}{\sqrt{(xU_\infty)/v}}\, dx$$

$$= \frac{0.664}{l\sqrt{U_\infty/v}}\int_0^l \frac{1}{\sqrt{x}}\, dx = \frac{0.664}{l\sqrt{U_\infty/v}}\left|2\sqrt{x}\right|_0^l$$

$$= \frac{1.328\sqrt{l}}{l\sqrt{U_\infty/v}}$$

$$= \frac{1.328}{\sqrt{(lU_\infty)/v}} = \frac{1.328}{\sqrt{Re_l}} \quad ...(12.28)$$

where Re_l is the Reynolds number based upon total length l of the plate.

12.4. vON-KARMAN INTEGRAL MOMENTUM EQUATION

The Blasius differential approach for the solution of boundary layer equations, though exact, is quite difficult and cumbersome to

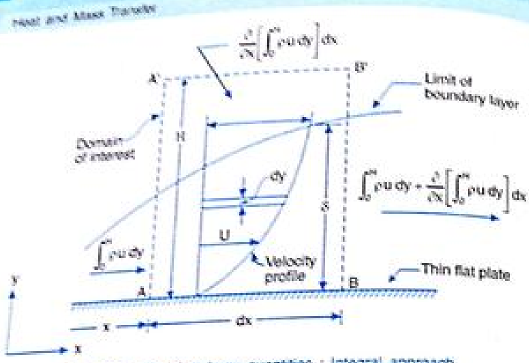

Fig. 12.7. Boundary layer quantities : Integral approach

obtain. Further, there are many practical situations with flow or geometry of a more complicated nature where the similarity parameter (stretching factor) η is either unknown or does not exist. Moreover, we are not really interested in the details of the velocity profile in the boundary layer, beyond knowing its slope at the wall. The slope helps to determine the shear stress at the wall, $\tau_w = \mu(\partial u/\partial y)y = 0$. For such cases, we employ the von-Karman integral approach which involves the application of Newton's second law to a finite control volume as opposed to infinitesimal element of Blasius. The boundary layer equations are integrated from the wall ($y = 0$) to boundary layer thickness ($y = \delta$). The resulting ordinary differential equation gives fairly accurate explicit equation for the wall shear stress.

Select a thin flat plate whose surface is parallel to the direction of free stream velocity; and consider a differential element (domain of interest) of boundary layer located at a distance x and $x + dx$ from the leading edge. The control valume is infinitesimal in the x-direction but finite in the y-direction; the control volume is sufficiently high and it encloses the edge of boundary layer, *i.e.*, $H > \delta$.

For unit width of plate, the flow rate entering through face AA' is

$$m = \int_0^H \rho\, u\, dy$$

and that leaving through face BB' is

$$= m + \frac{\partial m}{\partial x} dx$$

$$= \int_0^H \rho\, u\, dy + \frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx$$

No mass can enter the control volume through its solid wall AB. The continuity requirement then stipulates that the mass increment

$$\frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx$$

must represent the mass flow that enters the control volume through plane A'B' with free stream velocity U_∞. The corresponding x-momentum fluxes are :

influx through left face AA'

$$= \int_0^H \rho\, u^2\, dy$$

efflux through right face BB'

$$= \int_0^H \rho\, u^2\, dy + \frac{\partial}{\partial x}\left[\int_0^H \rho\, u^2\, dy\right] dx$$

influx through upper face

$$= U_\infty \frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx$$

In the absence of any pressure and gravity forces, the drag or shear force ($\tau_w\, dx$) at the plate surface must be balanced by the net momentum change for the control volume (There is no shear force at the upper face which is outside the boundary layer).

$$\therefore\ \tau_w\, dx = U_\infty \frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx - \frac{\partial}{\partial x}\left[\int_0^H \rho\, u^2\, dy\right] dx$$

$$= U_\infty \frac{\partial}{\partial x}\left[\int_0^\delta \rho\, u\, dy\right] dx - \frac{\partial}{\partial x}\left[\int_0^\delta \rho\, u^2\, dy\right] dx$$

The upper limit of itegration has been replaced by δ because the integrand is zero for $y > \delta$ ($u = U_\infty$ = constant for $y > \delta$, $U_\infty - u$ is zero outside the boundary layer).

Upon simplification :

$$\tau_w = \frac{\partial}{\partial x}\int \rho\,(U_\infty - u)\, u\, dy$$

$$= \rho U_\infty^2 \frac{\partial}{\partial x}\left[\int_0^\delta \left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty}\, dy\right] \quad ...(12.29)$$

which is the von-Karman mometum itegral equation for the hydrodynamic boundary layer. Evidetly, this integral equation expresses the wall shear stress τ_w as a function of the non-dimensional velocity distriution u/U_∞; u is the point velocity in the boundary layer and U_∞ is the velocity at the outer edge of boundary layer.

Experiments have shown that for the laminar boundary layer, the velocity distribution is parabolic and the velocity profiles at different locations along the plate are geometrically similar. This means that the dimensionless velocity u/U_∞ can be expressed at any location x as a function of the dimensionless distance from the wall, y/δ :

$$\frac{u}{U_\infty} = \phi\left(\frac{y}{\delta}\right)$$

$$= a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3 \quad ...(12.30)$$

The constants can be evaluated subject to the following compatibility (boundary) conditions:

At the wall surface ($y = 0$)

$$u = 0 \text{ and } \frac{d^2u}{dy^2} = 0$$

At the outer edge of boundary layer ($y = \delta$)

$$u = U_\infty \text{ and } \frac{du}{dy} = 0$$

The application of these boundary conditions gives :

$$0 = a$$

which eliminates a immediately

$$0 = 2c \text{ which eliminates } c \text{ as well}$$

$$1 = a + b + c + d$$

$$0 = b + 2c + 3d$$

The last two equalities are solved for b and d, and we obtain

$$b = \frac{3}{2} \text{ and } d = -\frac{1}{2}$$

That gives the velocity profile as :

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad ...(12.31)$$

We will now use this cubic velocity profile in the von-Karman integral equation to determine the boundary layer thickness and the average skin-friction coefficient for laminar flow over a flat plate.

Inserting velocity function as expressed by equation 12.31 into momentum integral equation 12.29 :

$$\tau_w = \rho U_\infty^2 \frac{\partial}{\partial x}\left[\int_0^\delta \left\{1-\frac{3}{2}\left(\frac{y}{\delta}\right)+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\} \times \left\{\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\} dy\right]$$

$$= \rho U_\infty^2 \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{9}{4}\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\left(\frac{y}{\delta}\right)^3+\frac{3}{2}\left(\frac{y}{\delta}\right)^4-\frac{1}{4}\left(\frac{y}{\delta}\right)^6 dy\right\}\right]$$

$$= \frac{39}{280}\rho U_\infty^2 \frac{\partial \delta}{\partial x} \qquad \text{...(12.32)}$$

At the solid surface, Newton's law of viscosity gives :

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \mu\left[\frac{\partial}{\partial y}\left\{\frac{3}{2}U_\infty\left(\frac{y}{\delta}\right)-\frac{1}{2}U_\infty\left(\frac{y}{\delta}\right)^3\right\}\right]_{y=0}$$

$$= \frac{3}{2}\frac{\mu U_\infty}{\delta} \qquad \text{...(12.33)}$$

Equating the two expressions for wall shear stress,

$$\frac{39}{280}\rho U_\infty^2 \frac{\partial \delta}{\partial x} = \frac{3}{2}\frac{\mu U_\infty}{\delta}$$

and this can be written in the differential form as :

$$\delta\, \partial\delta = \frac{140}{13}\frac{\mu}{\rho U_\infty}\partial x$$

Since δ is a function of x only, integration yields :

$$\frac{\delta^2}{2} = \frac{140}{13}\frac{\mu x}{\rho U_\infty} + C$$

The integration constant C is obtained from the boundary condition $\delta = 0$ at $x = 0$, and that gives $C = 0$. Therefore

$$\frac{\delta^2}{2} = \frac{140}{13}\frac{\mu x}{\rho U_\infty}$$

$$\delta^2 = \frac{140 \times 2}{13}\frac{\mu x}{\rho U_\infty}$$

This can be expressed in the non-dimensional form as

$$\frac{\delta}{x} = \sqrt{\frac{140\times 2}{13}}\sqrt{\frac{\mu}{x\rho U_\infty}} = \frac{4.64}{\sqrt{Re_x}} \qquad \text{...(12.34)}$$

where $Re_x = (x\,\rho\, U_\infty)/\mu$ is the Reynolds number based on distance x from the leading edge of the plate.

An estimate of wall shear can be made by substituting the value of boundary layer thickness (equation 12.34) in the expression for wall shear stress (equation 12.33).

$$\tau_w = \frac{3}{2}\frac{\mu U_\infty}{\delta}$$

$$= \frac{3}{2}\mu U_\infty \div \left(\frac{4.64}{\sqrt{Re_x}}\right)$$

$$= \frac{\rho U_\infty^2}{2} \times \frac{0.646}{\sqrt{Re_x}}$$

From the definition of local skin friction coefficient

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}(\rho U_\infty^2)} = \frac{0.646}{\sqrt{Re_x}} \qquad \text{...(12.35)}$$

Average value of skin friction coefficient

$$\overline{C}_f = \frac{1}{l}\int_0^l C_{fx}\, dx$$

$$= \frac{1}{l}\int_0^l \frac{0.646}{\sqrt{(\rho U_\infty)/\mu}}\frac{dx}{\sqrt{x}}$$

$$= 1.292\sqrt{\frac{\mu}{l\rho U_\infty}} = \frac{1.292}{\sqrt{Re_l}} \qquad \text{...(12.36)}$$

where $Re_l = (l\rho U_\infty)/\mu$ is Reynolds number based on total length l of the plate.

The average skin friction coefficient is quite often referred to as the *drag coefficient*.

The results derived above are not radically altered if some other velocity distribution is assumed for the laminar boundary layer. Fig. 12.8 shows three possible assumptions, ranging

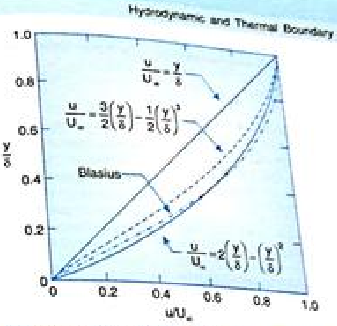

Fig. 12.8. Comparison of simple linear and third-degree polynomial fit with the exact boundary layer velocity profile

from a simple linear to a cubic profile; the Blasius exact profile is shown for reference.

The results for the boundary layer thickness and the average skin-friction coefficient yielded by different velocity profiles have been indicated in Table 12.2.

Table 12.2. Boundary layer parameters for different velocity profiles

Velocity profiles	Boundary conditions		$\frac{\delta}{x}\sqrt{Re_x}$	$C_{fx}\sqrt{Re_x}$
	At $y = 0$	At $y = \delta$		
1. $\frac{u}{U_\infty} = \frac{y}{\delta}$	$u = 0$	$u = U_\infty$	3.46	1.155
2. $\frac{u}{U_\infty} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$	$u = 0$	$u = U_\infty$ $\frac{\partial u}{\partial y} = 0$	5.47	1.462
3. $\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2$	$u = 0$ $\frac{\partial^2 u}{\partial y^2} = 0$	$u = U_\infty$ $\frac{\partial u}{\partial y} = 0$	4.64	1.292
4. $\frac{u}{U_\infty} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$u = 0$	$u = U_\infty$	4.78	1.310
5. Blasius exact solution			5.0	1.328

The above results indicate that :

(i) boundary layer thickness increases as square root of distance x from the leading edge and inversely as square root of free stream velocity

(ii) wall shear stress is inversely proportional to square root of x and directly proportional to 3/2 power of U_∞

(iii) local and average skin friction coefficients vary inversely as square root of both x and U_∞

> Note : With increase in length l of the plate, the laminar flow may change into turbulent flow. Undoubtedly the relations are valid for length of plate for which the boundary layer remains laminar in character.

EXAMPLE 12.1.

Glycerine at 30°C flows past a flat plate at 20 m/s. Workout the velocity components at a point P(x, y) in the flow field where

x = 2 m from the leading edge of the plate

y = 5 cm from the plate surface

For glycerine at 30°C, kinematic viscosity

$$\nu = 2.79 \times 10^{-3} \text{ m}^2/\text{s}$$

Solution : The Reynolds number is

$$Re_x = \frac{xU_\infty}{\nu} = \frac{2 \times 20}{2.79 \times 10^{-3}} = 14337$$

$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5 \times 2}{\sqrt{14337}}$$

$$= 0.0832 \text{ m} = 8.32 \text{ cm} > 5 \text{ cm}$$

Apparently the given point is within the boundary layer.

The velocity components are obtained from the Blasius solution (Table 12.1.)

At
$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} = 0.5\sqrt{\frac{20}{2.79 \times 10^{-3} \times 2}}$$
$$= 2.993$$

we have :

$$\frac{u}{U_\infty} = 0.846 \quad \text{and} \quad \frac{v}{U_\infty}\sqrt{Re_x} = 0.57$$

That gives :

$$u = 0.846 \times 20 = 16.92 \text{ m/s}$$

$$v = \frac{0.57 \times 20}{\sqrt{14337}} = 0.0952 \text{ m/s}$$

EXAMPLE 12.2.

Air at 25°C flows over a flat surface with a sharp leading edge at 1.5 m/s. Find the boundary layer thickness at 0.5 from the leading edge. Check the boundary layer assumption that v << u at the trailing edge. Further, estimate what fraction of the surface can legitimately be analysed using boundary layer theory.

For air at 25°C, kinematic viscosity

$$\nu = 15.53 \times 10^{-6} \text{ m}^2/\text{s}$$

Solution : The Reynolds number is

$$Re_x = \frac{xU_\infty}{\nu} = \frac{0.5 \times 1.5}{15.53 \times 10^{-6}} = 48294$$

The Reynolds number is low enough to permit the use of a laminar flow analysis. The from the Blasius exact solution.

$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5 \times 0.5}{\sqrt{48294}}$$

$$= 0.011376 \text{ m} = 1.1376 \text{ cm}$$

The boundary layer analysis is only valid if $(\delta/x) << 1$. In this case, $(\delta/x) = (1.1376/50) = 0.02275$. Further, at the outer edge of the boundary layer where $u/U_\infty = 1$.

$$\frac{v}{U_\infty}\sqrt{Re_x} = 0.8604 \qquad \text{(Table 12.1)}$$

$$\therefore \quad v = \frac{0.8604 \times 1.5}{\sqrt{48294}}$$

$$\frac{v}{U_\infty} = \frac{5.873 \times 10^{-3}}{\sqrt{1.5}} = 3.915 \times 10^{-3}$$

Thus $v << u$ as long as we are not near the leading edge.

The boundary layer assumptions break down at the leading edge and the analysis gives accurate results only when

$$x > 600\frac{\nu}{U_\infty}$$

$$x > \frac{600 \times 15.53 \times 10^{-6}}{1.5} > 0.006216$$

which is about 1.25% of the intial distance from the leading edge for a plate of 0.5 m length, the later 98.75% length of the surface can be legitimately analysed using the boundary layer theory.

EXAMPLE 12.3.

A plate 0.3 m long is placed at zero angle of incidence in a stream of 15°C water moving at 1 m/s. Find out the streamwise velocity component at the mid-point of the boundary layer, the maximum boundary layer thickness and the maximum value of the normal component of velocity at the trailing edge of the plate.

For water at 15°C :

$$\rho = 998.9 \text{ kg/m}^3$$

and $$\mu = 415.85 \times 10^{-2} \text{ kg/hr-m}$$

Solution : The boundary layer thickness is prescribed by the relation.

$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5x}{\sqrt{\rho x U_\infty/\mu}}$$

$$= 5\sqrt{\frac{x\mu}{\rho U_\infty}} = 5\sqrt{\frac{\nu x}{U_\infty}}$$

$$= \frac{5x}{\sqrt{\rho x U_\infty/\mu}}$$

$$= 5\sqrt{\frac{x\mu}{\rho U_\infty}} = 5\sqrt{\frac{\nu x}{U_\infty}}$$

Apparently the mid-point of the boundary layer $(y = \delta/2)$ occurs at

$$\eta = \left(y\sqrt{\frac{U_\infty}{\nu x}}\right) = 2.5$$

The streamwise velocity component is obtained from the Blasius solution (Table 12.1). At

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} = 2.5$$

we have : $u/U_\infty = 0.736$

That gives :

$$u = 0.736 \times 1 = 0.736 \text{ m/s}$$

(b) Maximum value of δ occurs at the trailing edge (x = 0.3 m)

$$Re_l = \frac{l\rho U_\infty}{\mu}$$

$$= \frac{0.3 \times 998.9 \times (1 \times 3600)}{415.85 \times 10^{-2}}$$

$$= 259423$$

$$\therefore \quad \delta(l) = \frac{5l}{\sqrt{Re_l}} = \frac{5 \times 0.3}{\sqrt{259423}}$$

$$= 2.945 \times 10^{-3} \text{ m} = 2.945 \text{ mm}$$

(c) The normal component of velocity attains its maximum value at the outer edge of the boundary layer where $u = U_\infty$.

At $u/U_\infty = 1$, we have :

$$\frac{v}{U_\infty}\sqrt{Re_l} = 0.860$$

$$\therefore \quad v = \frac{0.860U_\infty}{\sqrt{Re_l}} = \frac{0.86 \times 1}{\sqrt{259423}}$$

$$= 1.6885 \times 10^{-3} \text{ m/s}$$

EXAMPLE 12.4.

Atmospheric air at 25°C flows parallel to a flat plate at a velocity of 3 m/s. Use the exact Blasius solution to estimate the boundary layer thickness and the local skin friction coefficient at x = 1 m from the leading edge of the plate. How these values would compare with the corresponding values obtained from the approximate von-Karman integral technique ? Assume cubic velocity profile.

Solution : For air at 25°C,

$$\nu = 15.33 \times 10^{-6} \text{ m}^2/\text{s}$$

and therefore the Reynolds number is

$$Re_x = \frac{xU_\infty}{\nu} = \frac{1 \times 3}{15.33 \times 10^{-6}} = 193175$$

(i) Exact Blasius solution

$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5 \times 1}{\sqrt{193175}}$$

$$= 0.011376 \text{ m} = 1.1376 \text{ cm}$$

$$C_f = \frac{0.664}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{193175}}$$

$$= 1.511 \times 10^{-3}$$

(ii) Approximate solution with assumption of cubic velocity profile,

$$\delta = \frac{4.64x}{\sqrt{Re_x}} = \frac{4.64 \times 1}{\sqrt{193175}}$$

$$= 0.010557 \text{ m} = 1.0557 \text{ cm}$$

$$C_f = \frac{0.646}{\sqrt{Re_x}} = \frac{0.646}{\sqrt{193175}}$$

$$= 1.469 \times 10^{-3}$$

The approximate solution deviates from the exact solution by

$$\frac{1.1376 - 1.0557}{1.1376} \times 100 = 7.2\%$$

for the boundary layer thickness, and by

$$\frac{1.511 \times 10^{-3} - 1.469 \times 10^{-3}}{1.511 \times 10^{-3}} \times 100 = 2.72\%$$

for the local skin friction coefficient.

EXAMPLE 12.5.

The velocity distribution in the boundary layer of a flat plate is prescribed by the relation

$$\frac{u}{U_\infty} = \sin\left(\frac{\pi y}{2\delta}\right)$$

Use momentum integral equation to develop an expression for the boundary layer thickness.

Solution : Inserting the given velocity function into the momentum integral equation

$$\tau_w = \rho U_\infty^2 \frac{d}{dx}\left[\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy\right]$$

$$= \rho U_\infty^2 \frac{d}{dx}\left[\int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right) \left\{1 - \sin\left(\frac{\pi y}{2\delta}\right)\right\} dy\right]$$

$$= \rho U_\infty^2 \frac{d}{dx}\left[\int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right) dy - \int_0^\delta \sin^2\left(\frac{\pi y}{2\delta}\right) dy\right]$$

Making the substitution,

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

we get :

$$\tau_w = \rho U_\infty^2 \frac{d}{dx}\left[\int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right) dy - \frac{1}{2}\int_0^\delta \left\{1 - \cos\left(\frac{\pi y}{\delta}\right) dy\right\}\right]$$

$$= \rho U_\infty^2 \frac{d}{dx}\left[\left\{-\frac{2\delta}{\pi}\cos\left(\frac{\pi y}{2\delta}\right)\right\}_0^\delta - \frac{1}{2}\left\{y - \frac{\delta}{\pi}\sin\left(\frac{\pi y}{\delta}\right)\right\}_0^\delta\right]$$

$$= \rho U_\infty^2 \frac{d}{dx}\left[\frac{2\delta}{\pi} - \frac{\delta}{2}\right]$$

$$= 0.137\, \rho U_\infty^2 \frac{d\delta}{dx} \qquad \text{...(i)}$$

At the solid surface, Newton's law of viscosity gives :

$$\tau_w = \mu\left(\frac{du}{dy}\right)_{y=0}$$

$$= \mu \frac{d}{dy}\left[U_\infty \sin\left(\frac{\pi y}{2\delta}\right)\right]_{y=0}$$

$$= 1.57\, \frac{\mu U_\infty}{\delta} \qquad \text{...(ii)}$$

Equating the two expressions for wall shear stress,

$$0.137\, \rho U_\infty^2 \frac{d\delta}{dx} = 1.57\, \frac{\mu U_\infty}{\delta}$$

and this can be written in the differential form as

$$\delta\, d\delta = 11.46\, \frac{\mu}{\rho U_\infty} dx$$

Integration yields

$$\frac{\delta^2}{2} = 11.46 \frac{\mu}{\rho U_\infty} x \; ; \; \delta^2 = 22.92\, \frac{\mu x}{\rho U_\infty}$$

This can be expressed in the non-dimensional form as

$$\frac{\delta}{x} = \sqrt{22.92} \sqrt{\frac{\mu}{x \rho U_\infty}} = \frac{4.79}{\sqrt{Re_x}}$$

where $Re_x = (x\rho U_\infty)/\mu$ is the Reynolds number based on distance x from leading edge of the plate.

EXAMPLE 12.6.

A plate 0.5 m × 0.2 m has been placed longitudinally in a stream of crude oil which flows with undisturbed velocity of 6 m/s. Given that oil has a specific gravity 0.9 and kinematic viscosity 1 stoke, calculate the boundary layer thickness and shear stress at the middle of plate. Also calculate friction drag on one side of the plate.

Solution : Kinematic viscosity

$$\nu = 1 \text{ stoke} = 1 \times 10^{-4} \text{ m}^2/\text{s}$$

At the middle of plate,

$$x = 1/2 \times 0.5 = 0.25 \text{ m}$$

$$Re_x = \frac{x U_\infty}{\nu} = \frac{0.25 \times 6}{1 \times 10^{-4}} = 1.5 \times 10^4$$

Since the Reynolds number is less than 5×10^5, the boundary layer is of laminar character and the Blasius solution gives :

$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5 \times 0.25}{\sqrt{1.5 \times 10^4}}$$

$$= 1.02 \times 10^{-2} \text{ m} = 1.02 \text{ cm}$$

$$C_f = \frac{0.664}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{1.5 \times 10^4}}$$

$$= 0.542 \times 10^{-2}$$

By definition,

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

Shear stress τ_w

$$= \frac{1}{2}\rho U_\infty^2 \times C_f$$

$$= \left(\frac{1}{2} \times 0.9 \times 1000 \times 6^2\right) \times 0.542 \times 10^2$$

$$= 87.8 \text{ N/m}^2$$

(b) Reynolds number at the trailing edge of the plate,

$$Re_l = \frac{l U_\infty}{\nu} = \frac{0.5 \times 6}{1 \times 10^{-4}} = 3 \times 10^4$$

Even at the trailing edge, the boundary layer is laminar and therefore the average drag (friction) coefficient is

$$\bar{C}_f = \frac{1.328}{\sqrt{Re_l}} = \frac{1.328}{\sqrt{3 \times 10^4}}$$

$$= 0.766 \times 10^{-2}$$

∴ Friction (drag) force

$$= \bar{C}_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area of plate on one side}$$

$$= 0.766 \times 10^{-2} \times \left(\frac{1}{2} \times 0.9 \times 1000 \times 6^2\right) \times (0.5 \times 0.2)$$

$$= 12.40 \text{ N}$$

EXAMPLE 12.7.

Calculate the average shear stress and the overall drag coefficient for a flat surface past which air at 25°C blows with 1.5 m/s velocity. The flat surface has a sharp leading edge and its total length equals 0.5 m. Compare the average shear stress $\bar{\tau}_w$ with the shear stress τ_w at the trailing edge. At what point on the surface does $\bar{\tau}_w = \tau_w$?

For air at 25°C,

kinematic viscosity $\nu = 15.53 \times 10^{-6}$ m²/s

density $\rho = 1.183$ kg/m³.

Solution : The Reynolds number at the end of plate is

$$Re_l = \frac{l U_\infty}{\nu} = \frac{0.5 \times 1.5}{15.53 \times 10^{-6}} = 48294$$

Since the Reynolds number is less than 3×10^5, the boundary layer over the entire plate is laminar in character. Therefore, average skin friction coefficient (overall drag coefficient)

$$\bar{C}_f = \frac{1.328}{\sqrt{48294}} = 6.043 \times 10^{-3}$$

average shear stress,

$$\bar{\tau}_w = \frac{1}{2}\rho U_\infty^2 \times \bar{C}_f \qquad \text{...(i)}$$

$$= \frac{1}{2} \times 1.183 \times 1.5^2 \times 6.043 \times 10$$

$= 8.042 \times 10^{-3}$ kg/ms^2

$= 8.042 \times 10^{-3}$ N/m^2

(b) Skin friction coefficient at the trailing edge.

$$C_f = \frac{0.664}{\sqrt{48294}} = 3.0215 \times 10^{-3}$$

shear stress at the trailing edge,

$$\tau_w = \frac{1}{2}\rho U_\infty^2 \times C_f \quad ...(b)$$

$$= \frac{1}{2} \times 1.183 \times 1.5^2 \times 3.0215 \times 10^{-3}$$

$$= 4.021 \times 10^{-3} \text{ N/m}^2$$

$$\therefore \quad \frac{\tau_w}{\bar{\tau}_w} = \frac{4.021 \times 10^{-3}}{8.042 \times 10^{-3}} = \frac{1}{2}$$

This could be directly obtained from the relations (a) and (b)

$$\frac{\tau_w}{\bar{\tau}_w} = \frac{C_f}{\bar{C}_f} = \frac{0.664/\sqrt{Re_l}}{1.328/\sqrt{Re_l}} = \frac{1}{2}$$

3. Let $\tau_w = \bar{\tau}_w$ at distance x from the leading edge, then

$$\frac{0.664}{\sqrt{Re_x}} = \frac{1.328}{\sqrt{Re_l}}$$

$$\frac{0.664}{\sqrt{x}} = \frac{1.328}{\sqrt{l}}$$

$$\frac{0.664}{\sqrt{x}} = \frac{1.328}{\sqrt{0.5}} \; ; \quad x = 0.125 \text{ m}$$

The local shear stress equals the average value where $x/l = 1/4$

Comment : Shear stress which is initially infinite drops to its average value 1/4th of the way from leading edge, and further falls to one-half in the remaining 3/4th of the plate length.

EXAMPLE 12.8.

Air at 25°C and 1 bar flows over a flat plate at a speed 1.25 m/s. Calculate the boundary layer thicknesses at distances of 15 cm and 30 cm from the leading edge of the plate. What would be the mass entrainment (mass flow entering the boundary layer) between these two sections ? Assume parabolic velocity distribution

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

The viscosity of air at 25°C is stated to be 6.62×10^{-2} kg/hr m.

Solution : The density of air is calculated from the characteristic gas equation : $p = \rho RT$

Given :

$p = 1$ bar $= 1 \times 10^5$ N/m^2

$T = 25°C = (25 + 273) = 298$ K

$R = 287$ J/kg K

$$\therefore \quad \rho = \frac{p}{RT} = \frac{1 \times 10^5}{287 \times 298} = 1.169 \text{ kg/m}^3$$

The flow Reynolds number is,

$$Re_x = (x \, \rho \, U_\infty)/\mu$$

At $x = 15$ cm

$$Re_x = \frac{0.15 \times 1.169 \times (1.25 \times 3600)}{6.62 \times 10^{-2}} = 11919$$

At $x = 30$ cm ;

$$Re_x = \frac{0.3 \times 1.169 \times (1.25 \times 3600)}{6.62 \times 10^{-2}} = 23838$$

For the given parabolic velocity distribution, the boundary layer thickness is prescribed by the relation,

$$\delta = \frac{4.64x}{\sqrt{Re_x}}$$

$\therefore$ At $x = 15$ cm ;

$$\delta_1 = \frac{4.64 \times 0.15}{\sqrt{11919}} = 6.37 \times 10^{-3} \text{ m}$$

At $x = 30$ cm ;

$$\delta_2 = \frac{4.64 \times 0.30}{\sqrt{23838}} = 8.99 \times 10^{-3} \text{ m}$$

(b) At any position, the mass flow in the boundary layer is given by the integral

$$m_x = \int_0^\delta \rho \, u \, dy$$

where the velocity is given by

$$u = U_\infty \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}$$

Evaluating the integral with this velocity distribution,

$$m_x = \int_0^\delta \rho \left[U_\infty \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\right] dy$$

$$= \rho U_\infty \left[\frac{3}{4}\frac{y^2}{\delta} - \frac{1}{8}\frac{y^4}{\delta^3}\right]_0^\delta = \frac{5}{8}\rho U_\infty \delta$$

Thus the mass entrainment between the two sections is:

$$\delta_m = \frac{5}{8}\rho U_\infty (\delta_2 - \delta_1)$$

$$= \frac{5}{8} 1.168 \times 1.25 \left(8.99 \times 10^{-3} - 6.37 \times 10^{-3}\right)$$

$$= 2.393 \times 10^{-3} \text{ kg/s}$$

$$= 8.61 \text{ kg/hr}$$

12.5. THERMAL BOUNDARY LAYER

When a fluid flows past a heated or cold surface, a temperature field is set up in the fluid next to the surface. If the plate surface is hotter than the fluid, the temperature distribution will be as indicated in Fig. 12.9. Usually the temperature field encompasses a very small region of fluid, *i.e.*, the region of fluid being heated by the plate is confined to a thin layer near the surface. This zone or thin layer wherein the temperature field exists is called the ***thermal boundary layer***. The temperature gradient results due to heat exchange between the plate and the fluid.

The thickness δ_t of thermal boundary layer is arbitrarily defined as the distance y from the plate surface at which

$$\frac{t_s - t}{t_s - t_\infty} = 0.99 \quad ...(12.37)$$

The convection of energy reduces the outward conduction in the fluid and consequently the temperature gradient decreases away from the surface. Further, the temperature gradient is infinite at the leading edge of the plate and approaches zero as the layer develops downstream. Moreover in the turbulent boundary layer, the action of eddies flattens the temperature profile.

If the approaching free stream temperature t_∞ is above the plate surface temperature t_s, the thermal boundary layer will have the shape as depicted in Figure 12.10. The temperature of the fluid changes from a minimum at the plate surface to the temperature of the mainstream at a certain distance from the surface. At point A, the temperature of the fluid is the same as the surface temperature t_s. The fluid temperature increases gradually until it acquires the free stream temperature t_∞. The distance AB, measured perpendicularly to the plate surface, denotes the thickness of thermal boundary at a distance x from the leading edge of the plate.

The concept of thermal boundary layer is analogous to that of hyrodynamic boundary layer; the parameters affecting their growth

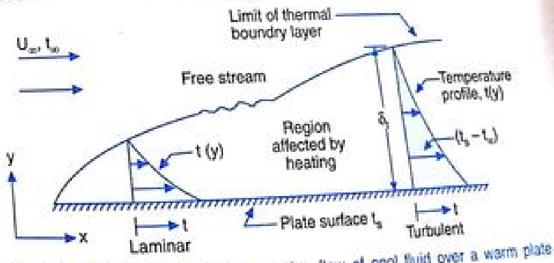

Fig. 12.9. Thermal boundry layer during flow of cool fluid over a warm plate

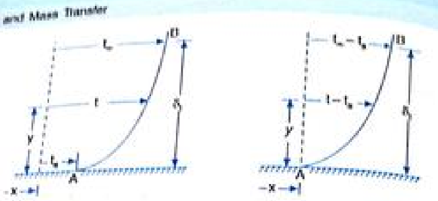

Fig. 12.10. Thermal boundry layer during flow of warm fluid over a cool plate

are, however, different. The velocity profile of the hydrodynamic boundary layer is dependent primarily upon the viscosity of the fluid. In a thermal boundary layer the temperature profile depends upon the flow velocity, specific heat, viscosity and thermal conductivity of the fluid. The thermo-physical properties of the fluid affect the relative magnitude of δ and δ_t, and the non-dimensional Prandtl number ($Pr = \mu\, c_p/k$) constitutes the governing parameter :

(*i*) When $Pr = 1$ $\delta_t = \delta$

(*ii*) When $Pr > 1$ $\delta_t < \delta$

(*iii*) When $Pr < 1$ $\delta_t > \delta$

12.6. ENERGY EQUATION FOR THERMAL BOUNDARY LAYER

For an element of dimensions ($dx \times dy \times$ unit depth) in the boundary layer, the quantities of energy entering and leaving have been indicated in Fig. 12.11. The rate of temperature change in the x-direction is being presumed small and as such conduction is to be considered only in the y-direction. Further,

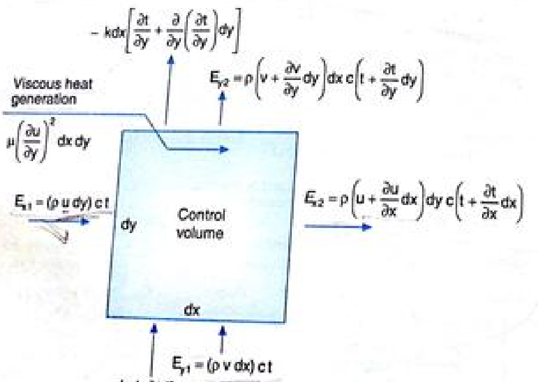

Fig. 12.11. Differential control volume for energy conservation in thermal boundry layer

the convective terms in the x-and y-directions have been written in terms of mass, temperature and specific heat which is assumed constant.

For the x–direction :

Energy influx, E_{x1}

$$= \text{mass} \times \text{specific heat} \times \text{temperature}$$
$$= (\rho\, u\, dy)\, c\, t$$

Energy efflux, E_{x2}

$$= \rho\left(u + \frac{\partial u}{\partial x}dx\right)dy\, c\left(t + \frac{\partial t}{\partial x}\right)dx$$

Neglecting the product of small quantities,

$$E_{x2} = \rho\, c\, dy\left[ut + u\frac{\partial t}{\partial x}dx + t\frac{\partial u}{\partial x}dx\right]$$

Net energy convection $= E_{x_1} - E_{x_2}$

$$= -\rho\, c\left[u\frac{\partial t}{\partial x} + t\frac{\partial u}{\partial x}\right]dx\, dy \qquad ...(12.38)$$

Likewise the net energy convection in the y-direction would be :s

$$= E_{y1} - E_{y2}$$
$$= (\rho\, v\, dx)\, c\, t - \rho\, c\, dx \times \left[vt + v\frac{\partial t}{\partial y}dy + t\frac{\partial v}{\partial y}dy\right]$$
$$= -\rho\, c\left[v\frac{\partial t}{\partial y} + t\frac{\partial v}{\partial y}\right]dx\, dy \qquad ...(12.39)$$

The conductive terms follow from the Fourier law, and the conduction heat rate in the y-direction is :

$$= -k\, dx\frac{\partial t}{\partial y} - \left[-k\, dx\left\{\frac{\partial t}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial t}{\partial y}\right)dy\right\}\right]$$
$$= k\frac{\partial^2 t}{\partial y^2}dx\, dy \qquad ...(12.40)$$

Due to relative motion of fluid in the boundary layer (fluid on the top face of the control volume moves faster than fluid on the bottom face), there will be viscous effects which will cause heat generation.

Viscous force

$$= \text{shear stress} \times \text{area upon which it acts}$$
$$= \mu\frac{\partial u}{\partial y}(dx \times 1)$$

This force will act through a distance s which can be determined by the relative velocity of fluid flow at the upper and lower faces of the element ; $s = (\partial u/\partial y)\, dy$

$\therefore$ Viscous heat generation

$$= \mu\frac{\partial u}{\partial y}dx \times \frac{\partial u}{\partial y}dy$$
$$= \mu\left(\frac{\partial u}{\partial y}\right)^2 dx\, dy \qquad ...(12.41)$$

With stipulations of steady state condition, the algebraic sum total of heat due to convection, conduction and viscous effect equals zero. Thus

$$-\rho c\left(u\frac{\partial t}{\partial x} + t\frac{\partial u}{\partial x} + v\frac{\partial t}{\partial y} + t\frac{\partial v}{\partial y}\right)dx\, dy + k\frac{\partial^2 t}{\partial y^2}dx\, dy + \mu\left(\frac{\partial u}{\partial y}\right)^2 dx\, dy = 0$$

or
$$-\rho c\left[u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} + t\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + k\frac{\partial^2 t}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 = 0 \qquad ...(12.42)$$

For a two-dimensional boundary layer flow, $(\partial u/\partial x) + (\partial v/\partial y) = 0$, and therefore the above equation can be recast as:

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{\rho c}\frac{\partial^2 t}{\partial y^2} + \frac{\mu}{\rho c}\left(\frac{\partial u}{\partial y}\right)^2 \qquad ...(12.4$$

which represents the differential ener equation for flow past a flat plate. If the h

generation due to viscous effects is neglected, the energy equation takes the form ;

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{\rho c}\frac{\partial^2 t}{\partial y^2}$$

$$= \alpha\frac{\partial^2 t}{\partial y^2} \qquad ...(12.44)$$

It may be noted that the energy equation is similar to the momentum equation. Further the kinematic viscosity ν and the thermal diffusivity α have the same dimensions.

The assumptions underlying equation 12.44 are :

(i) steady incompressible flow

(ii) negligible body forces, viscous heating and conduction in the flow direction

(iii) constant fluid properties evaluated at the film temperature, $t_f = (t_\infty + t_s)/2$

12.7. POHLHAUSEN SOLUTION FOR THE ENERGY EQUATION

The energy equation

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \alpha\frac{\partial^2 t}{\partial y^2}$$

can be recast into an ordinary differential equation by using the following variables :

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \text{ streching fatcor}$$

$$\psi = \sqrt{\nu x U_\infty}\, f(\eta) \text{ stream function}$$

and $$\theta = \frac{t_s - t}{t_s - t_\infty} = f(\eta) = f\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) \qquad ...(12.45)$$

The velocity components u and v were determined in section 12.3 and were found to have the following values

$$u = U_\infty \frac{df}{d\eta} \qquad ...(12.17)$$

$$v = \left[\frac{y}{2x}U_\infty\frac{df}{d\eta} - \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}\, f(\eta)\right] \qquad ...(12.18)$$

Now from the non-dimensional temperature parameter θ defined above ;

$$t = t_s + (t_\infty - t_s)\theta$$

$$\frac{\partial t}{\partial x} = (t_\infty - t_s)\frac{\partial \theta}{\partial x} = (t_\infty - t_s)\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= (t_\infty - t_s)\left\{\frac{-y}{2x^{3/2}}\sqrt{\frac{U_\infty}{\nu}}\right\}\frac{d\theta}{d\eta} \qquad ...(12.46)$$

Also, $$\frac{\partial t}{\partial x} = (t_\infty - t_s)\frac{\partial \theta}{\partial y} = (t_\infty - t_s)\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= (t_\infty - t_s)\sqrt{\frac{U_\infty}{\nu x}}\frac{d\theta}{d\eta} \qquad ...(12.47)$$

Further,

$$\frac{\partial^2 t}{\partial y^2} = \frac{\partial}{\partial y}\left[(t_\infty - t_s)\sqrt{\frac{U_\infty}{\nu x}}\frac{d\theta}{d\eta}\right]$$

$$= (t_\infty - t_s)\sqrt{\frac{U_\infty}{\nu x}}\frac{\partial}{\partial \eta}\left(\frac{d\theta}{d\eta}\right)\frac{d\eta}{dy}$$

$$= (t_\infty - t_s)\sqrt{\frac{U_\infty}{\nu x}}\frac{d^2\theta}{d\eta^2}\sqrt{\frac{U_\infty}{\nu x}}$$

$$= (t_\infty - t_s)\frac{U_\infty}{\nu x}\frac{d^2\theta}{d\eta^2} \qquad ...(12.48)$$

When the above obtained values of u, v, $(\partial t/\partial x)$, $(\partial t/\partial y)$ and $(\partial^2 t/\partial y^2)$ are substituted in the energy equation, we get :

$$U_\infty\frac{\partial f}{\partial \eta}(t_\infty - t_s)\left\{\frac{-y}{2x^{3/2}}\sqrt{\frac{U_\infty}{\nu}}\right\}\frac{d\theta}{d\eta}$$

$$+\left\{\frac{y}{2x}U_\infty\frac{df}{d\eta} - \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}\, f(\eta)\right\}$$

$$(t_\infty - t_s)\sqrt{\frac{U_\infty}{\nu}}\frac{d\theta}{d\eta}$$

$$= \alpha(t_\infty - t_s)\frac{U_\infty}{\nu x}\frac{d^2\theta}{d\eta^2} \qquad ...(12.49)$$

Upon simplification and rearrangement ; we obtain the following ordinary linear differential equation :

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{2}\frac{\nu}{\alpha}f(\eta)\frac{d\theta}{d\eta} = 0$$

or $$\frac{d^2\theta}{d\eta^2} + \frac{1}{2}Pr\, f(\eta)\frac{d\theta}{d\eta} = 0 \qquad ...(12.50)$$

where $Pr = \nu/\alpha$ is the Prandtl number.

The relevant boundary conditions for the temperature distribution $t = t_s$ at $y = 0$ and $t = t_\infty$ at $y = \infty$ take the following values in terms of the new variable :

$$\theta(\eta) = 0 \quad \text{at} \quad \eta = 0$$

and $$\theta(\eta) = 1 \quad \text{at} \quad \eta = \infty$$

Following is the solution obtained by Pohlhausen for the energy equation

$$\theta(\eta) = \left(\frac{d\theta}{d\eta}\right)_{\eta=0}$$

$$\int_0^\eta \exp\left[-\frac{Pr}{2}\int_0^\eta f(\eta)d\eta\right]d\eta \qquad ...(12.51)$$

The factor $(d\theta/d\eta)_{\eta=0}$ represents the dimensionless slope of the temperature profile at the surface where $\eta = 0$, and its value can be obtained by applying the boundary condition $\theta(\eta) = 1$ at $\eta = \infty$. Thus

$$1 = \left(\frac{d\theta}{d\eta}\right)_{\eta=0}$$

$$\int_0^\infty \exp\left[-\frac{Pr}{2}\int_0^\infty f(\eta)d\eta\right]d\eta \qquad ...(12.52)$$

The dimensionless slope is obviously a function of Prandtl number and the calculations made by Prandtl showed that

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = 0.332\ Pr^{1/3} \qquad ...(12.53)$$

for $0.6 < Pr$ 15.

The dimensionless temperature distribution for various values of Prandtl number has been shown plotted in Fig. 12.12; the curves for $Pr = 0.7$ is typical for air and several other gases.

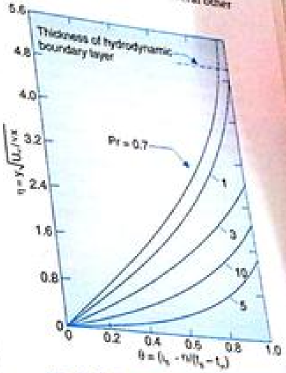

Fig. 12.12. Temperature distribution in the laminar layer on a heated flat plate.

The following results are of practical interest :

(i) The thickness of thermal boundary layer δ_t is taken to be the distance from the plate surface for which

$$\frac{t_s - t}{t_s - t_\infty} = 0.99$$

For this value of non-dimensional temperature parameter, there are several curves with different values of Prandtl number.

(a) When $Pr = 1$

$$\eta = \left(y\sqrt{\frac{U_\infty}{\nu}}\right) \cong 5.0 \text{ at } \frac{t_s - t}{t_s - t_\infty} = 0.99$$

At the outer edge of thermal boundary layer, $y = \delta_t$ and therefore

$$\delta_t\sqrt{\frac{U_\infty}{\nu x}} = 5.0$$

or $$\frac{\delta_t}{x} = \frac{5.0}{\sqrt{(x U_\infty)/\nu}} = \frac{5.0}{\sqrt{Re_x}} \quad ...(12.54)$$

The thickness of thermal boundary layer is thus proportional to $\sqrt{x}$. With increase in distance from the leading edge, the effects of heat transfer penetrate further into the free stream and the thermal boundary layer grows. Comparison of equations 12.25 and 12.54 reveals that when $Pr = 1$, the thermal and hydrodynamic boundary layers are of equal thickness and that the temperature and velocity surfaces are identical. Since Prandtl number for most gases are sufficiently close to unity $(0.6 < Pr < 1.0)$, the two boundary layers would be very close for the gases.

(i) When $Pr < 1$.

$$\eta = 5.0 \quad \text{at} \quad \frac{t_s - t}{t_s - t_\infty} = 0.99$$

$$y\sqrt{\frac{U_\infty}{\nu x}} > 5.0$$

$$\frac{\delta_t}{x} > \frac{5.0}{\sqrt{(x U_\infty)/\nu}} > \frac{5.0}{\sqrt{Re_x}} \quad ...(12.55)$$

Apparently the thermal boundary layer is thicker than the hydrodynamic boundary layer for the flow situations with $Pr < 1$. This has to be so because $Pr < 1$ implies that thermal diffusivity is more than the momentum diffusivity. A large thermal diffusivity means more penetration of the temperature effects and consequently a large value of the thermal boundary layer thickness.

(i) When $Pr > 1$,

$$\eta = 5.0 \quad \text{at} \quad \frac{t_s - t}{t_s - t_\infty} = 0.99$$

$$y\sqrt{\frac{U_\infty}{\nu x}} < 5.0$$

$$\frac{\delta_t}{x} < \frac{5.0}{\sqrt{(x U_\infty)/\nu}} < \frac{5.0}{\sqrt{Re_x}} \quad ...(12.56)$$

Apparently $\delta_t < \delta_h$ when $Pr > 1$.

Pohlhausen has suggested the following general correlation between Prandtl number and the relative values of thermal and hydrodynamic boundary layer thicknesses.

$$\delta_t = \delta_h \, (Pr)^{-0.33} \quad ...(12.57)$$

(ii) The heat flux at the surface may be written as

$$\frac{Q}{A} = h_x (t_s - t_\infty) = -k\left(\frac{\partial t}{\partial y}\right)_{y=0} \quad ...(12.58)$$

This expression is quite appropriate because at the plate surface, there is no fluid motion and the heat transfer can occur only through conduction.

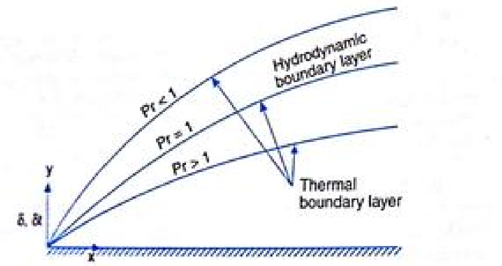

Fig. 12.13. Hydrodynamic and thermal boundary layers for different Prandtl numbers

The surface temperature gradient $(\partial t/\partial y)_{y=0}$ may be developed from the relation 12.58 according to which

$$\left(\frac{\partial t}{\partial y}\right)_{y=0} = -(t_s - t_\infty)\sqrt{\frac{U_\infty}{\nu x}}\left(\frac{d\theta}{d\eta}\right)_{\eta=0}$$

$$= -(t_s - t_\infty)\sqrt{\frac{U_\infty}{\nu x}} \times 0.332\,(Pr)^{0.33}$$

$$= -\frac{0.332}{x}(t_s - t_\infty)\sqrt{\frac{U_\infty x}{\nu}}\,(Pr)^{0.33}$$

$$= -\frac{0.332}{x}(t_s - t_\infty)(Re_x)^{0.5}\,(Pr)^{0.33}$$

Upon substitution in equation 12.58,

$$\frac{Q}{A} = h_x (t_s - t_\infty)$$

$$= 0.332\,\frac{k}{x}(t_s - t_\infty)(Re_2)^{0.5}\,(Pr)^{0.33}$$

Solving for h_x, we obtain

$$h_x = 0.332\,\frac{k}{x}(Re_x)^{0.5}\,(Pr)^{0.33} \quad ...(12.59)$$

and finally in the non-dimensional form

$$Nu_x = \frac{x\,h_x}{k} = 0.332\,(Re_x)^{0.5}\,(Pr)^{0.33} \quad ...(12.60)$$

Equations 12.59 and 12.60 relate local values of the convective coefficient and the Nusselt number; these apply at a specific value of distance x from the leading edge of the plate.

Taking averages over the interval $0 < x < l$, we can work out the average heat transfer coefficient and Nusselt number to be :

$$\bar{h} = \frac{1}{l}\int_0^l h_x\,dx$$

$$= 2 \times 0.332\,\frac{k}{l}(Re_l)^{0.5}\,(Pr)^{0.33}$$

$$= 2 \times \text{local film coefficient for a plate length } l \quad ...(12.61)$$

and $$\overline{Nu} = \frac{\bar{h}\,l}{k} = 0.664\,(Re_l)^{0.5}\,(Pr)^{0.33} \quad ...(12.62)$$

The bar over the quantity indicates its average value.

12.8. INTEGRAL ENERGY EQUATION

An approximate solution of the energy equation can be obtained by applying the von-Karman integral technique to the thermal-hydrodynamic boundary layer. Towards that end, we shall make an energy balance on a finite control volume (Fig. 12.14) and develop an expression for the heat conducted into the laminar, incompressible boundary layer at the surface of a plate.

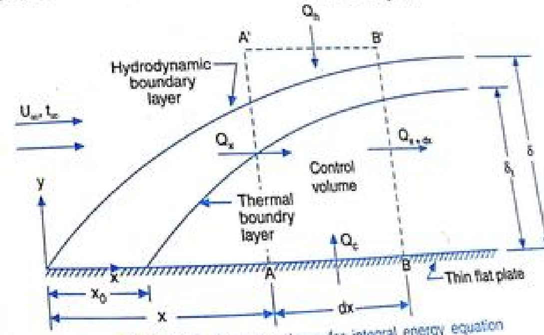

Fig. 12.14. Control volume for integral energy equation

Assumptions

(i) the thermo-physical properties ρ, c and k of the fluid remain constant within the operating range of the temperature

(ii) the heating of the plate commences at a distance x_0 from the leading edge of the plate. Within the initial length x_0, the plate temperature is the same as that of the fluid stream. Apparently there is only hydrodynamic boundary layer and no thermal boundary layer in this region. The thermal boundary layer initiates at $x = x_0$ and develops and grows beyond that.

For unit width of the plate, we have

Fluid mass entering through the left gap face AA′

$$= \int_0^H \rho\, u\, dy \qquad \text{...(12.63)}$$

Fluid mass leaving through the right face BB′

$$= \int_0^H \rho\, u\, dy + \frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx \qquad \text{...(12.64)}$$

From continuity considerations, the mass increment

$$\frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx \qquad \text{...(12.65)}$$

enters the control volume through its top face A′B′.

The corresponding heat fluxes through the faces are :

Heat influx through the face AA′

$$Q_x = \text{mass} \times \text{specific heat} \times \text{temperature}$$

$$= \int_0^H \rho\, u\, dy \times c \times t$$

$$= \rho c \int_0^H u\, t\, dy \qquad \text{...(12.66)}$$

Heat efflux through the face BB′

$$Q_{x+dx} = \rho c \int_0^H u\, t\, dy + \frac{\partial}{\partial x}\left[\rho c \int_0^H u\, t\, dy\right] dx \qquad \text{...(12.67)}$$

The upper face A′B′ is outside the boundary layer and there the temperature is constant at t_∞. Hence the energy influx is

$$Q_y = \frac{\partial}{\partial x}\left[\int_0^H \rho\, u\, dy\right] dx\, c\, t_\infty \qquad \text{...(12.6}$$

Heat is conducted into the lower face of the control volume at the rate

$$Q_c = -kA\left(\frac{\partial t}{\partial y}\right)_{y=0}$$

$$= -k\, dx\left(\frac{\partial t}{\partial y}\right)_{y=0} \qquad \text{...(12.6}$$

An energy balance for the element gives

$$\rho c \int_0^H u\, t\, dy + \frac{\partial}{\partial x}\left[\rho c\, t_\infty \int_0^H u\, dy\right] dx - k\, dx\left(\frac{\partial t}{\partial y}\right)_{y=0}$$

$$= \rho c \int_0^H u\, t\, dy + \frac{\partial}{\partial x}\left[\rho c \int_0^H u\, t\, dy\right] dx$$

Upon simplification and re-arrangement

$$\frac{d}{dx}\int_0^H (t_\infty - t)\, u\, dy = \frac{k}{\rho c}\left(\frac{dt}{dy}\right)_{y=0}$$

$$= \alpha\left(\frac{dt}{dy}\right)_{y=0} \qquad \text{...(12.7}$$

This is the integral equation for the boundary layer for constant properties and constant free stream temperature t_∞.

If the net viscous work done within the element

$$\mu \int_0^H \frac{\partial^2 u}{\partial y^2}\, dx\, dy \qquad \text{...(Equation 12.9)}$$

were also considered with the energy balance, the integral equation would have become

$$\frac{d}{dx}\int_0^H (t_\infty - t)\, u\, dy + \frac{\mu}{\rho c}\int_0^H \frac{\partial^2 u}{\partial y^2}\, dy$$

$$= \alpha\left(\frac{dt}{dy}\right)_{y=0} \qquad \text{...(12.71)}$$

The viscous dissipation term is very small unless the velocity of the flow field becomes very large, and is usually neglected.

We shall now use the cubic velocity and temperature distributions in the integral boundary layer energy equation, and develop an expression for the convective heat transfer coefficient for laminar flow over a flat plate that has an unheated starting length x_0.

(i) The cubic velocity profile within the boundary layer is of the form.

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad \text{...(12.71 a)}$$

(ii) The temperature distribution within the boundary layer satisfies the conditions :

$$t = t_s \quad \text{at} \quad y = 0$$

$$\frac{dt}{dy} = 0, \quad \text{at} \quad y = \delta_t$$

$$t = t_\infty \quad \text{at} \quad y = \delta_t$$

$$\frac{d^2t}{dy^2} = 0 \quad \text{at} \quad y = 0$$

These boundary conditions have the same form as those on u/U_∞. Therefore, when these are fitted to a cubic polynomial

$$\frac{\theta}{\theta_\infty} = a + b\left(\frac{y}{\delta_t}\right) + c\left(\frac{y}{\delta_t}\right)^2 + d\left(\frac{y}{\delta_t}\right)^3 \qquad \text{...(12.72)}$$

the temperaure distribution acquires the form:

$$\frac{\theta}{\theta_\infty} = \frac{t - t_s}{t_\infty - t_s}$$

$$= \frac{3}{2}\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \qquad \text{...(12.73)}$$

Inserting the appropriate values of velocity distribution and temperature distribution into the integral equation, we obtain :

$$\alpha\left(\frac{dt}{dy}\right)_{y=0} = \frac{d}{dx}\int_0^H (t_\infty - t)\, u\, dy$$

$$= U_\infty (t_\infty - t)\frac{d}{dx}\int_0^H \frac{u}{U_\infty}\left(\frac{t_\infty - t}{t_\infty - t}\right) dy$$

$$= U_\infty (t_\infty - t_s)\frac{d}{dx}\int_0^H \frac{u}{U_\infty}\left(1 - \frac{t - t_s}{t_\infty - t_s}\right) dy$$

$$= U_\infty (t_\infty - t_s)\frac{d}{dx}\left[\int_0^H \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\} \times \left\{1 - \frac{3}{2}\left(\frac{y}{\delta_t}\right) + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right\} dy\right]$$

$$= U_\infty (t_\infty - t_s)\frac{d}{dx}\left[\int_0^{\delta_t} \left\{\frac{3}{2\delta}y - \frac{9}{4\delta\delta_t}y^2 - \frac{1}{2\delta^3}y^3 + \left(\frac{3}{4\delta\delta_t^3} + \frac{3}{4\delta^3\delta_t}\right)y^4 - \frac{1}{4\delta^3\delta_t^3}y^6\right\} dy\right] \qquad \text{...(12.74)}$$

For most gases, the thermal boundary layer is thinner than the hydrodynamic boundary layer, $\delta_t < \delta$. Therefore, the upper limit on integration has been changed to δ_t because the integrand would be zero for $y > \delta_t$. Carrying out the integration and letting the thickness ratio $\delta_t/\delta = r$, the result simplifies to

$$\alpha\left(\frac{dt}{dy}\right)_{y=0} = U_\infty (t_\infty - t_s)\frac{d}{dx}\left[\delta\left(\frac{3}{20}r^2 - \frac{3}{280}r^4\right)\right] \qquad \text{...(12.75)}$$

Since $\delta_t < \delta$, $r < 1$; hence we may neglect the term involving r^4 and write

$$\alpha\left(\frac{dt}{dy}\right)_{y=0} = \frac{3}{20}U_\infty (t_\infty - t_s)\frac{d}{dx}(\delta r^2) \qquad \text{...(12.76)}$$

From the expression for temperature distribution

$$\frac{t - t_s}{t_\infty - t_s} = \frac{3}{2}\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

$$t = t_s + (t_\infty - t_s)\left[\frac{3}{2}\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]$$

$$\frac{dt}{dy} = (t_\infty - t_s)\left[\frac{3}{2\delta_t} - \frac{3}{2\delta_t^3}y^2\right]$$

$$\left(\frac{dt}{dy}\right)_{y=0} = \frac{3}{2}\frac{t_\infty - t_s}{\delta_t} = \frac{3}{2}\frac{t_\infty - t_s}{r\delta} \quad ...(12.77)$$

Hence,

$$\frac{3}{2}\alpha\frac{t_\infty - t_s}{r\delta} = \frac{3}{20}U_\infty (t_\infty - t_s)\frac{d}{dx}(\delta r^2)$$

$$\alpha = \frac{U_\infty}{10}(r\delta)\frac{d}{dx}(\delta r^2)$$

$$= \frac{U_\infty}{10}(r\delta)\left(\delta\, 2r\frac{dr}{dx} + r^2\frac{d\delta}{dx}\right)$$

$$= \frac{U_\infty}{10}\left[2\delta^2 r^2\frac{dr}{dx} + \delta r^3\frac{d\delta}{dx}\right] \quad ...(12.78)$$

From the relations already established for the hydrodynamic boundary layer,

$$\delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\nu}{U_\infty} \quad \text{and} \quad \delta^2 = \frac{280}{13}\frac{\nu x}{U_\infty}$$

When these values are substituted in equation 12.78, we get,

$$\alpha = \frac{U_\infty}{10}\left[2r^2 \times \frac{280}{13}\frac{\nu x}{U_\infty}\frac{dr}{dx} + r^3 \times \frac{140}{13}\frac{\nu}{U_\infty}\right]$$

$$\text{or } r^3 + 4r^2 x\frac{dr}{dx} = \frac{13}{14}\frac{\alpha}{\nu} \quad ...(12.79)$$

Incorporating the equality

$$\frac{d}{dx}(r^3) = 3r^2\frac{dr}{dx}$$

we can express the equation 12.79 as

$$r^3 + 4xr^2\frac{dr}{dx} = \frac{13}{14}\frac{\alpha}{\nu} \quad ...(12.80)$$

which is a linear differential equation of the first order in r^3, and the general solution is

$$r^3 = Cx^{-3/4} + \frac{13}{14}\frac{\alpha}{\nu} \quad ...(12.81)$$

The constant C when evaluated from the boundary condition

$$r^3 = \left(\frac{\delta_t}{\delta}\right)^3 = 0 \quad \text{at} \quad x = x_0$$

takes the value :

$$0 = Cx_0^{-3/4} + \frac{13}{14}\frac{\alpha}{\nu};$$

$$C = -\frac{13}{14}\frac{\alpha}{\nu}x_0^{3/4}$$

The final solution then becomes :

$$r^3 = -\frac{13}{14}\frac{\alpha}{\nu}x_0^{3/4}x^{-3/4} + \frac{13}{14}\frac{\alpha}{\nu}$$

$$= \frac{13}{14}\frac{\alpha}{\nu}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]$$

Therefore, ...(12.82)

$$r = \frac{\delta_t}{\delta} = \left(\frac{13}{14}\right)^{1/3}\left(\frac{\alpha}{\nu}\right)^{1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}$$

$$= \frac{0.976}{Pr^{1/3}}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \quad ...(12.83)$$

where the ratio ν/α is the Prandtl number.

If heating of the plate starts from the leading edge of the plate itself, *i.e.*, when the plate is heated over the entire length, $x_0 = 0$:

$$r = \frac{\delta_t}{\delta} = \frac{0.976}{Pr^{1/3}} \quad ...(12.84)$$

The local heat transfer coefficient h_x can be worked out from the equality,

$$\frac{Q}{A} = h_x(t_s - t_\infty) = -k\left(\frac{dt}{dy}\right)_{y=0}$$

$$h_x = \frac{-k\,(dt/dy)_{y=0}}{(t_s - t_\infty)}$$

From equation 12.77,

$$\left(\frac{dt}{dy}\right)_{y=0} = \frac{3}{2}\left(\frac{t_\infty - t_s}{\delta_t}\right)$$

$$\therefore \quad h_x = -\frac{3k}{2\delta_t}\left(\frac{t_\infty - t_s}{t_s - t_\infty}\right)$$

$$= \frac{3k}{2\delta_t} = \frac{3k}{2} \times \frac{1}{r\delta} \quad ...(12.85)$$

For a cubic polynomial for the velocity and temperature distribution

$$r = \frac{0.976}{Pr^{1/3}}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}$$

$$\delta = \frac{4.64x}{\sqrt{Re_x}}$$

Substitution in equation 12.85 yields :

$$h_x = \frac{3k}{2}\frac{\sqrt{Re_x}}{4.64x} \times \frac{Pr^{1/3}}{0.976\left[1 - (x_o/x)^{3/4}\right]^{1/3}}$$

$$= 0.332\,\frac{k}{x}(Re_x)^{1/2}(Pr)^{1/3} \times \frac{1}{\left[1 - (x_o/x)^{3/4}\right]^{1/3}} \quad ...(12.86)$$

and finally in the non-dimensional form

$$Nu_x = \frac{x\,h_x}{k} = \frac{0.332\,(Re_x)^{0.5}(Pr)^{0.33}}{\left[1 - (x_0/x)^{3/4}\right]^{1/3}} \quad ...(12.87)$$

When the plate is heated over the entire length, $x_0 = 0$

$$h_x = 0.332\,\frac{k}{x}(Re_x)^{0.5}(Pr)^{0.33} \quad ...(12.88)$$

$$\text{and } Nu_x = 0.332\,(Re_x)^{0.5}(Pr)^{0.33} \quad ...(12.89)$$

The foregoing analysis is valid only for laminar conditions. With increase in length of the plate, the flow may have transition from laminar to turbulent conditions. Apparently then a limit needs to be imposed on the plate length along the flow direction.

EXAMPLE 12.9.

A small thermo-couple is positioned in a thermal boundary layer near a flat plate past which water flows at 30°C and 0.15 m/s. The plate is heated to a surface temperature of 50°C and at the location of the probe, the thickness of thermal boundary layer is 15 mm. If the temperature profile as measured by the probe is well-represented by

$$\frac{t - t_s}{t_\infty - t_s} = 1.5\left(\frac{y}{\delta_t}\right) - 0.5\left(\frac{y}{\delta_t}\right)^3$$

determine (a) the heat flux from plate to water ; and (b) the heat transfer coefficient

Solution : At the mean film tmeperature

$$t_f = \frac{30 + 50}{2} = 40°C$$

the thermal conductivity of water is 0.633 W/m-deg

$$\frac{Q}{A} = -k(t_\infty - t_s)\frac{\partial}{\partial y}\left(\frac{t - t_s}{t_\infty - t_s}\right)_{y=0}$$

$$= -k(t_\infty - t_s)\left|\frac{1.5}{\delta_t} - \frac{3 \times 0.5}{\delta_t}y^2\right|_{y=0}$$

$$= \frac{1.5k(t_s - t_\infty)}{\delta_t}$$

$$= \frac{1.5 \times 0.633 \times (50 - 30)}{0.015}$$

$$= 1266 \text{ W/m}^2$$

(b) Heat transfer coefficient,

$$h = \frac{Q/A}{(t_s - t_\infty)}$$

$$= \frac{1.5k(t_s - t_\infty)}{\delta_t} \times \frac{1}{t_s - t_\infty}$$

$$= \frac{1.5k}{\delta_t} = \frac{1.5 \times 0.633}{0.05}$$

$$= 63.3 \text{ W/m}^2\text{-deg}$$

EXAMPLE 12.10.

Air at 25°C approaches a 0.9 m long by 0.6 m wide flat plate with an approach velocity 4.5 m/s. The plate is heated to a surface temperature of 135°C. Make calculations for :

(a) local heat transfer coefficient at a distance of 0.5 m from the leading edge ;

(b) total rate of heat transfer from the plate to the air

$$Nu = h\frac{l}{k}$$

$$= 0.664 (Re)^{0.5} \times (Pr)^{0.33}$$

$$= 0.664 (40000)^{0.5} \times (0.696)^{0.33}$$

$$= 117.79$$

$$h = Nu \times \frac{k}{l}$$

$$= 117.79 \times \frac{0.0289}{0.2}$$

$$= 17.02 \text{ W/m}^2 \text{ deg}$$

∴ Convective heat transfer from both sides of the plate,

$$Q_1 = h\,(2A) \times \Delta t$$

$$= 17.02 \times (2 \times 0.2 \times 0.1) \times (100 - 20)$$

$$= 54.46 \text{ W}$$

(b) From the correlation

$$Nu = \frac{hl}{k} = 0.664\,(Re)^{0.5} \times (Pr)^{0.33}$$

$$= 0.664 \left(\frac{\rho V l}{\mu}\right)^{0.5} \times \left(\frac{\mu c}{k}\right)^{0.33}$$

It follows that

the thermo-physical properties of air are :

$k = 0.0963$ kJ/m-hr-deg

$Pr = 0.701$

$\nu = 16 \times 10^{-6}$ m²/s

and $\rho = 1.165$ kg/m³

The value of $x = x_c$ at which flow changes from laminar to turbulent conditions is found from the expression for critical Reynolds number

$$Re_c = \frac{x_c U_\infty}{\nu}$$

$$\therefore \quad x_c = \frac{\nu Re_c}{U_\infty} = \frac{16 \times 10^{-6} \times 5 \times 10^5}{3} = 2.67 \text{ m}$$

(i) Assuming cubic velocity profile,

$$\delta = \frac{4.64}{\sqrt{Re_c}} = \frac{4.64 \times 2.67}{\sqrt{5 \times 10^5}} = 0.0175 \text{ m} = 17.5 \text{ mm}$$

$$\delta_t = \frac{0.976\delta}{(Pr)^{1/3}} = \frac{0.976 \times 0.0175}{(0.701)^{1/3}} = 0.01923 \text{ m} = 19.23 \text{ mm}$$

(ii) The local Nusselt number at $x = x_c$ is given by

$$Nu_c = 0.332\, (Re_c)^{0.5}\, (Pr)^{0.33} = 0.332 \times (5 \times 10^5)^{0.5} \times (0.701)^{0.33} = 208.47$$

$\therefore$ Local heat transfer coefficient at $x = x_c$ is,

$$h_c = \frac{Nu_c \times k}{x_c} = \frac{208.47 \times 0.0963}{2.67} = \textbf{7.519 kJ/m}^2\textbf{-hr-deg}$$

Average heat transfer coefficient is obtained as

$$\bar{h} = \frac{1}{x}\int_0^{x_c} h_x\, dx = 2\, h_c = 2 \times 7.519 = \textbf{15.038 kJ/m}^2\textbf{-hr-deg}$$

(iii) Convective heat flow from both sides of the plate to ambient air is

$$Q = \bar{h}\,(2A) \times \Delta T = 15.038\,(2 \times 2.67 \times 1) \times (40 - 2$$

$= 1606.06$ kJ/hr

(iv) Mass flow through the boundary layer from the leading edge to the point where transition occurs is given by :

$$m = \frac{5}{8}\rho U_\infty (\delta_2 - \delta_1)$$

$\delta_1 = 0$ at $x = 0$

and $\delta_2 = 0.0175$ m at $x = x_c = 2.67$ m

$$\therefore \quad m = \frac{5}{8} \times 1.165 \times 3 \times (0.0175 - 0) = 0.0382 \text{ kg/s} = 137.62 \text{ kg/hr}$$

EXAMPLE 12.13.

In a certain pharmaceutical process, castor oil at 35°C flows over a flat plate at 6 cm/s. The plate is 6 m long, is heated uniformly and maintained at a surface temperature of 95°C. Make calculations for the (a) hydrodynamic and thermal boundary layer thicknesses at the trailing edge of the plate, (b) total drag per unit width on one side of the plate, (c) local heat transfer coefficient at the end of the plate and (d) total heat flux from the surface per unit width.

At the mean film temperature

$$t_f = \frac{95 + 35}{2} = 60°C$$

the relevant fluid properties are :

Thermal diffusivity $\alpha = 7.2 \times 10^{-8}$ m²/s

Thermal conductivity $k = 0.213$ W/m K

Kinematic viscosity $\nu = 0.65 \times 10^{-4}$ m²/s

Density $\rho = 956.8$ kg/m³

Solution : The Reynolds number at the end of plate is,

$$Re_l = \frac{l U_\infty}{\nu} = \frac{6 \times 0.06}{0.65 \times 10^{-4}} = 5538.5$$

Since the Reynolds number is less than 3.5×10^5, the boundary layer over the entire plate is laminar in character.

(a) In the laminar range, the thickness of hydrodynamic boundary layer as prescribed by Blasius solution is

$$\delta = \frac{5x}{\sqrt{Re_x}}$$

$\therefore$ Thickness of hydrodynamic boundary layer at the trailing edge ($x = l$) is

$$\delta = \frac{5 \times 6}{\sqrt{5538.5}} = 0.4032 \text{ m}$$

When the plate is heated over the entire length, the hydrodynamic and thermal boundary layer thicknesses are related to each other by the expression,

$$\delta_t = \frac{0.976\delta}{(Pr)^{1/3}}$$

where, Prandtl number Pr

$$= \frac{\nu}{\alpha} = \frac{0.65 \times 10^{-4}}{7.2 \times 10^{-8}} = 902.78$$

$$\therefore \quad \delta_t = \frac{0.976 \times 0.4032}{(902.78)^{1/3}} = 0.0408 \text{ m}$$

(b) The average skin friction coefficient (drag coefficient) for the entire plate is

$$\bar{C}_f = \frac{1.328}{\sqrt{Re_l}} = \frac{1.328}{\sqrt{5538.5}} = 0.0178$$

$\therefore$ Friction (drag) force

$$= \bar{C}_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area of plate for one side}$$

$$= 0.0178 \times \left(\frac{1}{2} \times 958.8 \times 0.06^2\right) \times (6 \times 1)$$

= **0.184 N per metre width**

(c) The local Nusselt number at $x = l$ is given by :

$$Nu_l = 0.332\, (Re_l)^{0.5}\, (Pr)^{0.33} = 0.332\, (5538.5)^{0.5} \times (902.78)^{0.33} = 238.25$$

$\therefore$ Local heat transfer coefficent at the plate end is

$$h_l = \frac{Nu_l \times k}{l} = \frac{238.25 \times 0.213}{6} = 8.458 \text{ W/m}^2\text{K}$$

(d) The average heat transfer coefficient is obtained as

$$\bar{h} = \frac{1}{l}\int_0^l h_x\, dx = 2h_l = 2 \times 8.458 = 16.916 \text{ W/m}^2 \text{ K}$$

$\therefore$ Heat loss from one side of the plate is

$$Q = \bar{h}\, A\, \Delta t = 16.916 \times (6 \times 1) \times (95 - 35)$$

= **6089 W per metre width**

EXAMPLE 12.14.

A thin flat plate of length l = 1 m and breadth b = 0.45 m is exposed to a flow of air parallel to its surface. The velocity and temperature of the free stream flow of air are respectively $U_\infty = 2.5$ m/s and $t_\infty = 25°C$. If temperature at the surface of plate is $t_s = 95°C$, estimate the heat loss from 50 cm length of plate measured from the trailing edge.

Solution : At the mean film temperature

$$t_f = \frac{t_s + t_\infty}{2} = \frac{95 + 25}{2} = 60°C$$

the thermo-physical properties of air are :

$\rho = 1.060$ kg/m³

$k = 2.894 \times 10^{-2}$ W/mK

$\nu = 18.97 \times 10^{-6}$ m²/s

and $Pr = 0.696$

(i) For the first 50 cm of plate length ($x = 0.5$ m)

$$Re_x = \frac{x U_\infty}{\nu} = \frac{0.5 \times 2.5}{18.97 \times 10^{-6}} = 65800$$

$$Nu_x = 0.332\, (Re_x)^{0.5} \times (Pr)^{0.33} = 0.332 \times (65800)^{0.5} \times (0.696)^{0.33} = 75.617$$

$$h_x = Nu_x \frac{k}{x} = 75.617 \times \frac{2.894 \times 10^{-2}}{0.5} = 4.376 \text{ W/m}^2 \text{ K}$$

The average value of the heat transfer coefficient is twice this value

$$\bar{h} = 2 h_x = 2 \times 4.376 = 8.752 \text{ W/m}^2 \text{ K}$$

The heat loss from one side of plate is,

$$Q_1 = h\,A\,\Delta t = 8.752 \times (0.5 \times 0.45) \times (95 - 25) = 137.86 \text{ W}$$

(ii) For the entire 1 m length of the plate

$$Re_x = \frac{x\,U_\infty}{\nu} = \frac{1 \times 2.5}{18.97 \times 10^{-6}} = 131600$$

$$Nu_x = 0.332\,(Re_x)^{0.5} \times (Pr)^{0.33} = 0.332\,(131600)^{0.5} \times (0.696)^{0.33} = 106.86$$

$$h_x = Nu_x \frac{k}{x} = \frac{106.86 \times 2.894 \times 10^{-2}}{1} = 3.092 \text{ W/m}^2\text{ K}$$

$$\bar{h} = 2\,h_x = 2 \times 3.092 = 6.184 \text{ W/m}^2\text{ K}$$

$$Q_2 = \bar{h}A\,\Delta t = 6.184 \times (1 \times 0.45) \times (95 - 25) = 194.79 \text{ W}$$

Hence heat loss from 50 cm of plate length measured from the trailing edge is,

$$= Q_2 - Q_1 = 194.79 - 137.86 = \mathbf{56.93\ W}$$

EXAMPLE 12.15.

Air at 25°C flows past a flat plate at 2.5 m/s. The plate measures 60 cm × 30 cm and is maintained at a uniform temperature of 95°C. Calculate the heat loss from the plate if the air flows parallel to the 60 cm side. How would this heat loss be affected if the flow of air is made parallel to the 30 cm side ?

Solution : At the mean film temperature,

$$t_f = \frac{t_s + t_\infty}{2} = \frac{95 + 25}{2} = 60°\text{C}$$

the thermo-physical properties of air are :

$$\rho = 1.060 \text{ kg/m}^3$$
$$k = 2.894 \times 10^{-2} \text{ W/mK}$$
$$\nu = 18.97 \times 10^{-6} \text{ m}^2\text{/s}$$
and $Pr = 0.696$

(i) Air flows parallel to the 60 cm side

$$Re_x = \frac{x\,U_\infty}{\nu} = \frac{0.6 \times 2.5}{18.97 \times 10^{-6}} = 0.01791 \times 10^6$$

$$Nu_x = 0.332\,(Re_x)^{0.5} \times (Pr)^{0.33} = 0.332 \times (0.0791 \times 10^6)^{0.5} \times (0.696)^{0.33} = 82.686$$

$$h_x = Nu_x \frac{k}{x} = 82.686 \times \frac{2.894 \times 10^{-2}}{0.6} = 3.988 \text{ W/m}^2\text{ K}$$

The average value of the convective coefficient is twice this value.

$$\bar{h} = 2\,h_x = 2 \times 3.988 = 7.976 \text{ W/m}^2\text{ K}$$

The heat loss from one side of plate is

$$Q_1 = h\,A\,\Delta t = 7.976 \times (0.6 \times 0.3) \times (95 - 25) = \mathbf{100.50\ W}$$

(ii) Air flows parallel to the 30 cm side

$$Re_x = \frac{x\,U_\infty}{\nu} = \frac{0.3 \times 2.5}{18.97 \times 10^{-6}} = 0.0396 \times 10^6$$

$$Nu_x = 0.332\,(Re_x)^{0.5} \times (Pr)^{0.33} = 0.332 \times (0.0396 \times 10^6)^{0.5} \times (0.696)^{0.33} = 58.538$$

$$h_x = Nu_x \frac{k}{x} = 58.538 \times \frac{2.894 \times 10^{-2}}{0.3} = 5.64 \text{ W/m}^2\text{ K}$$

$$\bar{h} = 2\,h_x = 2 \times 5.64 = 11.28 \text{ W/m}^2\text{ K}$$

$$Q_2 = hA\,\Delta t = 11.28 \times (0.6 \times 0.3) \times (95 - 25) = \mathbf{142.128\ W}$$

Apparently more heat loss occurs when the air flows parallel to the shorter side and the percentage heat increase is

$$= \frac{Q_2 - Q_1}{Q_1} \times 100 = \frac{142.128 - 100.50}{142.128} \times 100 = \mathbf{41.42\%}$$

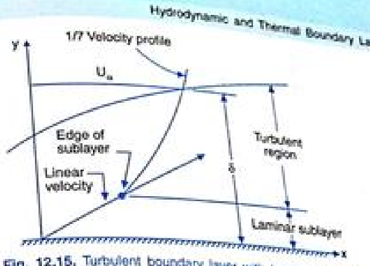

Fig. 12.15. Turbulent boundary layer with laminar sublayer

12.9. TURBULENT BOUNDARY LAYER FLOWS: FLAT PLATE

Based upon considerable experimental evidence, Prandtl suggested that the mean velocity distribution for turbulent boundary layer flow can be approximated by a single parametric relation of the form ;

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \qquad \text{...(12.90)}$$

which is known as the ***one-seventh power law.*** This relation lends itself readily to the mathematical analysis because of its extreme simplicity. However, the relation predicts infinite shear stress at the bounding surface as is evident from the infinite velocity gradient at the surface

$$\left|\frac{du}{dy}\right|_{y=0} = \left|\frac{1}{7}\frac{U_\infty}{\delta^{1/7}}\frac{1}{y^{6/7}}\right|_{y=0} \to \infty$$

This difficulty is circumvented by considering the velocity in the viscous sublayer to be linear and tangential to the seventh-root profile at the point where the laminar sublayer merges with the turbulent part of the boundary layer.

Blasius then suggests the following relation for viscous shear stress

$$\tau_w = 0.0225\,\rho U_\infty^2 \left[\frac{\mu}{\rho\delta U_\infty}\right]^{1/4} \qquad \text{...(12.91)}$$

for Reynolds number ranging from 5×10^5 to 10^7

Inserting velocity function expressed by equation 12.90 into momentum integral equation 12.30.

$$\tau_w = \rho U_\infty^2 \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left\{1 - \left(\frac{y}{\delta}\right)^{1/7}\right\} dy\right]$$

$$= \frac{7}{72}\rho U_\infty^2 \frac{\partial \delta}{\partial x} \qquad \text{...(12.92)}$$

By equating the expressions 12.91 and 12.92 for shear stress, a differential equation for boundary layer is obtained

$$\delta^{1/4}\,\partial\delta = 0.232\left(\frac{\mu}{\rho U_\infty}\right)^{1/4} \partial x$$

which upon integration yields

$$\frac{4}{5}\delta^{5/4} = 0.232\left(\frac{\mu}{\rho U_\infty}\right)^{1/4} x + C$$

Let boundary layer be assumed to be turbulent over the entire length of plate. The integration constant C is then obtained from the boundary condition $\delta = 0$ at $x = 0$, and that gives $C = 0$. Therefore

$$\delta^{5/4} = 0.29\left(\frac{\mu}{\rho U_\infty}\right)^{1/4} x$$

or

$$\delta = 0.371\left(\frac{\mu}{\rho U_\infty}\right)^{1/5} x^{4/5} \qquad \text{...(12.93)}$$

Apparently the boundary layer thickness for turbulent boundary layer increases as $x^{4/5}$, this growth is rapid than that in laminar boundary layer where δ varies as $x^{1/2}$. Expressed in the non-dimensional form, equation 12.93 can be rewritten as

$$\frac{\delta}{x} = \frac{0.371}{\left[(x\rho U_\infty)/\mu\right]^{1/5}} = \frac{0.371}{(Re_x)^{1/5}} \quad ...(12.94)$$

An estimate of wall shear stress can be made by substituting the value of boundary layer thickness in the Blasius expression for wall shear stress

$$\tau_w = 0.0225\rho U_\infty^2 \left(\frac{\mu}{\rho\delta U_\infty}\right)^{1/4}$$

$$= 0.0225\rho U_\infty^2 \left[\frac{\mu}{\rho\left\{0.371x/(Re_x)^{1/5}\right\}U_\infty}\right]^{1/4}$$

$$= \frac{0.0225}{(0.371)^{1/4}}\left[\frac{\mu}{x\rho U_\infty}(Re_x)^{1/5}\right]^{1/4}$$

$$= \frac{0.0288\,\rho U_\infty^2}{(Re_x)^{1/5}} \quad ...(12.95)$$

and thus the local skin friction coefficient is

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{0.0576}{(Re_x)^{1/5}} \quad ...(12.96)$$

Average value of skin friction coefficient

$$\bar{C}_f = \frac{1}{l}\int_0^l C_{fx}\,dx$$

$$= \frac{1}{l}\int_0^l \frac{0.0576}{(\rho U_\infty/\mu)^{1/5}}\frac{dx}{x^{1/5}}$$

$$= 0.0576\left(\frac{\mu}{\rho U_\infty}\right)^{1/5}\frac{1}{l}\int_0^l x^{-1/5}\,dx$$

$$= 0.072\left(\frac{\mu}{l\rho U_\infty}\right)^{1/5}$$

$$= \frac{0.072}{(Re_l)^{1/5}} \quad ...(12.97)$$

It may be remarked that equation 12.97 results when the velocity distribution follows the one-seventh power law which essentially is valid for $5 \times 10^5 < Re < 10^7$. Experiments indicate that for Reynolds number between 10^7 and 10^9, the velocity distribution deviates from the one-seventh power law and close fit with experimental data is obtained by the following empirical relationship suggested by Prandtl and Schlichting

$$\bar{C}_f = \frac{0.455}{(\log_{10} Re_l)^{2.58}} \quad ...(12.98)$$

Analysis made in the preceding paragraph is valid on the assumption that the boundary layer is turbulent from the leading edge onwards. However, irrespective of the laminar or turbulent characteristics of the main-stream flow, the boundary layer formation commences

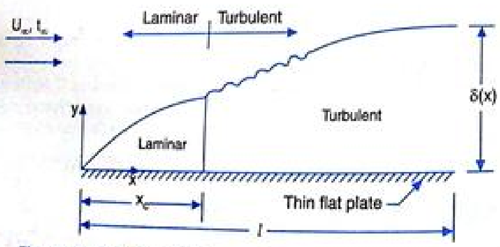

Fig. 12.16. Drag due to laminar and turbulent boundary layers

with a laminar boundary layer at the leading edge of the plate. Eventually the laminar boundary layer gets transformed to turbulent boundary layer after passing through a transition zone. The position of the point of transition depends upon the intensity of turbulence of the mainstream flow, on the roughness of the plate surface and is prescribed by the critical Reynolds number which normally ranges from 3×10^5 to 10×10^5.

Drag force for the region of turbulent boundary layer can be estimated from the relation :

$$F_{turb} = F_{total\ turb} - F_{turb\ to\ x_c}$$

where $F_{total\ turb}$ represents the drag which would occur if a turbulent boundary layer extends along the entire length of the plate, and $F_{turb\ to\ x_c}$ represents the drag due to fictitious turbulent boundary layer from the leading edge to distance x_c (Fig. 12.16)

If the plate is assumed to be long enough so that Reynolds number is greater than 10^7, then turbulent drag per unit width of the plate is

$$F_{turb} = \frac{0.455}{(\log_{10} Re_l)^{2.58}} \times \frac{1}{2}\rho U_\infty^2 (l \times 1)$$

$$- \frac{0.072}{(Re_c)^{0.2}} \times \frac{1}{2}\rho U_\infty^2 (x_c \times 1)$$

$$= \left[\frac{0.455l}{(\log_{10} Re_l)^{2.58}} - \frac{0.072x_c}{(Re_c)^{0.2}}\right]\frac{\rho U_\infty^2}{2} \quad ...(12.99)$$

Within the length x_c, laminar boundary layer prevails and its contribution towards drag force is

$$F_{laminar} = \left[\frac{1.328x_c}{\sqrt{Re_c}}\right]\frac{\rho U_\infty^2}{2} \quad ...(12.100)$$

and as such the total drag force equals :

$$F_{total} = \left[\frac{1.328x_c}{(Re_c)^{0.5}} + \frac{0.455l}{(\log_{10} Re_l)^{2.58}} - \frac{0.072x_c}{(Re_c)^{0.2}}\right]\frac{\rho U_\infty^2}{2} \quad ...(12.101)$$

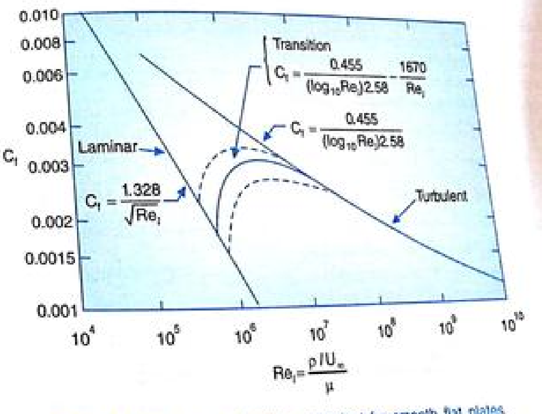

Fig. 12.17. Drag (skin friction) coefficient for smooth flat plates

and it represents the Nusselt number divided by the products of the Reynolds and Prandtl numbers, i.e.,

$$\frac{Nu_x}{Re_x\,Pr} = St_x = \frac{C_{fx}}{2} \qquad \text{...(12.107)}$$

The physical significance of Stanton number is

$$St = \frac{h_x\,\Delta t}{\rho c_p U_\infty \Delta t}$$

$$= \frac{\text{actual heat flux to the fluid}}{\text{heat flux capacity of the fluid flow}}$$

Equation 12.107 is called the **Reynolds analogy** and is an excellent example of the similar nature of energy and momentum transfer. This inter-relationship can be used directly to infer heat transfer data from measurement of shear stress.

Let us examine the exact result obtained from the laminar boundary layer on a flat plate

$$Nu_x = x\frac{h_x}{k} = 0.332\,(Re_x)^{1/2}\,(Pr)^{1/3} \qquad \text{...(a)}$$

$$C_{fx} = \frac{0.664}{(Re_x)^{1/2}} \qquad \text{...(b)}$$

Dividing both sides of expression (*a*) by the product $Re_x\,Pr^{1/3}$;

$$\frac{Nu_x}{Re_x\,Pr^{1/3}} = \frac{0.332}{(Re_x)^{1/2}} = \frac{C_{fx}}{2} \qquad \text{...(12.108)}$$

The left hand side of this equality can be recast as :

$$\frac{Nu_x}{Re_x\,Pr^{1/3}} = \frac{Nu_x}{Re_x\,Pr}\,Pr^{2/3}$$

$$= St_x\,Pr^{2/3} \qquad \text{...(12.109)}$$

The inter relationship between heat and momentum transfer then becomes

$$St_x\,Pr^{2/3} = \frac{C_{fx}}{2} \qquad \text{...(12.110)}$$

The above correlation for laminar boundary layer on a flat plate was applied by Colburn to a wide range of data for flow and configurations of all types and found to be quite accurate provided that (i) drag forces are wholly viscous in nature, i.e., no form or pressure drag and (ii) 0.5 < Pr < 50.

Equation 12.110 has been designated as **Colburn analogy**. It is to be noted that for Pr = 1, the Colburn and Reynolds analogies are the same.

The heat transfer parameters for turbulent boundary layer flow past a flat plate can be worked out by the application of Colburn analogy. The appropriate expressions for the skin friction coefficient, may be substituted into the Colburn equation to get the heat transfer coefficient. The choice of equation 12.96 gives

$$St_x\,Pr^{2/3} = \frac{C_{fx}}{2} = \frac{0.0576}{2}\,(Re_x)^{-1/5}$$

or $$\frac{Nu_x}{Pr\,Re_x}\,Pr^{2/3} = 0.0288\,(Re_x)^{-1/5}$$

or $$Nu_x = 0.0288\,(Re_x)^{4/5}\,(Pr)^{1/3}$$

or $$h_x = 0.0288\,\frac{k}{x}\,(Re_x)^{4/5}\,(Pr)^{1/3} \qquad \text{...(12.111)}$$

Taking averages over the interval $0 < x < l$

$$\bar{h} = \frac{1}{l}\int_0^l h_x\,dx$$

$$= \frac{1}{l}\int_0^l 0.0288\,\frac{k}{x}\,(Re_x)^{4/5}\,(Pr)^{1/3}\,dx$$

$$= 0.0288\,k\left(\frac{\rho U_\infty}{\mu}\right)^{4/5}(Pr)^{1/3} \times \frac{1}{l}\int_0^l x^{-1/5}\,dx$$

$$= 0.0288\,k\left(\frac{\rho U_\infty}{\mu}\right)^{4/5}(Pr)^{1/3} \times \frac{5}{4l}\,(l)^{4/5}$$

$$= 0.036\,\frac{k}{l}\left(\frac{l\rho U_\infty}{\mu}\right)^{4/5}(Pr)^{1/3}$$

$$= 0.036\,\frac{k}{l}\,(Re_l)^{4/5}\,(Pr)^{1/3} \qquad \text{...(12.112)}$$

and $$\overline{Nu} = \frac{\bar{h}l}{k}$$

$$= 0.036\,(Re_l)^{4/5} \times (Pr)^{1/3} \qquad \text{...(12.113)}$$

Combination of laminar and turbulent flow : An expression for the average heat transfer coefficient over a plate of length l when both laminar and turbulent boundary layers are present can be worked out by evaluating the integral ;

$$\bar{h} = \frac{1}{l}\left[\int_0^{x_c} h_x\, dx + \int_{x_c}^{l} h_x\, dx\right]$$

$$= \frac{1}{l}\left[\int_0^{x_c} 0.332\frac{k}{x}(Re_x)^{1/2}(Pr)^{1/3}\, dx + \int_{x_c}^{l} 0.0288\frac{k}{x}(Re_x)^{4/5}(Pr)^{1/3}\, dx\right]$$

$$= \frac{k}{l}(Pr)^{1/3}\left[0.332\left(\frac{\rho U_\infty}{\mu}\right)^{1/2} \int_0^{x_c} x^{-1/2}\, dx + 0.0288\left(\frac{\rho U_\infty}{\mu}\right)^{4/5}\int_{x_c}^{l} x^{-1/5}\, dx\right]$$

$$= \frac{k}{l}(Pr)^{1/3}\left[0.664(Re_x)^{1/2} + 0.036\left\{(Re_l)^{0.8} - (Re_c)^{0.8}\right\}\right] \quad ...(12.114)$$

Presuming that transition occurs at a critical Reynolds number of $Re_c = 5 \times 10^5$, we obtain :

$$\bar{h} = \frac{k}{l}(Pr)^{1/3}\left[0.664\left(5 \times 10^5\right)^{1/2} + 0.036(Re_l)^{0.8} - 0.036\left(5\times 10^5\right)^{0.8}\right]$$

$$= \frac{k}{l}(Pr)^{1/3}\left[0.036(Re_l)^{0.8} - 836\right] \quad ...(12.115)$$

and $$\overline{Nu} = \frac{\bar{h}l}{k}$$

$$= (Pr)^{1/3}\left[0.036(Re_l)^{0.8} - 836\right] \quad ...(12.116)$$

EXAMPLE 12.16.

A flat plate was positioned at zero incidence in a uniform flow stream of air. Assuming boundary layer to be turbulent over the entire plate, workout the ratio of skin-friction forces on the front and rear half part of the plate.

Solution : For turbulent boundary, the average drag (skin friction) coefficient is prescribed by the relation :

$$\bar{C}_f = \frac{0.072}{(Re_l)^{0.2}}$$

For the entire plate $R_{el} = l\,U_\infty/\nu$ and for the first half of the plate $Re_l = (l/2 \times U_\infty)/\nu$

Drag force, per unit width, for the entire plate is

$$F = \bar{C}_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area per unit width}$$

$$= \frac{0.072}{(lU_\infty/\nu)^{0.2}} \times \frac{1}{2}\rho U_\infty^2 \times l$$

Likewise, the drag force per unit width for the front half portion of the plate is :

$$F_1 = \frac{0.072}{(lU_\infty/2\nu)^{0.2}} \times \frac{1}{2}\rho U_\infty^2 \times \frac{l}{2}$$

$$= \frac{0.072}{(lU_\infty/\nu)^{0.2}} \times \frac{1}{2}\rho U_\infty^2 \times \frac{l}{2} \times (2)^{0.2}$$

$\therefore$ Drag force for the rear half of the plate is

$$F_2 = F - F_1$$

$$= \frac{0.072}{(lU_\infty/\nu)^{0.2}} \times \frac{1}{2}\rho U_\infty^2\left[1 - \frac{1}{2}(2)^{0.2}\right] \times l$$

Hence,

$$\frac{F_1}{F_2} = \left[\frac{\frac{1}{2}\times(2)^{0.2}}{1 - \frac{1}{2}(2)^{0.2}}\right]$$

$$= \frac{0.574}{1 - 0.574} = \mathbf{1.347}$$

EXAMPLE 12.17.

How does the boundary layer thickness for flow over a flat plate vary with distance from the leading edge for (i) laminar flow, and for (ii) turbulent flow.

The average drag coefficient for turbulent boundary layer flow past a thin plate is given by :

$$C_f = \frac{0.455}{(\log_{10} R_{el})^{2.58}}$$

where Re_l is the Reynolds number based on plate length. A plate 50 cm wide and 5 m long is kept parallel to the flow of water with free stream velocity 5 m/s. Calculate the drag force on both sides of the plate. For water, kinematic viscosity

$$\nu = 0.01 \text{ stokes.}$$

Solution : Reynolds number at the end of plate,

$$Re_l = \frac{lU_\infty}{\nu} = \frac{5 \times 3}{0.01 \times 10^4} = 15 \times 10^{-2}$$

$\therefore$ Average skin fricition coefficient

$$C_f = \frac{0.455}{(\log_{10} 15 \times 10^6)^{2.58}}$$

$$= 0.282 \times 10^{-2}$$

Drag force on one side of plate

$$F = C_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area of plate surface}$$

$$= 0.282 \times 10^{-2}\left(\frac{1}{2} \times 1000 \times 3^2\right) \times (5 \times 0.5)$$

$$= 31.69 \text{ N}$$

Hence drag force on both sides of plate is

$$= 31.69 \times 2 = \mathbf{63.38\ N}$$

EXAMPLE 12.18.

During test-run in a wind tunnel, air at 215 m/s velocity and 25°C temperature is made to flow past a smooth thin model airfoil which can be idealised as a flat plate. If the chord length of the airfoil is 15 cm, make calculations for the drag per unit width.

The relevant physical properties of air are :

$$\rho = 1.82 \text{ kg/m}^3$$
$$\nu = 15.53 \times 10^{-6} \text{ m}^2/\text{s}$$

Solution : The Reynolds number at the trailing edge is

$$Re_l = \frac{lU_\infty}{\nu} = \frac{0.15 \times 215}{15.53 \times 10^{-6}} = 2.077 \times 10^{4}$$

The boundary layer is turbulent since the Reynolds number is greater than 5×10^5, the average skin friction (drag) coefficient is

$$C_f = \left[\frac{0.072}{(Re_l)^{0.2}} - \frac{A}{Re_l}\right]$$

$$= \frac{0.072}{(2.077 \times 10^6)^{0.2}} - \frac{1670}{2.077 \times 10^6}$$

$$= 0.003107$$

Drag force $F = 2C_f \times \left(\frac{1}{2}\rho U_\infty^2\right) \times (l \times b)$

The factor of 2 accounts for both sides of the air foil

$$\therefore F = 2 \times 0.003107 \times \left(\frac{1}{2} \times 1.182 \times 215^2\right) \times (0.15 \times 1)$$

$$= \mathbf{25.42\ N} \text{ per metre width}$$

EXAMPLE 12.19.

For a particular engine, the underside of the crankcase can be idealised as a flat plate measuring 80 cm × 20 cm. The engine runs at 80 km/hr and the crankcase is cooled by the air flowing past it at the same speed. Make calculations for the loss of heat from the crank case surface (t_s = 75°C) to the ambient air (t_∞ = 25°C). Due to road induced vibrations, the boundary layer becomes turbulent from the leading edge itself.

Solution : At the mean film temperature

$$t_f = \frac{t_s + t_\infty}{2}$$

$$= \frac{75 + 25}{2} = 50°\text{C}$$

the thermo-physical properties of air are :

$$k = 2.824 \times 10^{-2} \text{ W/mK}$$
$$\nu = 17.95 \times 10^{-6} \text{ m}^2/\text{s}$$

and $Pr = 0.698$

Engine speed = 80 km/hr

$= (80 \times 1000/3600)$

$= 22.22$ m/s

$$Re_l = \frac{\rho U_\infty l}{\mu} = \frac{U_\infty l}{\nu} = \frac{0.8 \times 22.22}{17.95 \times 10^{-6}}$$

$= 0.99 \times 10^6$

For turbulent boundary layer : $\overline{Nu}$

$= 0.036\,(Re_l)^{0.8}\,(Pr)^{0.33}$

$= 0.036 \times (0.99 \times 10^6)^{0.8} \times (0.698)^{0.33}$

$= 2000.89$

$$\text{Heat transfer coefficient, } \bar{h} = \frac{\overline{Nu} \times k}{l}$$

$$= \frac{2000.89 \times 2.824 \times 10^{-2}}{0.8}$$

$= 70.63$ W/m^2 K

$\therefore$ Heat lost by the crankcase Q

$= \bar{h}\,A\,\Delta T$

$= 70.63 \times (0.8 \times 0.2) \times (75 - 25)$

$= 565.04$ W

EXAMPLE 12.20.

A flat plate 1 m × 1 m is placed in a wind tunnel. The velocity and temperature of free stream air are 80 m/s and 10°C respectively. The flow over the whole length of the plate is made turbulent by turbulizing grid placed upstream of the plate. Make calculations for the following parameters :

(a) thickness of hydrodynamic boundary layer at trailing edge of the plate

(b) heat flow from the surface of the plate.

The plate is maintained at 50°C and use the following thermo-physical properties of air

$\rho = 1.25$ kg/m^3

$k = 0.022$ W/m-deg

$\nu = 14.15 \times 10^{-6}$ m^2/s

$c_p = 1000$ J/kg K

Solution : Flow Reynolds number Re_l

$$= \frac{\rho U_\infty l}{\mu} = \frac{U_\infty l}{\nu}$$

$$= \frac{80 \times 1}{14.15 \times 10^{-6}}$$

$= 5.65 \times 10^6$

For turbulent boundary layer, the boundary layer thickness is prescribed by the relation.

$$\frac{\delta}{l} = \frac{0.371}{(Re_l)^{0.2}}$$

$$\therefore \quad \delta = \frac{0.371}{(5.65 \times 10^6)^{0.2}} \times 1$$

$= 0.01655$ m $= 16.55$ mm

(b) Prandtl number Pr

$$= \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k}$$

$$= \frac{1.25 \times 14.15 \times 10^{-6} \times 1000}{0.022}$$

$= 0.804$

$$\overline{Nu} = \frac{\bar{h} l}{k}$$

$= 0.036\,(Re_l)^{0.8}\,(Pr)^{0.33}$

$= 0.036\,(5.65 \times 10^6)^{0.8}\,(0.804)^{0.33}$

$= 8437$

$$\bar{h} = \overline{Nu}\,\frac{k}{l} = 8437 \times \frac{0.022}{1}$$

$= 185.61$ W/m^2-deg

$\therefore$ Heat flow from the plate (one side)

$= h\,A\,\Delta t$

$= 185.61 \times (1 \times 1) \times (50 - 10)$

$= 7424.5$ W

EXAMPLE 12.21.

Air at atmospheric pressure and 20°C flows past a flat plate with a velocity of 4 m/s. The plate is 30 cm wide, is heated uniformly throughout its entire length and is maintained at a surface temperature of 60°C. Make calculations for the following parameters at 40 cm distance from the leading edge :

(a) thickness of hydrodynamic and thermal boundary layers

(b) local and average friction coefficient

(c) local and average heat transfer coefficient and

(d) total drag force on the plate

Take the following thermo-physical properties of air at the mean film temperature of 40°C :

$\rho = 1.18$ kg/m^3

$\nu = 17 \times 10^{-6}$ m^2/s

$c = 1007$ J/kg-deg

and $k = 0.0272$ W/m-deg

Solution : Flow Reynolds number, Re_x

$$= \frac{x U_\infty}{\nu} = \frac{0.4 \times 4}{17 \times 10^{-6}}$$

$= 9.41 \times 10^{4}$

Since the Reynolds number is less than 5×10^5, The boundary layer is laminar in character. Assuming cubic velocity profile

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\therefore \quad \delta = 0.4 \times \frac{4.64}{\sqrt{9.41 \times 10^4}}$$

$= 0.00605$ m $= 6.05$ mm

When the plate is heated over the entire length of the plate, the hydrodynamic and thermal boundary layer thicknesses are related to each other by the expression

$$\delta_t = \frac{0.976\,\delta}{(Pr)^{0.33}}$$

where Prandtl number Pr

$$= \frac{\mu c_p}{k}$$

$$= \frac{\rho \nu c_p}{k} = \frac{1.18 \times 17 \times 10^{-6} \times 1007}{0.0272}$$

$= 0.742$

$$\therefore \quad \delta_t = \frac{0.976 \times 6.05}{(0.742)^{0.33}} = \frac{0.976 \times 6.05}{0.906}$$

$= 6.517$ mm

(b) Local friction coefficient at $x = 0.4$ m is given by

$$C_{fx} = \frac{0.664}{\sqrt{Re_x}}$$

$$= \frac{0.664}{\sqrt{9.41 \times 10^4}} = 0.00216$$

Average friction coefficient $\bar{C}_f$

$= 2 C_{fx} = 2 \times 0.00216 = 0.00432$

(c) The local Nusselt number at $x = l = 0.4$ m is given by

$$Nu_x = h_x \frac{x}{k}$$

$= 0.332\,(Re_x)^{0.5}\,(Pr)^{0.33}$

$= 0.332\,(9.41 \times 10^4)^{0.5}\,(0.742)^{0.33}$

$= 92.27$

Local heat transfer coefficient h_x

$$= Nu_x \times \frac{k}{x}$$

$$= 92.27 \times \frac{0.0272}{0.4}$$

$= 6.274$ W/m^2 K

$\therefore$ Average heat transfer coefficient

$\bar{h} = 2 h_x$

$= 2 \times 6.274 = 12.548$ W/m^2 K

(d) Friction (drag) force

$$= \bar{C}_f \times \frac{1}{2} \rho U_\infty^2$$

$\times$ area of plate for one side

$$= 0.0042 \times \left(\frac{1}{2} \times 1.18 \times 4^2\right) \times (0.4 \times 0.3)$$

$= 0.00476$ N

EXAMPLE 12.22.

A refrigerated truck is travelling on the high way at a speed of 80 km/hr in a desert area where the temperature is 70°C. The body of the truck is considered to be a rectangular box 3 m wide × 2 m high × 4.5 m long. The boundary layer is turbulent over the entire surface and temperature of the surface is uniform at 10°C. Estimate (i) heat loss from four surfaces, (ii) tonnage of the refrigeration required, and (iii) power required to overcome resistance acting on the four surfaces. Neglect any heat transfer and the frictional resistance from the front and back end of the truck ; and presume that for every 3500 W of heat loss we need one ton capacity of the refrigerating unit.

Solution : At the mean film temperature

$$t_f = \frac{70 + 10}{2} = 40°\text{C}$$

the thermophysical properties of air are :

$$\rho = 1.127 \text{ kg/m}^3;$$
$$\mu = 19.11 \times 10^{-6} \text{ kg/ms}$$
$$c_p = 1007 \text{ J/kg-deg}$$
and $$k = 27.1 \times 10^{-3} \text{ W/m-deg}$$

Using these properties, the pertinent parameters are :

$$Pr = \frac{\mu c_p}{k} = \frac{19.11 \times 10^{-6} \times 1007}{27.1 \times 10^{-3}} = 0.71$$

Velocity of truck,

$$V = (80 \times 1000)/3600 = 22.22 \text{ m/s}$$

$$Re = \frac{Vl\rho}{\mu} = \frac{22.22 \times 4.5 \times 1.127}{19.11 \times 10^{-6}} = 5.89 \times 10^6$$

Apparently the boundary layer is turbulent for which

$$Nu = 0.036 \, (Re)^{0.8} \, (Pr)^{0.33}$$

or $$\frac{hl}{k} = 0.036 \, (5.89 \times 10^6)^{0.8} \, (0.71)^{0.33} = 8381$$

$\therefore$ Heat transfer co-efficient, h

$$= 8381 \times 27.1 \times 10^{-3}/4.5 = 50.47 \text{ W/m}^2\text{-deg}$$

and heat lost from the four sides of the truck

$$= h \, A \, \Delta t$$
$$= 50.47 \times [2 \, (4.5 \times 2 + 4.5 \times 3)] \, (70 - 10)$$
$$= \mathbf{136269 \ W}$$

(b) The cooling capacity required in tons of refrigeration

$$= \frac{136269}{3500} = \mathbf{38.96 \ tons}$$

(c) Average skin friction coefficient,

$$\bar{C}_f = \frac{0.072}{(Re_l)^{0.2}} = \frac{0.072}{(5.89 \times 10^6)^{0.2}} = 0.00318$$

$$\text{Drag force} = \bar{C}_f \times \left(\frac{1}{2} \rho V^2\right) \times \text{area}$$

$$= 0.00318 \times \left(\frac{1}{2} \times 1.127 \times 22.22^2\right) \times \left[2(4.5 \times 2 + 4.5 \times 3)\right]$$

$$= 38.81 \text{ N}$$

$$\text{Power} = \text{drag force} \times \text{speed} = 39.81 \times 22.22 = 884.58 \text{ Nm/s} = \mathbf{884.58 \ W}$$

EXAMPLE 12.23.

Air at 20°C flows over a flat plate, 1.5 m long and 1.2 m wide and the plate surface is maintained at 80°C. If the rate of energy dissipation from one side of the plate is 3800 W, calculate the velocity at which the air must flow over the plate along its length.

Solution : Heat lost from the plate to air is given by

$$Q = h \, A \, \Delta T$$
$$3800 = h \times (1.5 \times 1.2) \times (80 - 20) = 108 \, h$$

$\therefore$ heat transfer coefficient, h

$$= 3800/108 = 35.18 \text{ W/m}^2\text{-deg}$$

Let it be presumend that the

(i) boundary layer is turbulent, and

(ii) transition occurs at critical Reynolds number of 5×10^5

The average heat transfer coefficient is then given by

$$h = \frac{k}{l}\left[0.036 \, (Re_l)^{0.8} - 836\right](Pr)^{0.33} \qquad ...(i)$$

At the mean film temperature

$$t_f = \frac{80 + 20}{2} = 50°\text{C}$$

the thermo-physical properties of air are :

$$\rho = 1.09 \text{ kg/m}^3$$
$$c_p = 1007 \text{ J/kg-deg}$$
$$\mu = 19.56 \times 10^{-6} \text{ kg/ms}$$
and $$k = 27.81 \times 10^{-3} \text{ W/m-deg}$$

$$\therefore \quad Pr = \frac{\mu c_p}{k} = \frac{19.56 \times 10^{-6} \times 1007}{27.81 \times 10^{-3}} = 0.708$$

Inserting the appropriate values in expression (i),

$$35.18 = \frac{27.81 \times 10^{-3}}{1.5} \times \left[0.036 (Re_l)^{0.8} - 836\right](0.708)^{0.33}$$
$$= 0.0165 \left[0.036 (R_el)^{0.8} - 836\right]$$

$\therefore$ Flow Reynolds number,

$$Re_l = \left[\left(\frac{35.18}{0.0165} + 836\right) \times \frac{1}{0.036}\right]^{1/0.8} = 1.397 \times 10^6$$

Obviously the assumption of turbulent flow is correct.

Now, $Re_l = V \, l \, \rho / \mu$

$\therefore$ Flow velocity,

$$V = \frac{Re_l \times \mu}{l \rho} = \frac{1.397 \times 10^6 \times 19.57 \times 10^{-6}}{1.5 \times 1.09} = \mathbf{16.71 \ m/s}$$

EXAMPLE 12.24.

Air at 10°C flows past a flat plate 1 m wide × 1.5 m long. The plate is maintained at 90°C temperature and dissipates 3.75 kW of energy. Determine the convective coefficient and the velocity at which air flows over the plate

Adopt the following correlations

$$Nu = \frac{hl}{k} = 0.664 \, (Re)^{0.5} \, (Pr)^{0.33}$$

for laminar flow

$$= \frac{hl}{k} = \left[0.036 \, Re^{0.8} - 836\right] \times (Pr)^{0.33}$$

for turbulent flow

At the mean temperature of 50°C, the thermo-physical properties of air are :

$$\rho = 1.09 \text{ kg/m}^3$$
$$k = 0.028 \text{ W/m-deg}$$
$$Pr = 0.703$$
$$c_p = 1007.5 \text{ J/kg-deg}$$
and $$\mu = 2.029 \times 10^{-5} \text{ kg/m-s}$$

Solution : Convection heat flow, Q

$$= h \, A \, \Delta t$$
$$3.75 \times 10^3 = h \times (2 \times 1 \times 1.5) \times (90 - 10)$$

$\therefore$ Convective coefficient h

$$= \frac{3.75 \times 10^3}{3.0 \times 80} = 15.625 \text{ W/m}^2\text{-deg}$$

$$Nu = \frac{hl}{k} = \frac{15.625 \times 1.5}{0.028} = 837.05$$

Assuming laminar flow along the plate

$$Nu = 0.664 \, (Re)^{0.5} \, (Pr)^{0.33}$$
$$837.05 = 0.664 \, (Re)^{0.5} \times (0.703)^{0.33}$$
$$Re = 2.01 \times 10^6$$

The Reynolds number is greater than the critical Reynolds number 2×10^5 ; the assumption made of laminar flow is wrong.

As such the fluid flow is turbulent. Then

$$Nu = \frac{hl}{k} = \left[0.036 \, Re^{0.8} - 836\right] \times Pr^{0.33}$$

Considering flow parallel to length of tube,

$$837.05 = \left[0.036 \, Re^{0.8} - 836\right] \times 0.703^{0.33}$$

or $$Re^{0.8} = \left[\frac{837.05}{(0.703)^{0.33}} + 836\right] \times \frac{1}{0.036} = 49341$$

or $$Re = \frac{\rho V l}{\mu} = (49341)^{1/0.8} = 7.354 \times 10^5$$

$\therefore$ Air flow velocity, V

$$= (7.354 \times 10^5) \times \frac{\mu}{\rho l} \quad \left(R_e = \frac{\rho V l}{\mu}\right)$$
$$= (7.354 \times 10^5) \times \frac{2.029 \times 10^{-5}}{1.09 \times 1.5} = 9.126 \text{ m/s}$$

EXAMPLE 12.25.

A smooth, thin model airfoil is to be tested for lift and drag in a wind tunnel. To obtain the desired

[illegible]

It may be assumed that flat plate theory is applicable and take

$\nu = 15.6 \times 10^{-6}\ m^2/s$

$k = 0.026\ W/m\text{-}deg$

and $Pr = 0.707$

Solution : $Re_L = \frac{V_\infty L}{\nu} = \frac{30 \times 0.8}{15.6 \times 10^{-6}}$

$= 1.026 \times 10^6 > 7.5 \times 10^5$

Obviously the boundary layer is turbulent and therefore

$Nu_L = 0.0288\,(Re_L)^{4/5}\,(Pr)^{1/3}$

$= 0.0288 \times (1.026 \times 10^6)^{4/5}\,(0.707)^{1/3}$

$= 1654.35$

$h_L = Nu_L \frac{k}{L} = 1654.35 \times \frac{0.026}{0.8}$

$= 53.76\ W/m^2\text{-}deg$

(b) $Re_L = \frac{30 \times 1.5}{15.6 \times 10^{-6}}$

$= 1.923 \times 10^6$ (turbulent)

When transition occurs at critical Reynolds number of 5×10^5, the average Nusselt number is given by

$$Nu = \left[0.036 \times (Re_L)^{4/5} - 836\right] Pr^{1/3}$$

$$= \left[0.036 \times \left(1.923 \times 10^6\right)^{4/5} - 836\right] (0.707)^{1/3}$$

$$h = Nu \times \frac{k}{L} = [illegible]$$

$= 46.44\ W/m^2\text{-}deg$

(c) Heat transfer per metre of width of plate (taking both sides into account)

$\frac{Q}{w} = h\,(2L)\,\Delta t$

$= 46.44 \times 2 \times 1.5 \times$ [illegible]

$=$ [illegible]

EXAMPLE 12.26

Atmospheric air at 25°C flows with a velocity 40 m/s past a flat plate 0.75 m long and 0.4 m wide. The surface of the plate is maintained at 275°C. Estimate (a) the total heat transferred and the total drag force on one side of the plate, (b) fraction of the heat transfer and drag force for the laminar portion of flow. What error in part (a) would result if the boundary layer is assumed to be turbulent from the leading edge?

Solution : At the mean film temperature

$$t_f = \frac{275 + 25}{2} = 150°C,$$

the thermo physical properties of air are

$\rho = 0.834\ kg/m^3$

$\mu = 23.84 \times 10^{-6}\ kg/ms$

$c_p = 1017\ J/kg\text{-}deg$

and $k = 34.59 \times 10^{-3}\ W/m\text{-}deg$

From these properties, the relevant parameters are :

$$Pr = \frac{\mu c_p}{k} = \frac{23.84 \times 10^{-6} \times 1017}{34.59 \times 10^{-3}}$$

$= 0.70$

$$Re_l = \frac{Vl\rho}{\mu} = \frac{40 \times 0.75 \times 0.834}{23.84 \times 10^{-6}}$$

$= 1.05 \times 10^6$

Presuming that transition occurs at critical Reynolds number of 5×10^5, the average Nusselt number is

$$Nu = \left[0.036 Re_l^{0.8} - 836\right] (Pr)^{0.33}$$

$$= \left[0.036 \times \left(1.05 \times 10^6\right)^{0.8} - 836\right] \times (0.70)^{0.33}$$

$= 1356$

$$h = Nu \times \frac{k}{l} = \frac{1356 \times 34.59 \times 10^{-3}}{0.75}$$

$= 62.56\ W/m^2\text{-}deg$

and heat transfer from one side of the plate is

$= h\,A\,\Delta t$

$= 62.56 \times (0.75 \times 0.4) \times (275 - 25)$

$= 4666\ W$

Further skin friction or drag coefficient is

$$C_D = \frac{0.072}{(Re_l)^{0.2}} - \frac{1670}{Re_l}$$

$$= \frac{0.072}{\left(1.05 \times 10^6\right)^{0.2}} - \frac{1670}{1.05 \times 10^6}$$

$= 0.0029$

and drag force on one side of the plate is

$= C_D \times \frac{1}{2}\rho V^2$

$\times$ area of plate surface

$$= 0.0029 \times \left(\frac{1}{2} \times 0.834 \times 40^2\right) \times (0.75 \times 0.4)$$

$= 0.58\ N$

(b) For the laminar portion it is necessary to locate the transition point. The distance x_c from the leading edge at which the transition takes place is

$$x_c = \frac{Re_c \times \mu}{\rho V}$$

$$= \frac{5 \times 10^5 \times 23.84 \times 10^{-6}}{0.834 \times 40} = 0.357$$

For laminar portion, average value of Nusselt number is

$Nu = 0.664 \times (Re_c)^{0.5} \times (Pr)^{0.33}$

$= 0.664 \times (5 \times 10^5)^{0.5} \times (0.7)^{0.33}$

$= 417$

$$h = Nu \times \frac{k}{x_c} = \frac{417 \times 34.59 \times 10^{-3}}{0.357}$$

$= 40.4\ W/m^2\text{-}deg$

and the heat transfer from the laminar portion is

$= 40.4 \times (0.357 \times 0.4) \times (275 - 25)$

$= 1443\ W$; 30.9 % of total

Further the drag coefficient for the laminar portion is

$$C_D = \frac{1.328}{\sqrt{Re_c}} = \frac{1.328}{\sqrt{5 \times 10^5}} = 0.00188$$

and hence the drag force from the laminar portion is

$$= 0.00188 \times \left(\frac{1}{2} \times 0.834 \times 40^2\right) \times (0.357 \times 0.4)$$

$= 0.179\ N$; 30.9% of total

The calculations made above clearly indicate that fractions of the heat transfer and drag force from the laminar portion of the flow are equal.

(c) If boundary layer is considered to be turbulent from the leading edge, then

$Nu = 0.036\,(Re_l)^{0.8} \times (Pr)^{0.33}$

$= 0.036 \times (1.05 \times 10^6)^{0.8} \times (0.70)^{0.33}$

$= 2100$

$$h = \frac{2100 \times 34.59 \times 10^{-3}}{0.75}$$

$= 96.85\ W/m^2\text{-}deg$

$Q = 96.85 \times (0.75 \times 0.4) \times (275 - 25)$

$= 7264\ W$

which is $(7264 - 4666)/4666 = 55\%$ higher than that found in part (a).

Further,

$$C_D = \frac{0.072}{(Re_l)^{0.2}} = \frac{0.072}{\left(1.05 \times 10^6\right)^{0.2}}$$

$= 0.0045$

Drag force $= 0.0045$

$$\times \left(\frac{1}{2} \times 0.834 \times 40^2\right) \times (0.75 \times 0.4)$$

$= 0.90\ N$

which is $(0.90 - 0.58)/0.58 = 55\%$ *higher than that found in part (a).*

EXAMPLE 12.27.

Atmospheric air at 25°C flows at 50 m/s velocity past a flat plate 0.6 m long with its surface maintained at 295°C. Under these conditions, the air may be treated as incompressible. Make calculations for heat transferred to air from the entire plate length taking into account both laminar and turbulent portions of boundary layer. Presume unit width of plate and the critical Reynolds number to be 5 × 10⁵. What percentage error would be introduced if the boundary layer is presumed to be of turbulent nature from the very leading edge of plate ?

Solution : At the mean film temperature

$$t_f = \frac{t_s + t_\infty}{2} = \frac{295 + 25}{2} = 160°C$$

the thermo-physical properties of air are:

$$k = 0.1310 \text{ kJ/mhrK}$$
$$\nu = 30.09 \times 10^{-6} \text{ m}^2/\text{s}$$
$$Pr = 0.682$$

For the entire plate length, the Reynolds number is

$$Re_l = \frac{\rho l U_\infty}{\mu} = \frac{l U_\infty}{\nu} = \frac{0.6 \times 50}{30.90 \times 10^{-6}} = 9.97 \times 10^5$$

The distance x_c from the leading edge at which transition takes place is

$$x_c = \frac{Re_c\, \nu}{U_\infty} = \frac{5 \times 10^5 \times 30.09 \times 10^{-6}}{50} = 0.3009 \text{ m}$$

(i) For the laminar boundary layer region, the average heat transfer coefficient is

$$\bar{h}_l = 0.664 \frac{k}{x_c} (Re_c)^{0.5} (Pr)^{0.33}$$
$$= 0.664 \times \frac{0.1310}{0.3009} \times \left(5 \times 10^5\right)^{0.5} \times (0.632)^{0.33}$$
$$= 180.05 \text{ kJ/m}^2\text{-hr-deg}$$

and heat transfer from the laminar portion is

$$Q_l = \bar{h}_l\, A\, \Delta t$$
$$= 180.05 \times (0.3009 \times 1) \times (295 - 25)$$
$$= 14628 \text{ kJ/hr}$$

(ii) The average heat transfer coefficient of the region of turbulent boundary layer is

$$\bar{h}_t = 0.036 \frac{k}{l - x_c}\left[(Re_l)^{0.8} - (Re_c)^{0.8}\right](Pr)^{0.33}$$
$$= 0.36 \times \frac{0.1310}{0.6 - 0.3009}\left[\left(9.97 \times 10^5\right)^{0.8} - \left(5 \times 10^5\right)^{0.8}\right] \times (0.682)^{0.33}$$
$$= 370.94 \text{ kJ/m}^2\text{-hr-deg}$$

and heat transfer from turbulent portion is

$$Q_t = \bar{h}_t\, A\, \Delta t$$
$$370.94 = (0.2991 \times 1) \times (295 - 25)$$
$$= 29956 \text{ kJ/hr}$$

Total heat transfer from the plate

$$Q = Q_l + Q_t$$
$$= 14628 + 29956 = 44584 \text{ kJ/hr}$$

The solution could also be obtained by using the direct formula for overall heat transfer coefficient

$$h = \frac{k}{l}\left[0.036 \times (Re_l)^{0.8} - 836\right] Pr^{0.33}$$
$$= \frac{0.1310}{0.6}\left[0.036 \times \left(9.97 \times 10^5\right)^{0.8} - 836\right] \times (0.682)^{0.33}$$
$$= 275.06 \text{ kJ/m}^2\text{-hr-deg}$$

$$\therefore \quad Q = 275.06 \times (0.6 \times 1) \times (295 - 25)$$
$$= 44560 \text{ kJ/hr}$$

(b) If the boundary layer is presumed to be turbulent from the very beginning,

$$\bar{h} = 0.036 \frac{k}{l} (Pr)^{0.33} (Re_l)^{0.8}$$
$$= 0.036 \frac{0.1310}{0.6} \times (0.682)^{0.33} \times (9.97 \times 10^5)^{0.8}$$
$$= 435.86 \text{ kJ/m}^2\text{-hr-deg}$$

$$\therefore \quad Q = 435.86 \times (0.6 \times 1) \times (295 - 25)$$
$$= 70609 \text{ kJ/hr}$$

$$\%\text{age error} = \frac{70609 - 44584}{44584} \times 100$$
$$= 58.37\ \% \text{ on the higher side}$$

EXAMPLE 12.28.

In a gas turbine system, hot gases at 1000°C flow at 75 m/s past the surface of a combustion chamber which is at a uniform temperature of 300°C. What would be the heat loss from the gases to the combustion chamber which can be idealised as a flat plate measuring 100 cm × 50 cm ? The flow is parallel to the 100 cm side and the transition from laminar to turbulent conditions is anticipated to be abruptly at a critical Reynolds number of $Re_c = 5 \times 10^5$

Take the following properties of gas :

$$\rho = 0.496 \text{ kg/m}^3$$
$$\nu = 93.5 \times 10^{-6} \text{ m}^2/\text{s}$$
$$k = 0.0744 \text{ W/mK}$$
$$\text{and} \quad Pr = 0.625$$

Solution : The average value of Nusselt number over a flat plate when both laminar and turbulent boundary layers are present is given by :

$$Nu = Pr^{0.33}\left[0.664(Re_l)^{0.5} + 0.036\left\{(Re_l)^{0.8} - (Re_c)^{0.8}\right\}\right]$$

When transition occurs at a critical Reynolds number of $Re_c = 5 \times 10^5$, the above expression simplifies to

$$\overline{Nu} = Pr^{0.33}\left[0.036(Re_l)^{0.8} - 836\right]$$

The Reynolds number at the plate end is

$$Re_l = \frac{l \rho U_\infty}{\mu} = \frac{l U_\infty}{\nu} = \frac{1 \times 75}{93.5 \times 10^{-6}} = 8.02 \times 10^5$$

Apparently the boundary layer becomes turbulent and the above correlation applies.

$$\overline{Nu} = (0.625)^{0.33}\left[0.036\left(8.02 \times 10^5\right)^{0.8} - 836\right]$$
$$= 914.33$$

$$\bar{h} = \overline{Nu} \times \frac{k}{l} = 914.33 \times \frac{0.744}{1}$$
$$= 68.02 \text{ W/m}^2 \text{ K}$$

Heat loss Q

$$= \bar{h}\, A\, \Delta t$$
$$= 68.02 \times (1 \times 0.5) \times (1000 - 300)$$
$$= 23809 \text{ W}$$

EXAMPLE 12.29.

A flat plate, 50 cm long × 75 cm wide and at 90°C, is located in water stream having a free stream velocity of 5 m/s and at 30°C. The transition from laminar to turbulent boundary layer flow occurs at Reynolds number equal to 4 × 10⁵. If flow is parallel to 50 cm side, calculate Nusselt number for the plate and thickness of thermal boundary layer at a distance of 20 cm from the leading edge.

At the mean temperature of 60°C, the thermo-physical properties of water are

$$\rho = 1000 \text{ kg/m}^3$$
$$c_p = 4200 \text{ J/kg K}$$
$$Pr = 13.7$$
$$k = 0.55 \text{ W/m-deg}$$
$$\text{and} \quad \mu = 1780 \times 10^{-6} \text{ kg/ms}$$

Solution : For the entire plate length, the Reynolds number is

$$Re_l = \frac{\rho l U_\infty}{\mu} = \frac{1000 \times 0.5 \times 5}{1780 \times 10^{-6}} = 1.4 \times 10^6$$

The distance x_c from the leading edge at which transition occurs is

$$x_c = \frac{Re_c\, \mu}{\rho U_\infty} = \frac{4 \times 10^5 \times 1780 \times 10^{-6}}{1000 \times 5} = 0.1424 \text{ m} = 14.25 \text{ cm}$$

The average value of heat transfer coefficient can be worked out by using the relation

$$\bar{h} = \frac{k}{l}(Pr)^{0.33}\left[0.664(Re_c)^{0.5} + 0.036\left\{(Re_l)^{0.8} - (Re_c)^{0.8}\right\}\right]$$
$$= \frac{0.55}{0.5}(13.7)^{0.33}\left[0.664\left(4 \times 10^5\right)^{0.5} + 0.036\left\{\left(1.4 \times 10^6\right)^{0.8} - \left(4 \times 10^5\right)^{0.8}\right\}\right]$$

$$= 2.658\,[1.664 + 632.45 + 0.036\,(82385 - 30314)]$$

$$= 6005\ \text{W/m}^2\,\text{K}$$

$$\therefore \quad \overline{Nu} = \frac{\bar{h}l}{k} = \frac{6005 \times 0.5}{0.55} = 5459$$

EXAMPLE 12.30.

An aeroplane flies with 250 km/hr speed at a height where the surrounding air has a temperature 1°C and pressure 68.5 cm of Hg. Consider the wing as an idealised flat plate with flow direction parallel to the width of 1.25 m.

(i) Set up equations for the local heat transfer coefficients in the laminar and turbulent regions of the boundary layer. Transition is anticipated to be at a critical Reynolds number of

$$Re_c = 5 \times 10^5$$

(ii) Compute the heat loss from the wing and the drag force being experienced by it.

Take the following properties of air :

$$\mu = 17.65 \times 10^{-6}\ \text{kg/ms}$$
$$k = 0.092\ \text{kJ/m-hr-deg}$$
$$Pr = 0.707$$

and Gas constant $R = 287$ J/kg-deg

Solution : The pressure of air at the height of flight is

$$P = \frac{68.5}{76} \times 1.0132 = 0.9132\ \text{bar}$$

Mean film temperature t_f

$$= \frac{1 + 23}{2} = 12°\text{C}$$

From the characteristic gas equation

$$\rho = \frac{P}{RT} = \frac{0.9132 \times 10^5}{287 \times (273 + 12)}$$
$$= 1.116\ \text{kg/m}^3$$

Flight speed, U_∞

$$= \frac{250 \times 1000}{3600} = 69.45\ \text{m/s}$$

The Reynolds number for the entire wing is,

$$Re_l = \frac{\rho l U_\infty}{\mu} = \frac{1.116 \times 1.25 \times 69.45}{17.65 \times 10^{-6}}$$
$$= 5.489 \times 10^6$$

(i) For the laminar region ; h_x

$$= 0.332\,\frac{k}{x}\,(Re_x)^{0.5}\,(Pr)^{0.33}$$

$$= 0.332\,k\left(\frac{\rho U_\infty}{\mu}\right)^{0.5}(Pr)^{0.33} \times \frac{1}{x^{0.5}}$$

$$= 0.332 \times 0.092\left(\frac{1.116 \times 69.45}{17.65 \times 10^{-6}}\right)^{0.5} \times (0.707)^{0.33} \times \frac{1}{x^{0.5}}$$

$$= \frac{57.08}{x^{0.5}}$$

For the turbulent zone,

$$h_x = 0.0288\,\frac{k}{x}\,(Re_x)^{0.8}\,(Pr)^{0.33}$$

$$= 0.0288\,k\left(\frac{\rho U_\infty}{\mu}\right)^{0.8}(Pr)^{0.33} \times \frac{1}{x^{0.2}}$$

$$= 0.0288 \times 0.092\left(\frac{1.116 \times 69.45}{17.65 \times 10^{-6}}\right)^{0.8} \times (0.707)^{0.33} \times \frac{1}{x^{0.2}}$$

$$= \frac{487.1}{x^{0.2}}$$

(ii) For the entire plate with considerations of laminar and turbulent boundary layers and with transition at 5×10^5.

$$\bar{h} = \frac{k}{l}\left[0.036(Re_x)^{0.8} - 836\right]Pr^{0.33}$$

$$= \frac{0.092 \times 10^2}{1.25}\left[0.036\left(5.489 \times 10^6\right)^{0.8} - 836\right] \times (0.707)^{0.33}$$

$$= 591.27\ \text{kJ/m}^2\text{-hr-deg}$$

Heat loss

$$= h\,A\,\Delta t \text{ from one side of the wing}$$
$$= 591.27 \times (1.25 \times 1) \times (23 - 1)$$
$$= 16260\ \text{kJ/hr}$$

Average friction coefficient,

$$\bar{C}_f = \left[\frac{0.072}{(Re_l)^{0.2}} - \frac{1670}{Re_l}\right]$$

$$= \frac{0.072}{\left(5.489 \times 10^6\right)^{0.2}} - \frac{1670}{5.489 \times 10^6}$$

$$= 2.925 \times 10^{-3}$$

Drag force

$$= \bar{C}_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area} \quad \text{(one side of the wing)}$$

$$= 2.925 \times 10^{-3} \times \left(\frac{1}{2} \times 1.116 \times 69.45^2\right) \times (1.25 \times 1)$$

$$= 10.21\ \text{N}$$

EXAMPLE 12.31.

A square flat plate is oriented parallel to atmospheric air which flows at 2.5 m/s. The air is at a temperature of $t_\infty = 20°$C while the plate surface is maintained at $t_s = 120°$C. Measurements show that the plate experiences a drag force of 11.75×10^{-3} N. Use the interrelationship between energy and momentum transfer to workout the heat transfer coefficient at the plate surface. Assume laminar flow and take the following properties of air :

$$\rho = 1.029\ \text{kg/m}^3$$
$$\nu = 21.09 \times 10^{-6}\ \text{m}^2/\text{s}$$
$$c_p = 1.009\ \text{kJ/kg K}$$
and $$Pr = 0.694$$

Solution : The drag force is

$$F = \bar{C}_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area}$$

$$= \frac{1.328}{\sqrt{Re_l}} \times \frac{1}{2}\rho U_\infty^2 \times (l \times l) \text{ as } b = l$$

$$= 0.664\left(\frac{\nu}{l U_\infty}\right)^{0.5} \times \frac{1}{2}\rho U_\infty^2 \times l^2$$

$$= 0.664\,\rho U_\infty^2\left(\frac{\nu}{U_\infty}\right)^{0.5} l^{3/2}$$

Inserting the appropriate values

$$11.75 \times 10^{-3} = 0.664 \times 1.029 \times 2.5^2\left(\frac{21.09 \times 10^{-6}}{2.5}\right)^{0.5} l^{3/2}$$

$$= 0.0124\ l^{3/2}$$

$$\therefore \quad l = \left(\frac{11.75 \times 10^{-3}}{0.0124}\right)^{2/3} = 0.964\ \text{m}$$

The Reynolds number at the plate end is,

$$Re_l = \frac{l U_\infty}{\nu} = \frac{0.964 \times 2.5}{21.09 \times 10^{-6}}$$
$$= 0.114 \times 10^6$$

$$\bar{C}_f = \frac{1.328}{\sqrt{Re_l}} = \frac{1.328}{\sqrt{0.114 \times 10^6}}$$
$$= 3.93 \times 10^{-3}$$

From Colburn relationship between energy and momentum transfer,

$$\bar{St}\,Pr^{2/3} = \frac{\bar{C}_f}{2} ; \quad \frac{\bar{h}}{\rho U_\infty c_p}(Pr)^{2/3} = \frac{\bar{C}_f}{2}$$

$$\bar{h} = \frac{\bar{C}_f}{2} \times \frac{\rho U_\infty c_p}{Pr^{2/3}}$$

$$= \frac{3.93 \times 10^{-3}}{2} \times \frac{1.029 \times 2.5 \times 1.009}{(0.694)^{2/3}}$$

$$= 6.497\ \text{W/m}^2\,\text{K}$$

Heat loss, Q

$$= \bar{h}\,A\,\Delta t$$
$$= 6.497 \times (0.964 \times 0.964) \times (120 - 20)$$
$$= 603.80\ \text{W}$$

EXAMPLE 12.32.

Air at 1 atm pressure flows over a plate 0.5 m long with a free stream velocity of 5 m/s. If the drag coefficient for one side of the plate is found to be 0.0037, make calculations for the value of convective heat transfer coefficient.

At the operating mean film temperature, take

$$\rho = 1.06\ \text{kg/m}^3$$
$$c = 1005\ \text{J/kgK}$$
$$k = 0.029\ \text{W/m-deg}$$
and $$Pr = 0.696$$

Solution : Invoking the relation for the average drag (friction) coefficient

$$\bar{C}_f = \frac{1.328}{\sqrt{Re_l}}$$

$$Re_l = \left[\frac{1.328}{\bar{C}_f}\right]^2 = \left[\frac{1.328}{0.0037}\right]^2 = 128823 \text{ (laminar)}$$

$$\overline{Nu} = \frac{\bar{h}l}{k} = 0.664\,(Re_l)^{0.5}\,(Pr)^{0.33} = 0.664\,(128823)^{0.5} \times (0.696)^{0.33} = 211.39$$

$$\therefore \quad \bar{h} = Nu \times \frac{k}{l} = 211.39 \times \frac{0.029}{0.5} = 12.26 \text{ W/m}^2\text{K}$$

Alternatively : From Colburn analojy

$$St\,Pr^{0.667} = \frac{\bar{C}_f}{2} = \frac{0.0037}{2} = 0.00185$$

$$St = \frac{0.00185}{(0.696)^{0.667}} = 0.002356$$

$$\frac{\bar{h}}{\rho c_p U_\infty} = 0.002356$$

$$\therefore \quad \bar{h} = 0.002356 \times 1.06 \times 1005 \times 5 = 12.55 \text{ W/m}^2\text{K}$$

EXAMPLE 12.33.

Air at 20°C and at atmospheric pressure is flowing past a flat plate at 3 m/s velocity. The plate is heated over its entire length to a uniform temperature of 60°C. Calculate the heat transfer from the first 30 cm length of the plate.

Also make calculations for the drag force exerted on the first 30 cm length of plate. Use the analogy between fluid friction and heat transfer.

The relevant thermo-physical properties of air are :

$$\rho = 1.128 \text{ kg/m}^3$$
$$c_p = 1.00 \text{ kJ/kg-deg}$$
$$k = 0.099 \text{ kJ/m-hr-deg}$$
and $$\mu = 19.13 \times 10^{-6} \text{ kg-s/m}$$

Solution : At distance $x = 0.3$ m form the leading edge, the Reynolds number is

$$Re_x = \frac{\rho x U_\infty}{\mu} = \frac{1.128 \times 0.3 \times 3}{19.13 \times 10^{-6}} = 53068$$

$$Pr = \frac{\mu c_p}{k} = \frac{(19.13 \times 10^{-6} \times 3600) \times 1}{0.099} = 0.696$$

The flow is laminar and the following correlation applies for the local Nusselt number

$$Nu_x = 0.332\,(Re_x)^{1/2}\,(Pr)^{1/3} = 0.332 \times (53068)^{1/2} \times (0.696)^{1/3} = 67.89$$

$$h_x = Nu_x \frac{k}{x} = 67.89 \times \frac{0.099}{0.3} = 22.40 \text{ kJ/m}^2\text{-hr-deg}$$

The average value of heat transfer coefficient is twice this value

$$\bar{h} = 2\,h_x = 2 \times 22.40 = 44.80 \text{ kJ/m}^2\text{-hr-deg}$$

The heat loss from one side of plate is,

$$Q = \bar{h}\,A\,\Delta t = 44.80 \times (0.3 \times 1) \times (60 - 20) = 537.6 \text{ kJ/hr}$$

(*ii*) The interrelationship between fluid friction and heat transfer is

$$St\,Pr^{2/3} = \frac{\bar{C}_f}{2}$$

$$\frac{\bar{h}}{\rho U_\infty c_p}(Pr)^{2/3} = \frac{\bar{C}_f}{2}$$

Substituting the appropriate values,

$$\frac{44.80}{1.128 \times 1 \times (3 \times 3600)} \times (0.697)^{2/3} = \frac{\bar{C}_f}{2} \quad \text{or} \quad \bar{C}_f = 5.77 \times 10^{-3}$$

Drag force (one side of plate),

$$F = \bar{C}_f \times \frac{1}{2}\rho U_\infty^2 \times \text{area} = 5.77 \times 10^{-3} \times \left(\frac{1}{2} \times 1.128 \times 3^2\right) \times (0.3 \times 1) = 7.81 \times 10^{-3} \text{ N}$$

EXAMPLE 12.34.

A kitchen in a restaurant has a large, flat burner plate for frying. Since a great deal of heat rises from the plate, the cook decides to let a small fan blow over the burner. The burner length is 1.25 m and it is positioned 1.5 m down a smooth level counter from the fan (total length is 2.75 m). If air at 30°C blows at 2 m/s over the burner plate which is at 120°C; make calculations for the heat transfer rate per square metre at x = 2.75 m.

Solution : At the mean film temperature

$$t_f = \frac{30 + 120}{2} = 75°\text{C}$$

thermo-physical properties of air are :

$$\nu = 20.55 \times 10^{-6} \text{ m}^2/\text{s}$$
$$k = 0.03 \text{ W/m-deg}$$
$$Pr = 0.693$$

At $x = 2.75$ m

$$Re_l = \frac{V_\infty l}{\nu} = \frac{2 \times 2.75}{20.55 \times 10^{-6}} = 2.676 \times 10^5 < 5 \times 10^5$$

Obviously the flow is laminar for the entire length and accordingly for unheated starting length x_i

$$Nu_x = 0.332\,(Re_x)^{1/2}\,(Pr)^{1/3}\left[1 - \left(\frac{x_i}{x}\right)^{3/4}\right]^{-1/3} \quad ...(i)$$

$$\therefore \quad Nu_l = 0.332\,(2.676 \times 10^5)^{0.5} \times (0.693)^{1/3}\left[1 - \left(\frac{1.5}{2.75}\right)^{3/4}\right]^{-1/3}$$

$$= \frac{152}{(0.365)^{1/3}} = 212$$

$$h_l = Nu_l \times \frac{k}{l} = 212 \times \frac{0.03}{2.75} = 2.313 \text{ W/m}^2\text{-deg}$$

$\therefore$ Heat transfer rate, $\left(\frac{Q}{A}\right)$

$$= h_l \times \Delta t = 2.313 \times (120 - 30) = 208.7 \text{ W/m}^2$$

(*b*) From the identity (i) given above

$$h_x = 0.332\,k\,(Pr)^{1/3}\left(\frac{V_\infty}{\nu}\right)^{1/2}\left[x^{-1/4}\left(x^{3/4} - x_i^{3/4}\right)^{-1/3}\right]$$

$$\therefore \quad \bar{h} = 0.332\,k\,(Pr)^{1/3}\left(\frac{V_\infty}{\nu}\right)^{1/2}\int_{x_i}^{l} x^{-1/4}\left(x^{3/4} - x_i^{3/4}\right)^{-1/3} dx$$

$$= 0.332\,k\,(Pr)^{1/3}\left(\frac{V_\infty}{\nu}\right)^{1/2}\frac{2}{1 - x_i}\left(l^{3/4} - x_i^{3/4}\right)^{2/3} \quad ...(ii)$$

But $$h_l = Nu_l \times \frac{k}{l}$$

$$= 0.332\,(Pr)^{1/3}\,(Re_l)^{1/2}\left[1 - \left(\frac{x_i}{l}\right)^{3/4}\right]^{-1/3} \times \frac{k}{l}$$

$$= 0.332\,k\,(Pr)^{1/3}\left(\frac{V_\infty}{\nu}\right)^{1/2} l^{1/2}\left[\frac{l^{3/4} - x_i^{3/4}}{l^{3/4}}\right]^{-1/3} \times \frac{1}{l}$$

$$= 0.332\,k\,(Pr)^{1/3}\left(\frac{V_\infty}{\nu}\right)^{1/2}\frac{l^{1/2}}{l^{-1/4}}\left[l^{3/4} - x_i^{3/4}\right]^{-1/3} \times \frac{1}{l}$$

$$= 0.332\,k\,(Pr)^{1/3}\left(\frac{V_\infty}{\nu}\right)^{1/2} l^{-1/4}\left[l^{3/4} - x_i^{3/4}\right]^{-1/3} \quad ...(iii)$$

From identities (*ii*) and (*iii*), we get

$$\frac{\bar{h}}{h_l} = 2\times\frac{l^{1/4}}{l-x_i}\frac{\left(l^{3/4}-x_i^{3/4}\right)^{2/3}}{\left(l^{3/4}-x_i^{3/4}\right)^{-1/3}}$$

$$= 2\times\frac{l^{1/4}}{l-x_i}\left(l^{3/4}-x_i^{3/4}\right)$$

$$= 2\times\frac{l^{1/4}}{l(1-x_i/l)}\,l^{3/4}\left[1-(x_i/l)^{3/4}\right]$$

$$\therefore \quad \bar{h} = 2\,\frac{1-(x_i/l)^{3/4}}{1-x_i/l}\,h_l$$

Substituting the relevant values,

$$\bar{h} = 2\times\frac{1-(1.5/2.75)^{3/4}}{1-1.5/2.75}\times 2.313$$

$$= 3.718\ \text{W/m}^2\text{-deg}$$

$\therefore$ Heat transfer rate per metre width

$3.718\times(1.25\times 1)\times(120-30) = 418.27$ W

12.11. LAMINAR TUBE FLOW

Velocity distribution : For analysing laminar flow of an incompressible fluid through a circular tube of radius *R*, we consider the equilibrium of forces on a small concentric cylindrical element of radius *r*, length *dx* and located at a distance *y* from the bottom internal surface of the pipe.

For a fully developed flow the velocity is same at the sections 1 and 2 and as such there is no momentum change. The pressure forces are then balanced by the viscous shear forces.

$$p\pi r^2 - \left(p+\frac{\partial p}{\partial x}dx\right)\pi r^2 = 2\pi r\,dx\,\tau$$

$$\left(-\frac{\partial p}{\partial x}dx\right)\pi r^2 = 2\pi r\,dx\times\tau$$

Dividing throughout by the volume of fluid element $\pi r^2\,dx$;

$$\frac{\partial p}{\partial x} = -\frac{2\tau}{r}$$

or in the limits

$$\frac{dp}{dx} = -\frac{2\tau}{r} \qquad ...(12.117)$$

Invoking Newton's law of viscosity for laminar flow,

$$\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$$

The minus sign is due to the fact that *r* is measured in a direction opposite of *y*.

$$\therefore \quad \frac{du}{dr} = \frac{r}{2\mu}\frac{dp}{dx} \qquad ...(12.118)$$

Since (dp/dx) is not a function of *r*, integration of equation 12.118 with respect to *r* yields the following expression for velocity distribution.

$$u = \frac{1}{4\mu}\left(\frac{dp}{dx}\right)r^2 + C$$

The constant of integration *C* can be evaluated from the boundary condition : $u = 0$ at $r = R$. That gives

$$C = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right)R^2$$

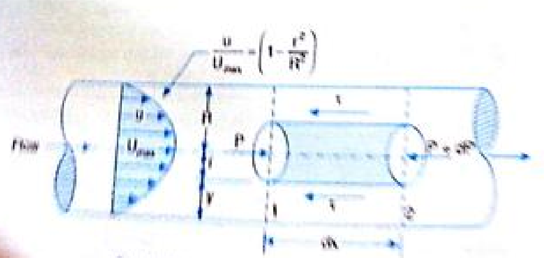

Fig. 12.18. Laminar flow through a tube.

$$\therefore \quad u = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)(R^2-r^2) \qquad ...(12.119)$$

Evidently the velocity profile varies parabolically and the surface of velocity distribution is a paraboloid of revolution. The minus sign is required because the pressure decreases in the flow direction. Position of maximum velocity can be obtained by putting derivative of velocity equal to zero., *i.e.*,

$$\frac{du}{dx} = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)(-2r) = 0 \text{ or } r = 0$$

Thus the maximum velocity occurs at the centre of pipe and its value is given by,

$$V_{max} = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)R^2 \qquad ...(12.120)$$

From equations 12.119 and 12.120, it follows, that :

$$\frac{u}{V_{max}} = 1-\left(\frac{r}{R}\right)^2 \qquad ...(12.121)$$

An estimate of average flow velocity *V* can be made by equating the volumetric flow to the integrated paraboloid flow, *i.e.*,

$$V(\pi R^2) = \int_0^R u\,(2\pi r)\,dr$$

$$= \int_0^R V_{max}\left(1-\frac{r^2}{R^2}\right)2\pi r\,dr$$

$$= 2\pi V_{max}\left[\frac{r^2}{2}-\frac{r^4}{4R^2}\right]_0^R$$

$$= 2\pi V_{max}\left(\frac{R^2}{4}\right)$$

$$\therefore \quad V = \frac{V_{max}}{2} = \frac{1}{8\mu}\left(-\frac{dp}{dx}\right)R^2 \qquad ...(12.122)$$

The pressure gradient dp/dx is usually expressed in terms of a friction factor *f*, defined as

$$-\frac{dp}{dx} = \frac{f}{d}\frac{\rho V^2}{2} \qquad ...(12.123)$$

where $\rho V^2/2$ is the dynamic pressure of the mean flow and *d* is the tube diameter.

Through equations 12.122 and 12.123, we get the friction factor as a simple function of Reynolds number;

$$f = \frac{64}{(\rho d V)/\mu} = \frac{64}{Re} \qquad ...(12.124)$$

which is valid for laminar tube flow, $Re < 2000$,

Further it follows from equation 12.123 that for a finite length of pipe between $x = x_1$ and $x = x_2$

$$\frac{p_1-p_2}{x_1-x_2} = f\frac{\rho V^2}{2d}$$

$$p_1 - p_2 = f\rho\frac{V^2}{2d}(x_1-x_2)$$

$$= \frac{64}{(\rho d V)/\mu}\rho\frac{V^2}{2d}l$$

$$= \frac{32\mu V l}{d^2} = \frac{128\mu Q l}{\pi d^4}$$

Equivalent pressure head becomes ;

$$\frac{p_1-p_2}{\rho} = \frac{32\mu V l}{\rho d^2} = \frac{128\mu Q l}{\pi\rho d^4} \qquad ...(12.125)$$

Evidently the head loss h_l over a length of pipe *l* varies directly as the first power of the rate of discharge *Q* and inversely as the fourth power of the pipe diameter.

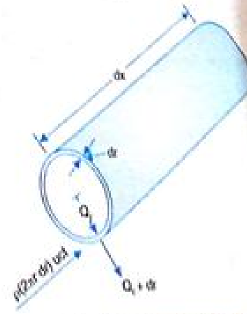

Fig. 12.19. Energy analysis in the tube flow

Equation 12.125 is usually known as the Hagen-Poiseuille equation and it is valid for a fully-developed flow; a flow in which the velocity profile does not vary along the pipe axis.

(iii) *Temperature distribution* : For estimating the temperature distribution, we consider the heat flow through an elementary ring of length dx and thickness dr. Neglecting axial conduction, Fig. 12.19 shows the radial conduction and axial enthalpy transport in the annular element.

Heat flow conducted into the annular element,

$$dQ_r = -k\,2\pi r\,dx\,\frac{\partial t}{\partial r}$$

and heat conducted out is

$$dQ_{r+dr} = -k\,2\pi(r+dr)\,dx\,\frac{\partial}{\partial r}\left(t+\frac{\partial t}{\partial r}dr\right)$$

The net heat convected out of the element is

$$dQ_{conv} = \rho(2\pi r\,dr)\,u c_p\,\frac{\partial t}{\partial x}dx$$

An energy balance on the annular element gives :

net heat conducted in

= net energy convected out

$$-k\,2\pi r\,dx\,\frac{\partial t}{\partial x} + k\,2\pi(r+dr)\,dx\left(\frac{\partial t}{\partial r}+\frac{\partial^2 t}{\partial r^2}dr\right)$$

$$= \rho(2\pi r\,dr)\,u c_p\,\frac{\partial t}{\partial x}dx$$

Upon simplification and rearrangement,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) = \frac{u}{\alpha}\frac{\partial t}{\partial x} \qquad ...(12.126)$$

Inserting the velocity distribution

$$u = V_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

we get

$$\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial t}{\partial x}V_{max}\left(1-\frac{r^2}{R^2}\right)r \qquad ...(12.127)$$

For a fully-developed thermal field and constant heat flux $\partial t/\partial x$ is constant. Equation 12.127 can be integrated with respect to r to yield :

$$r\frac{\partial t}{\partial r} = \frac{1}{\alpha}\frac{\partial t}{\partial x}V_{max}\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)+C_1$$

or $$\frac{\partial t}{\partial r} = \frac{1}{\alpha}\frac{\partial t}{\partial r}V_{max}\left[\frac{r}{2}-\frac{r^3}{4R^2}\right]+\frac{C_1}{r}$$

Integrating again, we get

$$t = \frac{1}{\alpha}\frac{\partial t}{\partial x}V_{max}\left(\frac{r^2}{4}-\frac{r^4}{16R^2}\right) + C_1\log_e r + C_2$$

The constants of integration C_1 and C_2 when evaluated from the boundary conditions

$$\frac{\partial t}{\partial r} = 0 \quad \text{at} \quad r = 0$$

$$t = t_c$$

(center line temperature) at $r = 0$

take the values $C_1 = 0$ and $C_2 = t_c$

$$\therefore \quad t = \frac{1}{\alpha}\frac{\partial t}{\partial x}V_{max}\left(\frac{r^2}{4}-\frac{r^4}{16R^2}\right)+t_c$$

or $$t - t_c = \frac{V_{max}R^2}{4\alpha}\frac{\partial t}{\partial x}\left[\left(\frac{r}{R}\right)^2-\frac{1}{4}\left(\frac{r}{R}\right)^4\right] \qquad ...(12.128)$$

The heat flux coefficient is obtained from the fundamental convection and conduction equations.

$$Q = h\,A\,(t_w - t_b) = -kA\left(\frac{\partial t}{\partial r}\right)_{r=0}$$

where t_w is the wall temperature, t_b is the average or bulk temperature of the fluid. The bulk temperature is the enthaply average temperature of the bulk of fluid. Mathematically it is defined as :

$$t_b = \frac{\int_0^R \rho(2\pi r\,dr)\,u c_p t}{\int_0^R \rho(2\pi r\,dr)\,u c_p} \qquad ...(12.129)$$

For an incompressible fluid having constant density and specific heat

$$t_b = \frac{\int_0^R u t r\,dr}{\int_0^R u r\,dr} \qquad ...(12.130)$$

Physically the bulk or mixing cup temperature represents the temperature that would be obtained if the fluid at a given cross-section were directed into an insulated chamber and allowed to mix thoroughly and attain equilibrium.

In engineering practice it is customary to use the arithmatic average value

$$t_b = \frac{t_{inlet} + t_{outlet}}{2}$$

in the calculation of average heat transfer coefficient

12.12. TURBULENT TUBE FLOW

It is known from flow measurements that a transition to turbulence occurs when the Reynolds number based on mean velocity and diameter exceeds 2500 in a certain pipe. Some of the important-relations for fully-developed turbulent flow through pipes and conduits are :

(i) The velocity distribution is of the power law form

$$\frac{u}{V_{max}} = \left(\frac{y}{R}\right)^{1/n} \qquad ...(12.131)$$

where u is the local average velocity, V_{max} is the velocity at centre line, R is pipe radius and $y = R - r$ is the distance from the wall.

(ii) The head loss equation

$$h_l = \frac{dp}{\rho} = f\frac{lV^2}{2g}$$

is equally valid for turbulent flow. The friction factor f is, however, determined experimentally rather than analytically as for laminar flow.

For turbulent flow, the friction factor is well represented by the following empirical relations:

$$f = 0.3164\,(Re)^{0.25} \text{ for } 20\times10^3 < Re < 80\times10^3 \qquad ...(12.132)$$

$$= 0.184\,(Re)^{-0.2} \text{ for } 10^4 < Re < 10^6 \qquad ...(12.133)$$

$$= 0.005 + 0.396\,(Re)^{-0.3} \text{ for } 2\times10^4 < Re < 2\times10^6 \qquad ...(12.134)$$

(iii) The wall shear stress is

$$\tau_w = \frac{f}{8}\rho V_{max}^2 = \frac{C_f}{2}\rho V_{max}^2 \qquad ...(12.135)$$

(iv) Reynolds analogy is of the form

$$\overline{St} = \frac{\bar{h}}{\rho C_p V} = \frac{f}{8} \qquad ...(12.136)$$

This relation is restricted to $Pr = 1$. For $0.5 < Pr < 100$, Colburn modification is applied and that gives

$$\overline{St}\,Pr^{2/3} = \frac{f}{8} \qquad ...(12.137)$$

Substituting for the friction factor f, we obtain the following equations for the average heat transfer coefficients

$$\overline{St}\,Pr^{2/3} = \frac{0.184}{8}(Re)^{-0.2}$$

$$\frac{\overline{Nu}}{Pr\,Re}Pr^{2/3} = \frac{0.184}{8}(Re)^{-0.2}$$

$$\therefore \quad \overline{Nu} = 0.023\,(Re)^{0.8}\,(Pr)^{0.33} \qquad ...(12.138)$$

and $$\bar{h} = \overline{Nu}\frac{k}{d} = 0.023\frac{k}{d}(Re)^{0.8}(Pr)^{0.33} \qquad ...(12.139)$$

This expression is valid for :

$1 \times 10^4 < Re < 1 \times 10^5$;

$0.5 < Pr < 100$;

$\frac{l}{d} > 60$

Many such correlations have been developed by different investigators and mention of those has already been made in chapter 11.

EXAMPLE 12.35.

The velocity and temperature distribution for laminar flow in a circular tube of 10 cm radius are approximated by the relations :

$u = (2.4r - 3r^2)$ m/s
$t = 90\,(1 - 2r)$ °C

where the distance r is measured from the surface of the tube. Find the average velocity and the mean bulk temperature of the fluid.

(b) The tube surface is maintained at a constant uniform temperature of 100° C and there occurs a heat loss of 1200 kJ/hr per metre length of the tube. Make calculations for the heat transfer coefficients based on the bulk mean temperature. The flow through the tube may be treated as laminar.

Solution : The average velocity V is obtained by equating the volumetric flow to the integrated flow through an elementary ring of radius r and thickness dr, i.e.,

$$V \pi R^2 = \int_0^R u\,(2\pi r)\,dr$$

$$V = \frac{2}{R^2}\int_0^R u\,r\,dr \qquad \ldots(i)$$

$$= \frac{2}{R^2}\int_0^R (2.4r - 3r^2)\,r\,dr$$

$$= \frac{2}{R^2}\left[2.4\frac{r^3}{3} - 3\frac{r^4}{4}\right]_0^R$$

$$= \frac{2}{R^2}\left[0.8R^3 - 0.75R^4\right]$$

$$= 1.6R - 1.5R^2$$

Substituting,

$R = 0.1$ m

$V = 1.6 \times 0.1 - 1.5 \times 0.1^2$
$= 0.145$ m/s

The mean bulk temperature is given by,

$$t_b = \frac{\int_0^R u\,t\,r\,dr}{\int_0^R u\,r\,dr}$$

From expression (i) above, the integral equals $\int_0^R u\,r\,dr = VR^2/2$. The integral in the numerator then needs to be evaluated

$$\int_0^R u\,t\,r\,dr = \int_0^R (2.4r - 3r^2) \times 90\,(1 - 2r)\,r\,dr$$

$$= 90\int_0^R (2.4r^2 - 7.8r^3 + 6r^4)\,dr$$

$$= 90\left[2.4\frac{r^3}{3} - 7.8\frac{r^4}{4} + 6\frac{r^5}{5}\right]_0^R$$

$$= 90\,(0.8R^3 - 1.95R^4 + 1.2R^5)$$

$$\therefore \quad t_b = \frac{90\,(0.8R^3 - 1.95R^4 + 1.2R^5)}{VR^2/2}$$

$$= \frac{180\,(0.8R - 1.95R^2 + 1.2R^3)}{V}$$

Substituting

$R = 0.1$ m and $V = 0.145$ m/s

$$t_b = \frac{180\,(0.8 \times 0.1 - 1.95 \times 0.1^2 + 1.2 \times 0.1^3)}{0.145}$$

$= 76.59$°C

(b) From the convective heat equation,

$Q = h\,A\,(t_s - t_b)$

$1200 = h \times (2\pi \times 0.1 \times 1) \times (100 - 76.59)$

$h = 81.62$ kJ/m²-hr-deg

EXAMPLE 12.36.

Calculate the change in heat transfer coefficient brought by (i) two-fold increase in the diameter of tube ; velocity of flow is maintained constant by a change in the rate of liquid flow (ii) two fold increase in the flow velocity.

There is no change in the temperature of liquid and the tube wall, and the flow through the tube is turbulent in character in both the cases.

Solution : The average heat transfer coefficient for turbulent tube flow is prescribed by the relation

$$\overline{Nu} = \frac{\bar{h}d}{k} = 0.023\,(Re)^{0.8}\,(Pr)^{0.33}$$

$$\bar{h} = 0.023\,\frac{k}{\nu}\,(Pr)^{0.33}\,\frac{(V)^{0.8}}{(d)^{0.2}}$$

(i) When the flow velocity and the fluid properties remain unchanged

$$\bar{h} \propto \frac{1}{d^{0.2}}$$

$$\therefore \quad \frac{h_2}{h_1} = (d_1/d_2)^{0.2} = (1/2)^{0.2} = 0.87$$

Thus the coefficient of heat transfer decreases to 0.87 times or the decrease is by 1/0.87 = 1.15 times.

(ii) When the tube diameter and the fluid properties remain unchanged

$\bar{h} \propto (V)^{0.8}$

or $h_2/h_1 = (V_2/V_1)^{0.8} = (2)^{0.8} = 1.741$

Thus the coefficient of heat transfer will increase to 1.741 times or the increase is by 1/1.741 = 0.574 times.

EXAMPLE 12.37.

A fluid is passed through a 20 cm diameter pipe whose surface is maintained at a constant temperature. At the inlet section, the temperature of the pipe wall exceeds the fluid temperature by 25°C. How much would the fluid temperature increase over a 2.5 m length of the pipe ?

The inter relationship between energy and momentum transfer for turbulent flow through a pipe is prescribed by the Reynolds analogy

$$S_t = \frac{h}{\rho c_p V} = \frac{f}{8}$$

The friction factor f is approximated as 0.02

Solution : From the energy balance,

$$Q = h\,A\,(t_s - t_b) = m\,c_p\,(t_o - t_i)$$

$$\therefore h\,\pi\,d\,l\left[t_s - \frac{t_o + t_i}{2}\right] = \frac{\pi}{4}d^2 V \rho\,c_p\,(t_o - t_i)$$

Dividing thorughout by $\rho\,V\,c_p\,\pi\,d$,

$$\frac{hl}{\rho V c_p}\left[\frac{(t_s - t_o) + (t_s - t_i)}{2}\right] = \frac{d}{4}(t_o - t_i)$$

$$\text{or} \quad \frac{h}{\rho V c_p}\,\frac{l}{2}\left[(t_s - t_o) + (t_s - t_i)\right]$$

$$= \frac{d}{4}\left[(t_s - t_i) + (t_s - t_o)\right]$$

From the Reynolds analogy $h/\rho\,V\,c_p = f/8$

$$\therefore \frac{f}{8} \times \frac{l}{2}\left[(t_s - t_o) + (t_s - t_i)\right]$$

$$= \frac{d}{4}\left[(t_s - t_i) - (t_s - t_o)\right]$$

$$\frac{f}{4}\left[(t_s - t_o) + (t_s - t_i)\right]$$

$$= \frac{d}{l}\left[(t_s - t_i) - (t_s - t_o)\right]$$

Substituting the given values for $f = 0.02$ and $(t_s - t_i) = 25$°C

$$\frac{0.02}{4}\left[(t_s - t_o) + 25\right] = \frac{0.2}{2.5}\left[25 - (t_s - t_o)\right]$$

$0.005\,(t_s - t_o) + 0.125 = 2 - 0.08\,(t_s - t_o)$

$0.085\,(t_s - t_o) = 1.875$ or $(t_s - t_o) = 22.06$°C

$\therefore$ Rise in temperature of the fluid,

$(t_o - t_i) = (t_s - t_i) - (t_s - t_o)$
$= 25 - 22.06 = 2.94$°C

EXAMPLE 12.38.

A circular tube having a diameter 15 cm and length 150 cm carries air at 20°C and flowing with an average velocity of 20 m/s. The tube is heated externally so that its surfaces are maintained at 160 °C temperature. Determine the amount of heat transferred from the tube surface to the air flowing inside it.

Consider the flow to be turbulent and use the Reynolds interrelationship between the energy and momentum transfer.

Take the following properties of air :

$\nu = 22.10 \times 10^{-6}$ m²/s

and $k = 3.127 \times 10$ W/mK

Solution : The flow Reynolds number is

$$Re = \frac{\rho d V}{\mu} = \frac{dV}{\nu} = \frac{0.15 \times 20}{22.10 \times 10^{-6}}$$

$$= 0.136 \times 10^6$$

Friction factor f

$$= \frac{0.184}{(Re)^{0.2}} = \frac{0.184}{(0.136 \times 10^6)^{0.2}}$$

$$= 0.0173$$

From heat balance,

$$Q = A\,h\,(t_s - t_b) = mc_p\,(t_o - t_i)$$

$$\therefore \pi\,d\,l\,h\left[t_s - \frac{(t_o + t_i)}{2}\right]$$

$$= \frac{\pi}{4}d^2 V \rho\,c_p\,(t_o - t_i)$$

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Dividing throughout by $\rho V c_p \pi d$

$$\frac{h}{\rho V c_p}\left[\frac{(t_s - t_1) + (t_s - t_2)}{2}\right] = \frac{d}{4}(t_2 - t_1)$$

or $$\frac{h}{\rho V c_p}\frac{1}{2}\left[(t_s - t_1) + (t_s - t_2)\right]$$

$$= \frac{d}{4}\left[(t_s - t_1) - (t_s - t_2)\right]$$

From the Reynolds analogy $h/\rho V c_p = f/8$

$$\therefore \frac{f}{8} \times \frac{1}{2}\left[(t_s - t_1) + (t_s - t_2)\right]$$

$$= \frac{d}{4}\left[(t_s - t_1) - (t_s - t_2)\right]$$

$$\frac{f}{4}\left[(t_s - t_1) + (t_s - t_2)\right]$$

$$= \frac{d}{l}\left[(t_s - t_1) - (t_s - t_2)\right]$$

Now, $f = 0.0173$

and $t_s - t_1 = 160 - 20 = 140°C$

$$\frac{0.0173}{4}\left[(t_s - t_2) + 140\right]$$

$$= \frac{0.15}{1.5}\left[140 - (t_s - t_2)\right]$$

Solution gives: $(t_s - t_2) = 128.39°C$

$(t_2 - t_1) = (t_s - t_1) - (t_s - t_2)$

$= 140 - 128.39 = 11.61°C$

$t_2 = t_1 + 11.61$

$= 20 + 11.61 = 31.61°C$

Again, from the Reynolds analogy :

$$S_t = \frac{f}{8}; \quad \frac{Nu}{Re\,Pr} = \frac{f}{8}$$

In the absence of data for Pr, we presume it to be unity

$$\therefore \quad Nu = \frac{f}{8}Re = \frac{0.0173}{8} \times 0.136 \times 10^6$$

$$= 294.1$$

$$h = Nu\frac{k}{d} = \frac{294.1 \times 3.127 \times 10^{-2}}{0.15}$$

$$= 61.31 \text{ W/m}^2\text{ K}$$

Heat loss from the tube surface to the air,

$Q = h\,A\,\Delta t$

$$= 61.31 \times (\pi \times 0.15 \times 1.5) \times \left(160 - \frac{31.61 + [illegible]}{2}\right)$$

$= 5812.72$ W

EXAMPLE 12.39.

A straight tube having a diameter of 40 mm carries water with a velocity of 10 m/s. The temperature of the tube surface is 50°C and the flowing water is heated from the inlet temperature t_1 = 15°C to an outlet temperature t_2 = 25°C. Determine the coefficient of heat transfer from the tube surface to water and the length of tube. Take the physical properties of water at its mean bulk temperature.

Solution : At the mean bulk temperature,

$$t_b = \frac{t_1 + t_2}{2} = \frac{15 + 25}{2} = 20°C$$

the relevant physical properties of water are:

$\nu = 1.006 \times 10^{-6}$ m²/s

$k = 59.86 \times 10^{-2}$ W/mK

$c_p = 4138$ J/kg K

and $Pr = 0.702$

$$Re = \frac{\rho d V}{\mu} = \frac{d V}{\nu} = \frac{0.04 \times 10}{1.006 \times 10^{-6}}$$

$$= 3.976 \times 10^5$$

The flow is turbulent since Reynolds number is greater than 2500. With an incompressible liquid in turbulent flow through tubes and conduits, the following correlation applies for the heat transfer coefficient

$$\overline{Nu} = \frac{\bar{h} d}{k} = 0.023\,(Re)^{0.8}\,(Pr)^{0.33}$$

$$= 0.023 \times (3.976 \times 10^5)^{0.8} \times (0.702)^{0.33}$$

$$= 617.41$$

and the heat transfer coefficient

$$\bar{h} = \frac{617.41 \times 59.86 \times 10^{-2}}{0.04}$$

$$= 9239.6 \text{ W/m}^2\text{K}$$

The length of tube is determined from the heat balance equation

$$Q = \bar{h}\,\pi\,d\,l\,(t_s - t_b) = m c_p (t_2 - t_1)$$

The rate of water flow and the amount of heat transferred to it is

$$m = \frac{\pi}{4} d^2 V \rho$$

$$= \frac{\pi}{4}(0.04)^2 \times 10 \times 998$$

$$= 12.53 \text{ kg/s}$$

$Q = m c_p (t_2 - t_1)$

$= 12.53 \times 4138 \times (25 - 15)$

$= [illegible]$ W

$\therefore$ Tube length, l

$$= \frac{[illegible]}{9239.6 \times \pi \times 0.04 \times (50 - 20)}$$

$= [illegible]$ m

SALIENT POINTS

1. Hydrodynamic boundary layer is the narrow region of fluid near a solid body (immersed in the flowing fluid) where the velocity of fluid varies from zero to free stream velocity.
2. The nominal thickness δ_h of hydrodynamic boundary layer is the vertical distance from the surface of solid body where the flow velocity is approximately 99% of free stream velocity, i.e., $u = 0.99\,U_\infty$
3. The pattern of flow in the boundary layer is judged by Reynolds number $R_e = \frac{x U_\infty}{\nu}$ where x is the distance along the plate and measured from its leading edge. In general
   (i) When $R_e < 2 \times 10^5$, the boundary layer is laminar. The boundary layer becomes turbulent when R_e exceeds 5×10^5.
   (ii) The transition from laminar to turbulent pattern of flow occurs at values of Reynolds number between 2×10^5 to 5×10^5.
4. For the laminar boundary layer the velocity distribution is parabolic and is presumed to conform to the relation $\frac{u}{U_\infty} = f\left(\frac{y}{\delta}\right)$.
5. Blasius exact solution suggests the following expressions for laminar boundary layers:
   (i) Boundary layer thickness $\delta = \frac{5x}{\sqrt{R_{ex}}}$
   (ii) Local skin friction coefficient
   $$C_{fx} = \frac{0.646}{\sqrt{R_{ex}}}$$
   (iii) Drag coefficient $C_D = \frac{1.328}{\sqrt{R_{el}}}$

where R_{ex} is the Reynolds number based on distance x from the leading edge of plate and R_{el} is the [illegible]

6. For turbulent boundary on a flat plate
   (i) $\delta = \frac{0.371\,x}{(R_{ex})^{0.2}}$
   (ii) $C_{fx} = \frac{0.0592}{(R_{ex})^{0.2}}$
   (iii) $C_D = \frac{0.072}{(R_{el})^{0.2}}$

   These expressions are valid on the assumption that
   – the boundary layer is turbulent from the leading edge onwards
   – the velocity distribution follows the one-seventh power law
   $$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$
7. Irrespective of the laminar or turbulent characteristics of main flow, the boundary layer commences with a laminar boundary layer at the leading edge. Eventually, the laminar boundary layer transforms to turbulent boundary layer after passing through transition. Presuming abrupt transition at $R_e = 5 \times 10^5$, the average skin or drag coefficient is
   $$C_D = \frac{0.455}{(\log_{10} R_{el})^{2.58}} - \frac{1700}{R_{el}}$$
8. When a fluid flows past a heated or cold surface, a temperature field is set up in the fluid next to the surface. The zone or thin layer wherein the temperature field exists is called the thermal boundary layer.

9. The thermal boundary layer thickness δ_t is defined as the distance from the surface at which the temperature difference between the surface and fluid reaches 99 per cent of free stream value. That is when

$$\frac{t_s - t}{t_s - t_\infty} = 0.99$$

10. The relationship between the thermal and hydrodynamic boundary layer thickness is governed by the non-dimensional Prandtl number $P_r = \frac{\mu c_p}{k}$.

$$\delta_t = \delta_h (P_r)^{-1/3}$$

(i) $\delta_t = \delta_h$ when $P_r = 1$

(ii) $\delta_t < \delta_h$ when $P_r > 1$

(iii) $\delta_t > \delta_h$ when $P_r < 1$

11. The local and average heat transfer coefficients are prescribed by the relations

$$N_{ux} = \frac{h_x x}{k} = 0.332 (R_{ex})^{0.5} (P_r)^{0.33}$$

$$\bar{N}_u = \frac{\bar{h} l}{k} = 0.664 (R_{el})^{0.5} (P_r)^{0.33}$$

12. The local Nusselt number N_{ux} and the friction coefficient are correlated by the expression

$$St_x \, P_r^{2/3} = \frac{C_{fx}}{2}$$

The above identity is known as Reynolds-Colburn analogy between the fluid friction and heat transfer in laminar flow. Further,

$$St_x = \frac{N_{ux}}{R_{ex} \, P_r}$$

13. For turbulent boundary layer flow part a flat plate

(i) $N_{ux} = \frac{h_x x}{k} = 0.0288 (R_{ex})^{0.8} (P_r)^{0.33}$

(ii) $\bar{N}_u = \frac{\bar{h} l}{k} = 0.036 (R_{el})^{0.8} (P_r)^{0.33}$

(iii) With combination of laminar and turbulent boundary layer and presuming that transition occurs at critical Reynolds number of $R_{ec} = 5 \times 10^5$,

$$\bar{N}_u = \frac{\bar{h} l}{k} = [0.036 (R_{el})^{0.8} - 836] P_r^{0.33}$$

14. For laminar flow in a tube, the convection heat transfer coefficient is estimated from the relation

$$N_u = \frac{h \, d_o}{k} = 4.364$$

where d_o is the tube diameter.

15. For turbulent flow in a tube,

(i) $\frac{u}{V_{max}} = \left(\frac{y}{R}\right)^{\frac{1}{n}}$

(ii) Friction factor $f = 0.3164 (R_e)^{0.25}$

(iii) Wall shear stress τ_w

$$= \frac{f}{8} \rho V_{max}^2 = \frac{C_f}{2} \rho V_{max}^2$$

(iv) Reynolds analogy is of the form

$$\bar{S}t \, P_r^{2/3} = \frac{f}{8}$$

(v) $\bar{N}_u = 0.023 (R_e)^{0.8} (P_r)^{0.33}$

and $\bar{h} = \bar{N}_u \frac{k}{d}$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. (i) What is meant by hydrodynamic boundary layer?
   (ii) Set up the force or momentum equation for the hydrodynamic boudnary layer. List the assumptions made.
2. (i) Define the hydrodynamic boundary layer thickness.
   (ii) Write the relation between laminar boundary layer thickness and local Reynolds number and state how the thickness varies with
   - local Reynolds number
   - distance from leading edge of the plate, and
   - increasing velocity of fluid flow.
3. Define the local friction coefficient and average friction coefficient for hydrodynamic boundary layer.

4. Explain the essential features of Blasius method of solving laminar boundary layer equations for a flat plate. Derive expressions for boundary layer thickness and local skin friction coefficients from this solution.
5. Apply the order of magnitude analysis to establish a functional relationship for the boundary layer thickness. Presume zero pressure gradient and constant fluid properties.
6. Derive the two-dimensional momentum equation for the hydrodynamic boundary layer on a flat plate. Also mention the boundary conditions when the fluid is approaching the plate with a free stream velocity U_∞.

   Using $\frac{u}{U_\infty} = 0.99$ at $\eta = \left(y\sqrt{\frac{U_\infty}{\nu x}}\right) = 5$ as the solution of hydrodynamic boundary layer equation, show that boundary layer thickness at a distance x from the leading edge is given by

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$$

7. Define the local and average skin friction (drag) coefficient for a flat smooth plate at zero incidence.

   Establish the following relations for laminar boundary layer over a flat plate.

   (i) local skin friction coefficient $C_{fx} = \frac{0.664}{\sqrt{Re_x}}$

   (ii) average drag coefficient $\bar{C}_f = \frac{1.328}{\sqrt{Re_x}}$

   Presume the value $\left(\frac{df}{d\eta}\right)_{\eta=0} = 0.332$

8. Derive the von-Karman momentum integral equation for flow past a flat plate in the form:

$$\tau_w = \rho U_\infty^2 \frac{d}{dx}\left[\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy\right]$$

   Based upon this equation calculate the boundary layer thickness, wall shear stress and the skin friction coefficient for laminar flow over a flat plate. Assume the following velocity distribution

$$\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

   where u is the velocity at a distance y from the surface and at $y \to \delta$, $u \to U_\infty$.

9. Write the velocity profile which is normally used in the approximate analysis of laminar boundary layer equation.
10. Calculate the friction drag on a plate 15 cm wide and 45 cm long placed longitudinally in a stream of oil (specific gravity 0.925 and kinematic viscosity 0.9 stokes) flowing with a free stream velocity of 6 m/s. Also find the thickness of boundary layer and shear stress at the trailing edge.

    (Ans. 17.30 N, 0.013 m, 63.77 N/m²)

11. Air at 20°C and 1 atm flows with a velocity 7.5 m/s past a flat plate placed at zero angle of incidence. The plate surface is maintained at a uniform temperature of 120°C. Calculate (i) thickness of boundary layer 0.6 m from the leading edge, (ii) position of the point of transition to turbulent flow if the critical Reynolds number is 5×10^5, and (iii) drag force on one side of the plate over the first 0.6 m length. Assume unit width of plate.

    (Ans. 0.731 m, 1.335 m, 0.056 N)

12. Consider a flat plate 3 wide and 1.8 m long placed in an air stream having a velocity of 160 km/hr under standard atmospheric conditions.

    (i) Assuming a laminar boundary layer exists over the entire length of plate, what is the skin friction drag and how thick is the boundary layer 1.2 m aft of the leading edge?

    (ii) Assuming a turbulent boundary layer exists over the entire length of plate, what is the skin friction drag and how thick is the boundary layer 1.2 m aft of the leading edge?

    (iii) If the critical Reynolds number for transition from laminar to turbulent flow is 1×10^6, how far aft of the plate leading edge will the transition point be ? If an immediate transition from laminar to turbulent flow is assumed, what is the skin friction drag on the plate ?

    For air $\rho = 1.2$ kg/m³

    and $\nu = 15 \times 10^{-6}$ m²/s

    (Ans. (i) 7.36 N, 3.183 m
    (ii) 42.50 N, 2.18 cm; (iii) 35.00

13. What is meant by thermal boundary la State the relationship between thermal hydrodynamic boundary layer thicknes
14. Derive the two-dimensional energy eq for thermal boundary layer on a flat

Mention the boundary conditions, when the plate is heated and its surface is maintained at constant temperature t_s and the ambient temperature is t_∞.

15. Using Pohlhausen solution for temperature gradient

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = 0.332\ (Pr)^{0.33}$$

set up a relationship between the Nusselt number, the Reynolds number and the Prandtl number for forced convection over a flat plate

(b) Show that

$$\delta_t < \delta \quad \text{for} \quad Pr > 1$$
$$\delta_t = \delta \quad \text{for} \quad Pr = 1$$
$$\delta_t > \delta \quad \text{for} \quad Pr < 1$$

where δ_t and δ are the thermal and hydrodynamic boundary layer thicknesses at a certain location x from the leading edge.

16. Set up the two-dimensional energy equation for thermal boundary layer on a flat plate. State the relevant boundary condition, when the plate is heated and its surface is maintained at constant temperature t_s whilst the ambient temperature is t_∞.

17. Establish the following relation for laminar flow over a flat plate

$$\frac{\delta_t}{\delta} = \frac{1}{(Pr)^{0.33}}$$

where δ_t and δ denote the thermal and hydrodynamic boundary layer thicknesses, respectively and Pr stands for the Prandtl number. Following assumptions may be made:

(i) heat is transferred only by conduction into the fluid closely adjoining the wall

(ii) temperature distribution is similar to that of velocity distribution which is given by

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

(iii) $\dfrac{\delta}{x} = \dfrac{4.64}{\sqrt{Re_x}}$ and $Pr > 1$

18. Assuming a linear velocity distribution

$$\frac{u}{U_\infty} = \frac{y}{\delta}$$

for the laminar boundary layer of a fluid flowing past a flat plate, it can be shown that boundary layer thickness is given by

$$\frac{\delta}{x} = \frac{3.464}{\sqrt{Re_x}}$$

Using this result and the assumption that the temperature distribution is linear across a boundary layer of thickness δ_t, show that for a plate maintained at uniform temperature

$$\frac{\delta_t}{\delta} = \frac{1}{(P_r)^{0.33}}$$

Proceed further to show that the local Nusselt number at a distance x along the plate is given by

$$Nu_x = 0.288\ (Re_x)^{0.5}\ (Pr)^{0.33}$$

19. Develop dimensionless expressions for boundary layer thickness and local and average Nusselt numbers for laminar boundary layer flow over a plane isothermal surface. The velocity and temperature profiles within the boundary layer may be approximated as :

(i) $\dfrac{u}{U_\infty} = \dfrac{t_s - t}{t_s - t_\infty} = a + by$

(ii) $\dfrac{u}{U_\infty} = \dfrac{t_s - t}{t_s - t_\infty} = a + by + cy^2$

(iii) $\dfrac{u}{U_\infty} = \dfrac{t_s - t}{t_s - t_\infty} = a + by + cy^2 + dy^3$

Use the integral approach.

20. Experimental results for heat transfer over a thin flat plate were found to be correlated by an expression of the form

$$Nu_x = 0.332\ (Re_x)^{0.5}\ (Pr)^{0.33}$$

where Nu_x is the local value of Nusselt number at a position x measured from the leading edge of the plate. Obtain an expression for the ratio of the average heat transfer coefficient to the local coefficient.

21. Atmospheric air at 20°C flows past a flat plate at 2.5 m/s. If the plate is 1 m wide and its surface is maintained at 80°C, make calculations for the following parameters at a distance of 0.75 m from the leading edge

(i) hydrodynamic boundary layer thickness and the local friction coefficient

(ii) thermal boundary layer thickness and the local heat transfer coefficient.

(Ans. (i) 1.16 cm and 0.00205; (ii) 1.307 cm and 3.5 W/m² K)

22. A thin flat plate has been placed longitudinally in a stream of air at 20°C and which flows

with undisturbed velocity of 7.5 m/s. The surface of plate is maintained at a uniform temperature of 120°C. Calculate the heat transfer coefficient 0.8 m from the leading edge of the plate. Also calculate the rate of heat transfer from one side of the plate to the air over the first 0.8 m length. Assume unit width of the plate.

(Ans. 5.954 W/m² K, 762.2 W)

23. Atmospheric air at 50°C flows past a 0.5 m long flat plate placed with zero angle of incidence. The air flow velocity is 5 m/s and the plate surface is maintained at 30°C temperature. What is the Reynolds number along the plate ? Is the boundary layer laminar or turbulent ? What is your criterion of laminar or turbulent boundary layer ? Calculate the average heat transfer coefficient over the surface under these conditions.

24. Wind at 75 km per hour is blowing on a winter day in a direction parallel to the short side of a flat roof measuring 8 m × 18 m. The temperature at the surface of the roof is 10°C and the air temperature is 5°C. By neglecting the end effects and assuming no separation of air from the roof, compute the heat loss.

25. Water at a velocity of 2.5 m/s flows parallel to the surface of a horizontal smooth and thin flat plate whose dimension in the flow direction is 1.25 m. The temperature of the plate surface is kept uniformly at 140°C and the temperature of the main water stream is 20°C. Calculate the thermal and hydrodynamic boundary layer thicknesses and the local frictional coefficient at the mid point of the plate. Also workout the rate of heat transfer from the plate to the water. Assume unit width of the plate.

26. A thin flat plate of length 1.25 m is exposed to air flow parallel to its surface. The velocity and temperature of the free stream air flow are 80 m/s and 10°C, respectively. A turbulence grid placed upstream of the plate causes the boundary layer to be turbulent over the entire length of the plate. Determine (i) hydrodynamic boundary layer and thermal boundary layer thicknesses at the trailing edge of the plate, (ii) average coefficient of heat transfer from the surface of the plate.

27. Explain the inter-relations that can be used directly to infer heat transfer data from measurement of shear stress.

28. Explain the Reynolds analogy and Colburn analogy. Under what conditions these two analogies become the same.

29. Air at atmospheric pressure and 5°C flows at 160 km/hr across the wing of an air plane (to be approximated as a flat plate). Plot a curve showing the thickness of a laminar boundary layer as a function of the distance x from the leading edge up to the distance at which transition to turbulent boundary layer is likely to occur. Use either the Blasius exact solution or the result of the application of the momentum integral method.

30. Calculate the average Nusselt number for turbulent flow over a flat plate where the transition occurs abruptly when the local Reynolds number is 4×10^5.

(Ans. [illegible])

31. (a) Obtain a relation between the heat transfer coefficient and the friction coefficient.

(b) Discuss the significance of bulk temperature in case of fully-developed laminar flow in a tube.

32. Determine the ratio of heat transfer coefficient from the wall of a tube to water (h_w) and to air (h_a). The tubes are of equal diameter and the fluids are in turbulent state at equal Reynolds number and Prandtl number. What will be the numerical value of this ratio if water temperature t_w = 200°C and air temperature t_a = 25°C ?

(Ans. $\dfrac{h_w}{h_a} = \dfrac{k_w}{k_a}$, 23.16)

B. Fill in the blanks with appropriate word/words :

1. The nominal thickness of .... layer represents the distance from the surface to the point where velocity is 99 percent of its asymptotic limit.
2. The results for laminar boundary layer for flow past a flat plate are based on velocity distribution which has .... variation.
3. The .... is defined as the ratio of wall shear stress to the dynamic head caused by the free stream velocity.
4. A zone or layer in the fluid flowing past a heated or cold surface where in the temperature field exists is called the ....
5. The thermal boundary layer thickness is .... than the hydrodynamic boundary

layer thickness for fluids having Prandtl less than unity.

4. At Prandtl number equal to .... the temperature distribution will be identical to the velocity distribution.

7. .... number correlates the relative thickness of the hydrodynamic and thermal boundary layers.

8. The expression $St_x = \frac{N_{ux}}{R_{ex} P_r}$ is known as .... analogy between fluid friction and heat transfer in laminar flow.

9. The .... is defined as the ratio of flux of enthalpy at a cross-section to the product of the mass flow rate and the specific heat of the fluid.

10. The .... for the fully developed laminar tube flow is constant and is independent of the Reynolds number and Prandtl number.

Answers : 1. hydrodynamic boundary; 2. parabolic; 3. skin friction coefficient; 4. thermal boundary layer; 5. larger; 6. unity; 7. Prandtl; 8. Reynolds-Colburn; 9. bulk temperature or mixing up temperature; 10. Nusselt number.

C Multiple choice questions :

1. The temperature gradient in the fluid flowing over a heated plate will be
   (a) zero at the plate surface
   (b) positive at the surface
   (c) very steep at the surface
   (d) zero at the top of thermal boundary layer

2. Thermal boundary layer is a region where
   (a) heat dissipation is negligible
   (b) inertia and convection terms are of the same order of magnitude
   (c) convection and dissipation terms are of the same order of magnitude
   (d) convection and conduction terms are of the same order of magnitude

3. For laminar conditions, the thickness of thermal boundary layer increases with its distance from the leading edge in proportion to
   (a) x (b) $x^{1/2}$
   (c) $x^{1/3}$ (d) $x^{1/4}$

4. The hydrodynamic and thermal boundary layer are identical at Prandtl number equal to
   (a) 0.5 (b) 1
   (c) 10 (d) 50

5. For a fluid having Prandtl number equal to unity, the hydrodynamic boundary layer thickness δ and thermal boundary layer thickness δ_t are related as
   (a) $\delta = \delta_t$ (b) $\delta > \delta_t$
   (c) $\delta < \delta_t$ (d) $\delta_t = \delta^{1/3}$

6. The ratio of hydrodynamic to thermal boundary layer thickness varies as
   (a) root of Prandtl number
   (b) one-third power of Prandtl number
   (c) two-third power of Stanton number
   (d) four-fifth power of Nusselt number

7. The ratio of the thickness of thermal boundary layer δ_t to thickness of hydrodynamic boundary δ_h is given by

$$\frac{\delta_t}{\delta_h} = (\text{Prandtl number})^n$$

The index n is
   (a) $-1/3$ (b) $-2/3$
   (c) 1 (d) -1

8. In a convective heat transfer situation, Reynolds number is very large but the Prandtl number is so small that the product $(R_e P_r)$ is less than one. In such a condition
   (a) thermal boundary layer does not exist
   (b) viscous boundary layer thickness equals the thermal boundary layer thickness
   (c) viscous boundary layer thickness is less than the thermal boundary layer thickness
   (d) viscous boundary layer thickness is greater than the thermal boundary layer thickness

9. Consider development of laminar thermal boundary layer for a moving non-reacting fluid in contact with a flat plate of length l along the flow direction. The average value of heat transfer coefficient can be obtained by multiplying the local heat transfer coefficient at the trailing edge by the factor
   (a) 0.75 (b) 1.0
   (c) 1.5 (d) 2.0

10. Reynolds analogy states that
   (a) $S_t = \frac{C_{fx}}{4}$ (b) $S_t = \frac{C_{fx}}{2}$
   (c) $S_t = \sqrt{C_{fx}}$ (d) $S_t = 2\, C_{fx}$

   where S_t is the Stanton number and C_{fx} is the skin friction coefficient

11. Colburn analogy is valid for Prandtl number lying in the range
   (a) $0.5 < P_r < 50$ (b) $0.1 < P_r < 0.5$
   (c) $10 < P_r < 100$ (d) $1.5 < P_r < 10$

12. The average heat transfer coefficient for turbulent tube flow is prescribed by the relation

$$\bar{N}_u = 0.023\, (R_e)^{0.8}\, (P_r)^{0.33}$$

If the fluid properties and the flow velocity remain unchanged, a two-fold increase in the tube diameter will decrease the coefficient of heat transfer by the factor
   (a) 1.15 (b) 1.32
   (c) 1.86 (d) 2.1

Answers :

1. (d)	2. (c)	3. (b)	4. (b)	5. (a)
6. (b)	7. (a)	8. (c)	9. (d)	10. (b)
11. (a)	12. (a)			

HINTS AND COMMENTS

6(b): The general correlation between Prandtl number and the relative values of thermal and hydrodynamic boundary layer thickness is given by

$$\frac{\delta_t}{\delta_h} = P_r^{-0.33}$$

Obviously when $P_r = 1$, the hydrodynamic and thermal boundary layers are of equal thickness.

5(a), 8(b): The hydrodynamic boundary boundary layer thickness δ and the thermal boundary layer thickness δ_t are correlated by the expression

$$\delta_t = \delta \times (P_r)^{-1/3}$$

where P_r is the Prandtl number.

As such
(a) $\delta_t = \delta$ when $P_r = 1$
(b) $\delta_t < \delta$ when $P_r > 1$
(c) $\delta_t > \delta$ when $P_r < 1$

12(a):

$$\overline{Nu} = 0.023\, (R_e)^{0.8}\, (P_r)^{0.33}$$

or

$$\frac{hd}{k} = 0.023 \left(\frac{\rho V d}{\mu}\right)^{0.8} \left(\frac{\mu c_p}{k}\right)^{0.33}$$

As the fluid properties and the flow velocity remain unchanged,

$$h \propto \frac{1}{d^{0.2}} = \frac{1}{(2)^{0.2}} = \frac{1}{1.15}$$

As such, a two-fold increase in the tube diameter will decrease the coefficient of heat transfer by the factor 1.15.

Chapter

13 Condensation and Boiling

Learning objectives : A study of the subject matter dealt with in this chapter will enable the readers to

- define and make distinction between film condensation and dropwise condensation
- make Nusselt analysis of laminar film condensation on flat verticle plates
- state the emperical correlations for film condensation on tubes and bank of tubes and inside horizontal tubes
- understand different regimes of boiling heat transfer, and comment on critical heat flux in nucleate boiling

Condensation and boiling are the convective heat transfer processes that are associated with change in the phase of a fluid. Condensation refers to a change from the vapour to a liquid phase and boiling involves change from liquid to vapour phase of a fluid substance. These processes are very common in power plants (boilers and condensers), refrigeration systems (evaporators and condensers), process heating and cooling, melting of metals in furnaces and heat exchangers used in refineries and sugar mills etc where heating of the process fluid occurs by the condensation of steam. During a phase change, there is either liberation (condensers) or absorption (boilers and evaporators) of latent heat; the rate of energy transfer is quite rapid even though the accompanying temperature differences may be quite small. These phenomenons are quite difficult to describe due to change in fluid properties (density, specific heat, thermal conductivity and viscosity) and due to additional considerations of surface tension, latent heat of vaporisation, surface characteristics and features of two phase flow.

This chapter will be devoted to examine the condensation and boiling processes with regard to their fundamental mechanism and the means of describing and predicting heat transfer rates.

13.1. CONDENSATION

Fluid in a gaseous or vapour phase changes to a liquid state with the liberation of heat from the vapour.

When a vapour is in contact with a surface whose temperature t_s is lower than the saturation temperature t_{sat} corresponding to the vapour pressure, the condensation sets in and the vapour changes to liquid phase. The condensation of vapour liberates latent heat and there is heat flow to the surface. The liquid condensate may get somewhat sub-cooled by contact with the cooled surface and that may eventually result in more vapour to condense on the exposed surface or upon the previously formed condensate.

Depending upon the behaviour of condensate upon the cooled surface, the condensation process has been categorised into the following distinct modes :

(i) **Film condensation** : The liquid condensate wets the solid surface, spreads out and forms a continuous film over the entire surface. The liquid flows down the cooling surface under the action of gravity and the layer continuously grows in thickness because of newly condensing vapours. The continuous film offers thermal resistance and restricts further transfer of heat between the vapour and the surface.

Film condensation usually occurs when a vapour, relatively free from impurities, is allowed to condense on a clean surface.

(ii) **Dropwise condensation** : The liquid condensate collects in droplets and does not wet the solid cooling surface. The droplets develop in cracks and pits on the surface, grow in size, break away from the surface, knock off other droplets and eventually run off the surface without forming a film. A part of the condensation surface is directly exposed to the vapour without an insulating film of condensate liquid. Evidently there is no film barrier to heat flow and higher heat transfer rates are experienced; heat transfer fluxes of the order of 750 kW/m^2 have been obtained with dropwise condensation.

Dropwise condensation has been observed to occur either on highly polished surfaces, or on surfaces contaminated with impurities like fatty acids and organic compounds. Dropwise condensation gives coefficient of heat transfer generally five to ten times larger than with film condensation. Because of potential performance gain, dropwise condensation is provoked artificially by surface coatings, called promoters, that inhibit wetting. Silicons, teflons and an assortment of waxes and fatty acids are often used for this purpose. These substances are either applied to the heat transfer surface or introduced into the vapour. However, the phenomenon is highly unstable as these coatings gradually lose their effectiveness due to oxidation, fouling or outright removal and the surfaces become wetted when exposed to condensing vapour over an extended length of time. Consequently film condensation is generally encountered in industrial applications and is usually planned for condenser design calculations.

13.2. LAMINAR FILM CONDENSATION ON A VERTICAL PLATE

Consider the process of film condensation occuring on the surface of a flat vertical plate as depicted in Fig. 13.2. The coordinate axes of the system have been so chosen that the origin 'o' is at the upper end of the plate, the x-axis lies along the vertical surface (positive

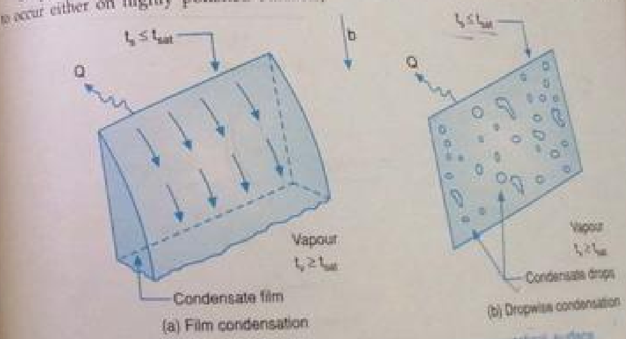


Fig. 13.1. Film and dropwise condensation on a vertical surface

direction of x measured downward) and the y-axis is perpendicular to it. The thickness of liquid film which is zero at the upper end of plate gradually increases as further condensation occurs at the liquid-vapour interface and attains its maximum value at the lower end of the plate. The vertical plate has height L, width b, and δ denotes the thickness of film at a distance x from the origin.

An estimate of the heat transfer coefficient for the liquid film can be made by setting up expressions for the velocity distribution, the mass flow rate and heat flux through the layer. The analysis is based on the findings of Nusselt (1916) and it makes the following assumptions:

- The liquid film is in good thermal contact with the cooling surface and therefore temperature at the inside of the film is taken equal to the surface temperature t_s. Further, the temperature at the outer surface of the film (interface of liquid and vapour) equals the saturation temperature t_{sat} at the prevailing pressure.
- The condensate film is so thin that a linear temperature variation exists between the plate surface and the vapour conditions.
- The physical parameters (the thermal conductivity k, the dynamic viscosity μ and the density ρ) of the condensate film are independent of temperature.
- Vapour density is small compared to that of the condensate.
- The vapour delivers only the latent heat (i.e., there is no under cooling of condensate) and the heat flow across the plate is by conduction
- An element of fluid mass within the film is influenced only by the viscous shear and the gravitational forces; the forces of inertia appearing in the condensate film are disregarded.
- There is no velocity gradient at the liquid-vapour interface and obviously the viscous shear at the phase interface is negligible.
- The condensing vapour is entirely clean and free from gases, air and non-condensing impurities.

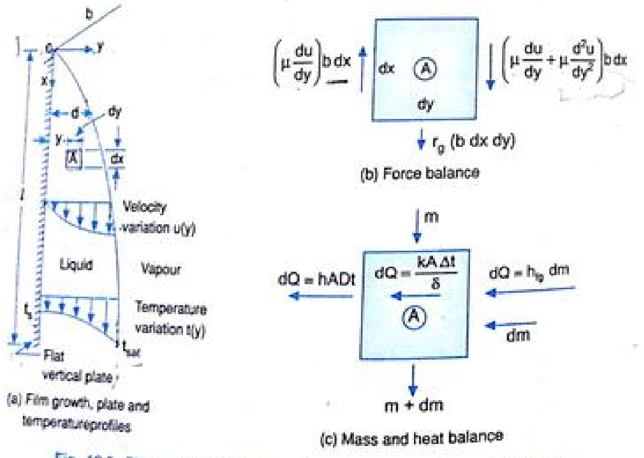


Fig. 13.2. Film condensation on a flat vertical plate: Nusselt analysis

- Drainage of condensate film along the vertical surface is by the action of gravity and is through a laminar motion.
- Radiation between the vapour and the liquid film; horizontal component of velocity at any point in the liquid film; and curvature of the film are considered negligibly small.

• Velocity distribution

An equation for the velocity distribution u as a function of distance y from the wall surface can be setup by considering the equilibrium between the gravity and viscous forces on an elementary volume ($b\ dx\ dy$) of the liquid film.

Gravitational force on the element

$$= \rho g\ (b\ dx\ dy)$$

Viscous shearing stress on the element face at y

$$= \mu \frac{du}{dy}$$

Change in shearing stress in distance dy

$$= \frac{d}{dy}\left(\mu \frac{du}{dy}\right) dy = \mu \frac{d^2u}{dy^2} dy$$

Equating the gravity force to the net sheat force,

$$\rho g\ (b\ dx\ dy) =$$

$$\mu \frac{du}{dy}(b\ dx) - \left(\mu \frac{du}{dy} + \mu \frac{d^2y}{dy^2} dy\right)(b\ dx)$$

Upon simplication,

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{\mu} \quad \text{...(13.1)}$$

Integrating twice :

$$\frac{du}{dy} = -\frac{\rho g}{\mu} y + c_1 \quad \text{and}$$

$$u = -\frac{\rho g}{\mu} \frac{y^2}{2} + c_1 y + c_2$$

The relevant boundary conditions are

$$u = 0 \quad \text{at} \quad y = 0$$

and $$\frac{du}{dy} = 0 \quad \text{at} \quad y = \delta$$

These conditions determine that

$$c_2 = 0 \quad \text{and} \quad c_1 = \frac{\rho g}{\mu} \delta$$

Therefore the velocity distribution through the film is prescribed by the following parabolic relationship

$$u = \frac{\rho g}{\mu}\left(\delta y - \frac{y^2}{2}\right) \quad \text{...(13.2)}$$

The mean flow velocity V of the liquid fim at a distance x from the top edge can be determined from the expression

$$V = \frac{1}{\delta}\int_0^\delta u\,dy$$

$$= \frac{1}{\delta}\int_0^\delta \frac{\rho g}{\mu}\left(\delta y - \frac{y^2}{2}\right) dy$$

$$= \frac{\rho g \delta^2}{3\mu} \quad \text{...(13.3)}$$

• Mass flow rate

The downward flow of the liquid at any elevation x (i.e. over the layer of thickness δ) is :

mass flow rate = mean flow velocity × flow area × density

$$m = \frac{\rho g \delta^2}{3\mu} \times b\delta \times \rho$$

$$= \frac{g b \rho^2 \delta^3}{3\mu} \quad \text{...(13.4)}$$

The mass flow is thus a function of x; this is so because the film thickness δ is essentially dependent upon x.

An increase in the mass flow rate of condensation during downward flow of condensate from x to $x + dx$ can be worked out by differentiating equation 13.4 with respect to x or δ.

$$dm = \frac{d}{d\delta}\left(\frac{g b \rho^2 \delta^3}{3\mu}\right)$$

$$= \frac{g b \rho^2}{\mu} \delta^2\, d\delta \quad \text{...(13.5)}$$

* **Heat flux**

The heat flow rate into the film, dQ, equals the rate of energy release due to condensation at the surface. Thus,

$$dQ = h_{fg}\, dm$$

$$= h_{fg} \frac{g b \rho^2}{\mu} \delta^2 d\delta \qquad ...(13.6)$$

where h_{fg} is the latent heat of condensation.

Nusselt presumed that the heat released during condensation flows only by conduction through the film.

$$dQ = \frac{k\,(b\,dx)}{\delta}(t_{sat} - t_s) \qquad ...(13.7)$$

Combining equations 13.6 and 13.7, we get :

$$\delta^3 d\delta = \frac{k\mu}{\rho^2 g h_{fg}}(t_{sat} - t_s)\, dx$$

Integration yields an expression for the thickness of condensate layer

$$\frac{\delta^4}{4} = \frac{k\mu}{\rho^2 g h_{fg}}(t_{sat} - t_s)x + c$$

Substitution of the boundary condition $\delta = 0$ at $x = 0$ yields $c = 0$

$$\therefore \quad \delta = \left[\frac{4k\mu(t_{sat} - t_s)x}{\rho^2 g h_{fg}}\right]^{0.25} \qquad ...(13.8)$$

Evidently the film thickness increases as the fourth root of the distance down the surface; the increase is rather rapid at the upper end of the vertical surface and slow thereafter.

* **Film heat transfer coefficient**

Nusselt had presumed that heat flow from the vapour to the surface is by conduction through the liquid film, *i.e.*,

$$dQ = \frac{k(b\,dx)}{\delta}(t_{sat} - t_s)$$

The heat flow can also be expressed as

$$dQ = h_x\,(b\,dx)\,(t_{sat} - t_s)$$

where h_x is the local heat transfer coefficient. It follows from these expressions that

$$h_x = \frac{k}{\delta}\,. \qquad ...(13.9)$$

Thus at a definite point on the heat transfer surface, the film coefficient h_x is directly proportional to thermal conductivity k and inversely proportional to thickness of film at that point.

Substituting the value of film thickness from equation 13.8,

$$h_x = \left[\frac{k^3 \rho^2 g h_{fg}}{4\mu x(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.10)$$

Local heat transfer coefficient at the lower edge at plate; *i.e.*, at $x = l$.

$$h_l = \left[\frac{k^3 \rho^2 g h_{fg}}{4\mu l(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.11)$$

Undoubtedly the rate of condensation heat transfer is higher at the upper end of the plate than at the lower end.

By integrating the local value of conductance (equation 13.10) over the entire length l of the plate, we get the average heat transfer coefficient ;

$$\bar{h} = \frac{1}{l}\int_0^l h_x\, dx$$

$$= \frac{1}{l}\int_0^l \left[\frac{k^3 \rho^2 g h_{fg}}{4\mu x(t_{sat} - t_s)}\right]^{0.25} dx$$

$$= \left[\frac{k^3 \rho^2 g h_{fg}}{4\mu (t_{sat} - t_s)}\right]^{0.25} \times \frac{1}{l}\int_0^l x^{-0.25} dx$$

$$= \left[\frac{k^3 \rho^2 g h_{fg}}{4\mu (t_{sat} - t_s)}\right]^{0.25} \times \left[\frac{x^{0.75}}{0.75}\right]_0^l$$

$$= \frac{4}{3}\left[\frac{k^3 \rho^2 g h_{fg}}{4\mu l(t_{sat} - t_s)}\right]^{0.25} = \frac{4}{3}h_l \qquad ...(13.12)$$

where h_l is the local heat transfer coefficient at the lower edge of the plate.

Then it follows from equation 13.9 that

$$\bar{h} = \frac{4}{3}\frac{k}{\delta_l} \qquad ...(13.13)$$

where δ_l is the film thickness at the lower end of the plate. Obviously the average heat transfer coefficient is 4/3 times the local heat transfer coefficient at the trailing edge of the plate.

Equation 13.12. is usually written in the form,

$$\bar{h} = 0.943\left[\frac{k^3 \rho^2 g h_{fg}}{\mu l(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.14)$$

The Nusselt solution derived above is an approximate one because of the assumptions admitted in the statement of the problem. Experimental results have shown that the Nusselt equation is conservative; it yields results which are approximately 20% lower than the measured values. Accordingly, use of a value of 1.13 in place of the coefficient 0.943 has been recommended by McAdams.

$$\bar{h} = 1.13\left[\frac{k^3 \rho^2 g h_{fg}}{\mu l(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.15)$$

The results concerning development of film condensation along a vertical flat plate indicate that the thickness of film increases with increase in plate height, (equation 13.8). Since the thermal resistance increases with film thickness, a decrease in heat transfer coefficient is expected and that is evident from the relations 13.10 and 13.15. The dependence of these parameters on height of plate has been shown graphically in Fig. 13.3. Further the convection coefficient increases with temperature difference $(t_{sat} - t_s)$. This may be attributed to an increase in the film thickness which results from increase in condensate rate at high temperature differences.

Whilst computing the average coefficient vide equation 13.15, all the fluid properties are evaluated at the film temperature

$$t_f = (t_{sat} + t_s)/2$$

and the vapour latent heat of condensation is evaluated at t_{sat}. Having thus determined the average condensation coefficient, the following expressions are used to obtain the total heat transfer and the total condensation rate.

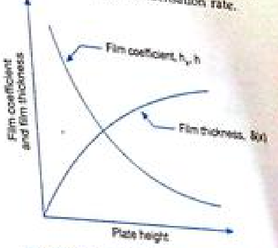


Fig. 13.3. Variation of film thickness and film coefficient with plate height

$$Q = \bar{h}A(t_{sat} - t_s) \qquad ...(13.16)$$

and $$m = \frac{Q}{h_{fg}} = \frac{\bar{h}A(t_{sat} - t_s)}{h_{fg}} \qquad ...(13.17)$$

For a vertical surface, the flow area A is $b\,\delta$.

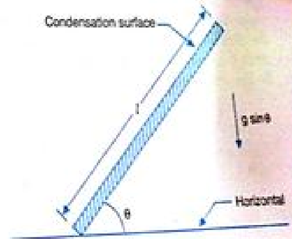


Fig. 13.4. Geometry of an inclined flat condensation surface

For *inclined flat surfaces*, the gravitational acceleration in the basic Nusselt equation is replaced by a projection of the gravity acceleration vector on the y-axis : $g_y = g \sin\theta$

where θ is the inclination angle with the horizontal. This yields the following expression for film condensation on a flat inclined surface;

$$\bar{h}_{inc} = 0.943\left[\frac{k^3\rho^2 h_{fg}\,(g \sin\theta)}{\mu\, l\,(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.18)$$

Evidently,

$$\bar{h}_{inc} = \bar{h}_{ver} \times (\sin\theta)^{0.25} \qquad ...(13.19)$$

Equation 13.19 must be used with caution for small values of inclination θ; its accuracy becomes poor as the angle of inclination approaches the horizontal.

13.3. TURBULENT FILM CONDENSATION

The character of condensate film can range from laminar to highly turbulent. The liquid flows in laminar film at the upper end of the plate, then becomes undulating in the middle section and finally flows in a turbulent state. When turbulence sets in, the condensate film no longer offers as high a thermal resistance as it does in laminar film. Heat is then transferred not only by conduction but also by eddy diffusion which is a characteristic of turbulence. Obviously the turbulent condensate film results in increased convective coefficients. Turbulent films are usually encountered when the film thickness becomes appreciable, *i.e.*, when the condensation rates are large or when the condensation surface has substantial length.

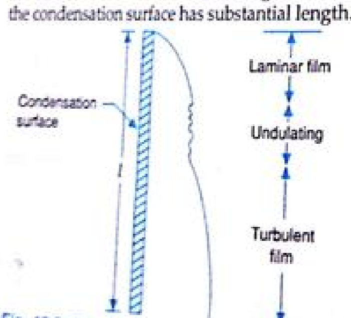


Fig. 13.5. Regions of film condensation on a vertical surface

The parameter indicating the commencement of turbulent flow is the Reynolds number conventionally defined as $Re = V d_{eq}\rho/\mu$ where the characteristic length d_{eq} is the equivalent diameter $d_{eq} = 4A/P$. For a vertical surface the flow area A is $b\delta$ and the wetted perimeter P of the solid interface is simply the width b of the plate.

That gives :

$$d_{eq} = \frac{4A}{P} = \frac{4b\delta}{b} = 4\delta$$

and

$$Re = \frac{V d_{eq}\rho}{\mu} = \frac{V \times 4\delta \times \rho}{\mu}$$

Since $m = \rho AV = \rho(b\delta)V$, we may write :

$$Re = \frac{4m}{\mu b}$$

Expressing the mass flow rate m in terms of the heat transfer,

$$m = \frac{Q}{h_{fg}} = \frac{\bar{h}A(t_{sat} - t_s)}{h_{fg}}$$

we get ;

$$Re = \frac{4\bar{h}A(t_{sat} - t_s)}{h_{fg}\,\mu b}$$

$$= \frac{4\bar{h}l(t_{sat} - t_s)}{\mu h_{fg}} \qquad ...(13.20)$$

The transition from laminar to turbulent flow occurs at a critical Reynolds number of 1800. For turbulent film condensation on vertical surfaces, Kirkbride has suggested the following correlation for the average heat transfer coefficient.

$$\bar{h} = 0.0077(Re)^{0.4}\left[\frac{k^3\rho^2 g}{\mu^2}\right]^{1/3} \qquad ...(13.21)$$

which is valid for $Re > 1800$.

The distinguishing features between the laminar and turbulent film condensation are conveyed by equations 13.14 and 13.21 as:

(*i*) In the laminar film, the average film coefficient decreases with distance l. This is due to gradual increase in the thickness of laminar film.

(*ii*) In the turbulent region, the average film coefficient increases with distance l. This is due to eddies which promote convection. Heat is then transferred not only by conduction but also by eddy diffusion.

13.4. CONVECTIVE COEFFICIENT FOR FILM CONDENSATION ON TUBES

(*i*) *Vertical tubes* : When the tube diameter is large compared to the film thickness, the convective coefficient for condensation on vertical tubes are calculated by the Nusselt correlation for film condensation on a vertical flat surface. Thus

$$\bar{h}_v = 0.943\left[\frac{k^3\rho^2 g\, h_{fg}}{\mu\, l\,(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.14)$$

(*ii*) *Single horizontal tube* : The average coefficient for film condensation of a pure saturated vapour on the outside of a single horizontal tube is determined from the correlation :

$$\bar{h}_h = 0.725\left[\frac{k^3\rho^2 g\, h_{fg}}{\mu D\,(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.22)$$

where D is the outside diameter of the tube.

It is interesting to note the relative effectiveness of horizontal and vertical tubes as condensing surfaces:

$$\frac{\bar{h}_h}{\bar{h}_v} = \frac{0.725}{0.943}\left(\frac{l}{D}\right)^{0.25}$$

$$= 0.768\left(\frac{l}{D}\right)^{0.25} \qquad ...(13.23)$$

When $\bar{h}_h = \bar{h}_v$, solution of the above relation yields $l/D = 2.86$. Thus for a cylindrical tube with a length-to-diameter ratio of 2.86, equal amount of heat transfer occurs from both horizontal and vertical orientations. As the ratio l/D increases, the greater heat transfer and so high condensation rate is possible with a horizontal tube. For most steam condensers, the ratio l/D for a tube is in the range 50 to 100 or even more.

(*iii*) *Banks of horizontal tubes* : For a vertical tier of n-horizontal tubes, the average convection coefficient for film condensation is :

$$\bar{h} = 0.725\left[\frac{k^3\rho^2 g\, h_{fg}}{\mu D_e\,(t_{sat} - t_s)}\right]^{0.25} \qquad ...(13.24)$$

where the equivalent tube diameter D_e of the tube bank is the sum of outside-tube diameter in a vertical column of the tube bank pattern.

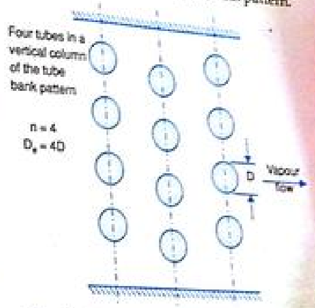


Fig. 13.6. Concept of equivalent tube diameter of a horizontal bank tube

For n-tubes in a vertical column of the tube bank pattern, $D_e = nD$ where D is the diameter of a single tube in the bank.

A reduction in the film coefficient with increasing n may be attributed to an increae in the average film thickness for each successive tube due to accumulation of drip from the upper tubes. Obviously it is advantageous to stagger the tubes (Fig. 13.7) as the accumulation of drip from the upper rows is at least partially offest by the splashing effects, *i.e.*, by the agitation caused by the drip as it falls from one tube to another.

(*iv*) *Condensation inside tubes* : Condensation of vapour occurs on the inside surface of cylindrical tubes used in condensers of refrigeration and air-conditioning systems and in many of the chemical and petrochemical

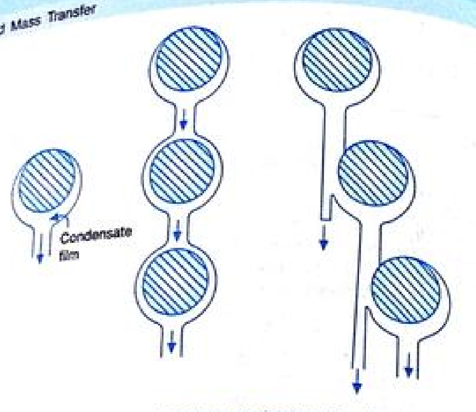


Fig. 13.7. Film condensation on horizontal cylinders

industries. The flowing vapour, however, strongly influences the character and thickness of the condensate film and that affects the heat transfer. The magnitude of this effect is dependent upon the horizontal or vertical disposition of the tubes and also upon the upward or downward flow direction.

Following empirical correlations have been suggested to workout the heat transfer when steam condenses on the inside surface of tubes:

For saturated steam :

$$Q = (3951 + 116\ V)$$

$$A\,(t_{sat} - t_s)\left(\frac{1.21}{l}\right)^{1/3}\ \text{W} \qquad ...(13.25)$$

For superheated steam :

$$Q = (4068 + 59\ V)$$

$$A\,(t_{sat} - t_s)\left(\frac{1.21}{l}\right)^{1/3}\ \text{W} \qquad ...(13.26)$$

where l is the cooled length, A is the surface area based on inside diameter of the tube and V is the entrance velocity.

EXAMPLE 13.1

A condensation experiment for steam on plate type vertical condenser has been setup for a particular fluid with a given temperature difference. The same setup was subsequently used with another fluid with thermo-physical properties given as :

$$\frac{\rho_2}{\rho_2} = 0.65\ ; \quad \frac{k_2}{k_1} = 2$$

$$\frac{\mu_2}{\mu_1} = 1.2\ ; \quad \frac{h_{fg_1}}{h_{fg_2}} = 0.5$$

If temperature difference is reduced to 8 percent, make calculations for the percentage change in the convection coefficient.

Solution : The average heat transfer coefficient for vapour condensation on a vertical plate is give by

$$h_1 = 0.943\left[\frac{k_1^3\,\rho_1^2\,g\,h_{fg1}}{\mu_1\,l_1\,(t_{sat} - t_s)_1}\right]^{0.25}$$

and
$$h_2 = 0.943\left[\frac{k_2^3\,\rho_2^2\,g\,h_{fg2}}{\mu_1\,l_2\,(t_{sat} - t_s)_2}\right]^{0.25}$$

Substituting

$$\rho_2 = 0.65\,\rho_1\ ; \quad k_2 = 2\,k_1$$

$$\mu_2 = 1.2\,\mu_1\ ; \quad h_{fg2} = 0.5\,h_{fg1}$$

$$(t_{sat} - t_s)_2 = 0.8(t_{sat} - t_s)_1$$

and $l_1 = l_2$, we get

$$h_2 = 0.943\left[\frac{2^3\,k_1^3 \times 0.65^2\,\rho_1^2 \times g \times 0.5\,h_{fg1}}{1.2\mu_1 \times l_1 \times 0.8(t_{sat} - t_s)_1}\right]^{0.25}$$

$$= 0.943\left[\frac{k_1^3\,\rho_1^2\,g\,h_{fg1}}{\mu_1\,l_1\,(t_{sat} - t_s)_1}\right]^{0.25} \times \left[\frac{2^3 \times 0.65^2 \times 0.5}{1.2 \times 0.8}\right]^{0.25}$$

$$= h_1 \times 1.152$$

i.e., there is an increase of 15.2% in the convection coefficient.

EXAMPLE 13.2

(a) A plate condender was designed to be kept vertical. How would the condensation coefficient be effected if due to site constraints, it has to be kept at 60° to the horizontal ?

(b) A plate condenser of dimensions $l \times b$ has been designed to be kept with side l in the vertical position. However due to oversight during erection and installation, it was fixed with side b vertical. How would this affect the heat transfer ? Assume laminar conditions and same thermo-physical properties in both cases and take $b = l/2$.

(c) Determine the length of a 25 cm outer diameter tube if the condensate formed on the surface of the tube is to be same whether it is kept vertical or horizontal.

Solution : (*a*) For a vertical flat plate

$$h_{ver} = 0.943\left[\frac{k^3\,\rho^2\,g\,h_{fg}}{\mu\,l\,(t_{sat} - t_s)}\right]^{0.25}$$

For inclined flat surfaces, the gravity acceleration g is replaced by $g \sin\theta$ where θ is the inclination angle with the horizontal. Then

$$h_{inc} = 0.943\left[\frac{k^3\,\rho^2\,(g\sin\theta)\,h_{fg}}{\mu\,l\,(t_{sat} - t_s)}\right]^{0.25}$$

Evidently :

$$h_{inc} = h_{ver}\,(\sin\theta)^{0.25}$$

$$= h_{ver}\,(\sin 60)^{0.25} = 0.9647$$

This implies 3.53% reduction in condensation coefficient

(*b*) With side l vertical :

$$h_1 = 0.943\left[\frac{k^3\,\rho^2\,g\,h_{fg}}{\mu\,l\,(t_{sat} - t_s)}\right]^{0.25}$$

With side b vertical:

$$h_2 = 0.943\left[\frac{k^3\,\rho^2\,g\,h_{fg}}{\mu\,b\,(t_{sat} - t_s)}\right]^{0.25}$$

$$\frac{h_1}{h_2} = \left(\frac{b}{l}\right)^{0.25} = \left(\frac{l/2}{l}\right)^{0.25} = 0.8409$$

Comments : The condensation coefficient and accordingly heat flow increases when the shorter side is kept vertical. For better condensation, the condensers should be installed with shorter side vertical.

(*c*) For laminar film condensation on a vertical tube

$$h_v = 0.943\left[\frac{k^3\,\rho^2\,g\,h_{fg}}{\mu\,l\,(t_{sat} - t_s)}\right]^{0.25} \qquad ...(i)$$

$$h_h = 0.725\left[\frac{k^3\,\rho^2\,g\,h_{fg}}{\mu\,d\,(t_{sat} - t_s)}\right]^{0.25} \qquad ...(ii)$$

From expression (*i*) and (*ii*),

$$\frac{h_h}{h_v} = \frac{0.725}{0.943}\left(\frac{l}{d}\right)^{0.25} = 0.7688\left(\frac{l}{d}\right)^{0.25}$$

For equal amount of condensation, the heat transfer rate and accordingly condensation coefficient should be same for the horizontal and vertical orientations. In that case

$$\frac{l}{d} = \left(\frac{1}{0.7688}\right)^{1/0.25} = 2.86$$

$$\therefore \quad l = 2.86\,d = 2.86 \times 25 = 71.5\ \text{cm}$$

EXAMPLE 13.3

(a) For condensing conditions, compare the condensation rate when a 6.5 cm diameter and

1.25 m long pipe is kept (i) horizontally and (ii) vertically. Assume that other conditions remain same.

(b) For condensing conditions, compare the values of convective heat transfer coefficients over a pipe of diameter with that of two pipes having the same total circumference when (i) both pipes are horizontal and parallel and (ii) the pipes lie one over the other. Assume that other conditions remain same.

Solution : (a) For vertical position

$$h_v = 0.943\left[\frac{k^3 \rho^2 g\, h_{fg}}{\mu\, l\,(t_{sat} - t_s)}\right]^{0.25} \quad ...(i)$$

and for horizontal position

$$h_h = 0.725\left[\frac{k^3 \rho^2 g\, h_{fg}}{\mu\, d\,(t_{sat} - t_s)}\right]^{0.25} \quad ...(ii)$$

From identities (i) and (ii)

$$\frac{h_h}{h_v} = \frac{0.725}{0.943}\times\left(\frac{l}{d}\right)^{0.25}$$

$$= \frac{0.725}{0.943}\times\left(\frac{1.25}{0.065}\right)^{0.25} = 1.61$$

Obviously horizontal positioning provides 61% more heat transfer. This may be attributed to larger film thickness with increase in length. Accordingly condensers are generally of horizontal type.

(b) The average heat transfer coefficient for vapour condensation on a horizontal tube is given by

$$h = 0.725\left[\frac{k^3 \rho^2 g\, h_{fg}}{\mu\, D\,(t_{sat} - t_s)}\right]^{0.25}$$

Case (i) : With same total circumference $\pi D = 2 \times \pi d$ and therefore $d = D/2$. When diameter is reduced to half the value, the convective coefficient becomes $(2)^{0.25} = 1.189$.

EXAMPLE 13.4

A vertical cooling fin, approximating a flat plate 40 cm in height is exposed to steam at atmospheric pressure. If surface of the fin is held at 80°C, make calculations for the following parameters :

(i) film thickness at the bottom edge of the fin,
(ii) overall heat transfer coefficient,
(iii) heat transfer rate and the condensate mass flow rate.

Assume unit width of the fin and check the flow Reynolds number for the assumption of laminar flow conditions.

(b) Estimate the minimum height of the plate necessary for condensate to become just turbulent.

Solution : For saturated vapour at atmospheric pressure ; the saturation temperature

$$t_{sat} = 100°\ C$$

and the latent heat of vaporisaton

$$h_{fg} = 2257 \times 10^3\ J/kg$$

For saturated water at the mean film temperature,

$$t_f = \frac{t_{sat} + t_s}{2}$$

$$= \frac{100 + 80}{2} = 90°\ C$$

The relevant fluid properties are :

$$\rho = 965.3\ kg/m^3$$
$$k = 67.995 \times 10^{-2}\ W/mK$$
and $$\mu = 3.153 \times 10^{-4}\ kg/ms$$

(i) The film thickness at a distance x from the top edge is,

$$\delta = \left[\frac{4 k \mu (t_{sat} - t_s)\, x}{\rho^2 g\, h_{fg}}\right]^{0.25}$$

$$= \left[\frac{4 \times 67.995 \times 10^{-2} \times 3.153 \times 10^{-4} \times (100 - 90)x}{(965.3)^2 \times 9.81 \times 2257 \times 10^3}\right]^{0.25}$$

$$= 1.4278 \times 10^{-4}\ x^{0.25}$$

At the bottom edge of the fin $x = 0.4$ m

$$\therefore \quad \delta_l = 1.4278 \times 10^{-4} \times (0.4)^{0.25}$$
$$= 1.135 \times 10^{-4}\ m = 0.1135\ mm$$

(ii) Overall heat transfer coefficient,

$$\bar{h} = \frac{4}{3}\frac{k}{\delta_l} = \frac{4}{3}\times\frac{67.995\times 10^{-2}}{1.135\times 10^{-4}}$$
$$= 9066\ W/m^2K$$

Applying Mc Adam's correction for steam condensation on flat vertical plates,

$$\bar{h} = 1.2 \times 9066 = 10879.2\ W/m^2K$$

(iii) Heat transfer rate Q

$$= \bar{h}A\,(t_{sat} - t_s)$$
$$= 10879.2 \times (0.4 \times 1) \times (100 - 90)$$
$$= 43588.8\ W$$

Steam condensation rate

$$m = \frac{Q}{h_{fg}} = \frac{43588.8}{2257 \times 10^3}$$
$$= 0.0193\ kg/s$$

Checking the film Reynolds number, we get

$$Re = \frac{4m}{\mu b} = \frac{4 \times 0.0193}{3.153 \times 10^{-4} \times 1}$$
$$= 244.8 < 1800$$

Thus the assumption of laminar flow has been correct.

(b) From the correlations

$$\bar{h} = 0.943\left[\frac{k^3 \rho^2 g\, h_{fg}}{\mu\, l\,(t_{sat} - t_s)}\right]^{1/4}$$

and $$Re = \frac{4m}{\mu b} = \frac{4\bar{h}\, l\,(t_{sat} - t_s)}{\mu\, h_{fg}}$$

it is noted that $Re \propto l^{3/4}$ or $l \propto Re^{4/3}$ in the laminar regime.

$$\therefore l_{transition} = 40\left(\frac{1800}{244.8}\right)^{4/3}$$

$$= 571.55\ cm \approx 5.72\ m$$

EXAMPLE 13.5

Saturated steam at atmospheric pressure condenses on the outer surface of a vertical tube of length 1 m and outer diameter 75 mm. The tube wall is maintained at a uniform surface temperature of 40°C by the flow of cooling water inside the tube. Estimate the steam condensation rate and the heat transfer rate to the tube. What water flow rate will result in a 5°C temperature difference of water between the outlet and inlet of pipe? Also calculate the flow Reynolds number to check the assumption of laminar flow conditions.

Solution : For saturated vapour at atmospheric pressure,

$$t_{sat} = 100\ °C$$
and $$h_{fg} = 2258.76\ kJ/kg$$

For saturated water at the mean film temperature

$$t_f = \frac{t_{sat} + t_s}{2}$$
$$= \frac{100 + 40}{2} = 70°\ C$$

the relevant fluid properties are ;

$$\rho = 977.8\ kg/m^3$$
$$k = 2.430\ kJ/m\text{-}hr\text{-}deg$$
$$\mu = 4.06 \times 10^{-4}\ kg/ms$$

The film thickness at the bottom edge of the tube is,

$$\delta = \left[\frac{4 k \mu (t_{sat} - t_s) l}{\rho^2 g\, h_{fg}}\right]^{0.25}$$

Inserting the appropriate values in consistent units,

$\delta =$

$$\left[\frac{4 \times 2.403\,(4.06 \times 10^{-4} \times 3600) \times (100 - 40) \times 1}{(977.8)^2 \times (9.81 \times 3600 \times 3600) \times 2258.76}\right]^{0.25}$$

$$= 2.354 \times 10^{-4}\ m = 0.2354\ mm$$

The average heat transfer coefficient is,

$$\bar{h} = \frac{4}{3}\frac{k}{\delta} = \frac{4}{3}\times\frac{2.403}{2.354\times 10^{-4}}$$
$$= 13610\ kJ/m^2\text{-}hr\text{-}deg$$

Applying Mc Adam's correction for steam condensing on vertical plates or cylinders.

$$\bar{h} = 1.2 \times 13610$$
$$= 16332\ kJ/m2\text{-}hr\text{-}deg$$

(i) Heat transfer,

$$Q = \bar{h}\, A\,(t_{sat} - t_s)$$
$$= 16332 \times (\pi \times 0.075 \times 1) \times (100 - 40)$$
$$= 230771\ kJ/hr$$

and the steam condensation rate is

$$m = \frac{Q}{h_{fg}} = \frac{230771}{2258.76} = 102.17\ kg/hr$$

Checking the flow Reynolds number, we get

$$Re = \frac{4m}{\mu b}$$

$$= \frac{4 \times 102.17}{\left(4.06 \times 10^{-4} \times 3600\right) \times (\pi \times 0.075)}$$

$$= 1187 < 1800$$

Apparently the film flow is laminar in character.

(ii) The heat released during condensation is picked up by water flowing inside the tube. Therefore,

$$Q = m_w\, c_p\, (t_0 - t_i)\ ;$$

$$230771 = m_w \times 4.186 \times 5$$

$\therefore$ Flow rate of water

$$m_w = \frac{55136.67}{5} = \mathbf{11025.8\ kg/hr}$$

EXAMPLE 13.6

A 0.5 m square plate is exposed to dry saturated steam at 0.08 bar. If surface of the plate is to be maintained at 18.5°C, make calculations for the (a) film thickness, local heat transfer coefficient and mean flow velocity of condensate at 25 cm from the top of plate, (b) average heat transfer coefficient for the entire plate and (c) total steam condensate rate and the total heat transfer rate to the plate.

What change, if any, would result in the average heat transfer coefficient if the plate is inclined at 60°C to the vertical plane ?

Solution : For saturated vapour at 0.08 bar

$$t_{sat} = 41.5°C$$

and $h_{f_g} = 2402$ kJ/kg

For saturated water at the mean film temperature,

$$t_f = \frac{t_{sat} + t_s}{2}$$

$$= \frac{41.5 + 18.5}{2} = 30°C$$

The relevant fluid properties are :

$$\rho = 995.1\ kg/m^3$$

$$k = 2.22\ kJ/m\text{-}hr\text{-}deg$$

$$\mu = 8.01 \times 10^{-4}\ Ns/m^2$$

The film thickness at a distance x from the top edge is,

$$\delta = \left[\frac{4 k \mu (t_{sat} - t_s) x}{\rho^2 g\, h_{fg}}\right]^{0.25}$$

Inserting the appropriate values in consistent units,

$\delta =$

$$\left[\frac{4 \times 2.22 \times \left(8.01 \times 10^{-4} \times 3600\right) \times (41.2 - 18.8) x}{(995.1)^2 \times (9.81 \times 3600 \times 3600) \times 2402}\right]^{0.25}$$

$$= 2.0837 \times 10^{-4}\ x^{0.25}$$

(a) At $x = 0.25$ m from the top edge

$$\delta = 2.0837 \times 10^{-4} \times (0.25)^{0.25}$$

$$= 1.475 \times 10^{-4}\ m = 0.1475\ mm$$

$$h_x = \frac{k}{\delta} = \frac{2.22}{1.475 \times 10^{-4}}$$

$$= 15051\ kJ/m^2\text{-}hr\text{-}deg$$

Mean condensate flow velocity,

$$V_m = \frac{\rho g \delta^2}{3\mu}$$

$$= \frac{995.1 \times 9.81 \times \left(1.475 \times 10^{-4}\right)^2}{3 \times 8.01 \times 10^{-4}}$$

$$= 0.08838\ m/s$$

(b) At the bottom edge of the plate,

$$x = 0.5$$

$$\delta = 2.087 \times 10^{-4} \times (0.5)^{0.25}$$

$$= 1.775 \times 10^{-4}\ m$$

Average heat transfer coefficient

$$\bar{h} = \frac{4}{3}\frac{k}{\delta} = \frac{4}{3} \times \frac{2.22}{1.775 \times 10^{-4}}$$

$$= 20068\ kJ/m^2\text{-}hr\text{-}deg$$

Applying Mc Adam's correction factor,

$$\bar{h} = 1.2 \times 20068$$

$$= 24082\ kJ/m^2\text{-}hr\text{-}deg$$

Heat flow rate,

$$Q = \bar{h}\ A\ \Delta t$$

$$= 24082 \times (0.5 \times 0.5) \times (41.5 - 18.5)$$

$$= 138471.5\ kJ/hr$$

Steam condensation rate is,

$$m = \frac{Q}{h_{f_g}} = \frac{26804.29}{574}$$

$$= \frac{138471.5}{2402} = 57.648\ kg/m$$

Checking the film Reynolds number, we get

$$Re = \frac{4m}{\mu b} = \frac{4 \times (57.648 / 3600)}{8.01 \times 10^{-4} \times 0.5}$$

$$= 159.73 < 1800$$

Apparently the flow remains laminar in character.

For a flat surface sloping θ^0 from the vertical, the gravitational acceleration changes to $g \cos\theta$. With all other parameters remaining constant, the average heat transfer coefficient takes the value :

$$\bar{h}_{inc} = h_v \cos\theta$$

$$= 24082 \cos 60$$

$$= \mathbf{12041\ kJ/m^2\text{-}hr\text{-}deg}$$

EXAMPLE 13.7

Saturated steam at 2 bar condenses on a cylindrical vertical drum having an outside diameter of 25 cm and a temperature of 90°C. Calculate how long must the drum be to condense 50 kg of steam per hour. Also estimate the thickness of condensate layer.

Solution : For saturated vapour at 2 bar, the saturation temperature t_{sat} = 120°C and the latent heat vaporisation h_{f_g} = 2705 kJ/kg

For saturated water at the mean film temperature,

$$t_f = \frac{t_{sat} + t_s}{2}$$

$$= \frac{120 + 90}{2} = 105°C$$

The relevant fluid properties are :

$$\rho = 954.7\ kg/m^3$$

$$k = 2.46\ kJ/m\text{-}hr\text{-}deg$$

$$\mu = 2.70 \times 10^{-4}\ Ns/m^2$$

(i) The film thckness at the bottom edge of the drum is,

$$\delta = \left[\frac{4 k \mu (t_{sat} - t_s) l}{\rho^2 g\, h_{fg}}\right]^{0.25}$$

Inserting the appropriate values in consistent units,

$\delta =$

$$\left[\frac{4 \times 2.46 \times \left(2.70 \times 10^{-4} \times 3600\right) \times (119.6 - 90.4) \times l}{(954.7)^2 \times (9.81 \times 3600 \times 3600) \times 2705}\right]^{0.25}$$

$$= 1.727 \times 10^{-4}\ l^{0.25}$$

The average heat transfer coefficient is

$$\bar{h} = \frac{4}{3}\frac{k}{\delta} = \frac{4}{3} \times \frac{2.46}{\left(1.727 \times 10^{-4}\right) l^{0.25}}$$

$$= \frac{18959.5}{l^{0.25}}$$

Applying Mc Adam's correction for steam condensing on vertical surfaces

$$\bar{h} = 1.2 \times \frac{18959.5}{l^{0.25}} = \frac{22751.4}{l^{0.25}}$$

Heat flow rate,

$$Q = \bar{h}\ A\ (t_{sat} - t_s) = m\ h_{f_g}$$

Thus,

$$\frac{22751.4}{l^{0.25}} \times (\pi \times 0.25 \times l) \times (119.6 - 90.4)$$

$$= 50 \times 2705$$

$$l^{0.75} = \frac{50 \times 2705}{22751.4 \times (\pi \times 0.25) \times 29.2}$$

$$= 0.259$$

$\therefore$ Length of the cylindrical drum l

$$= \mathbf{0.166\ m}$$

(ii) The film thickness at the bottom edge of the drum becomes,

$$\delta = 1.727 \times 10^{-4}\ l^{0.25}$$

$$= 1.727 \times 10^{-4}\ (0.166)^{0.25}$$

$$= 1.102 \times 10^{-4}\ m = \mathbf{0.1102\ mm}$$

EXAMPLE 13.8

The outer surface of a vertical tube which is of length 1.25 m and outer diameter 50 mm is exposed to saturated steam at atmospheric pressure. If the tube surface is maintained at 80°C by the flow of cooling water through it, determine the rate of

heat transfer to the coolant and the rate at which steam is condensed at the tube surface.

If the tube is held in horizontal position, will there be any change in mass of condensate? If yes, find the value.

Solution : For saturated steam at atmospheric pressure,

$$t_{sat} = 100°C$$

and $h_{f_g} = 2285 \times 10^3$ J/kg

At the mean film temperature,

$$t_f = \frac{t_{sat} + t_s}{2}$$

$$= \frac{100 + 80}{2} = 90°C$$

The property values of saturated water are :

$$\rho = 965.3 \text{ kg/m}^3$$

$$k = 0.68 \text{ W/mK}$$

$$\mu = 1.135 \text{ kg/m-hr}$$

$$= 3.139 \times 10^{-4} \text{ kg/ms}$$

(i) The average heat transfer coefficient for steam condensing on vertical tubes is given by :

$$\bar{h} = 0.943 \left[\frac{k^3 \rho^2 g h_{f_g}}{\mu l (t_{sat} - t_s)}\right]$$

$$= 0.943 \left[\frac{0.68^3 \times 965.3^2 \times 9.81 \times 2285 \times 10^3}{3.139 \times 10^{-4} \times 1.25 \times (100 - 80)}\right]^{0.25}$$

$$= 5046.8 \text{ W/m}^2\text{K}$$

Heat flow rate,

$$Q = \bar{h} A (t_{sat} - t_s)$$

$$= 5064.8 \times (\pi \times 0.05 \times 1.25) \times (100 - 80)$$

$$= 19879 \text{ W}$$

Steam condensation rate,

$$m = \frac{Q}{h_{f_g}} = \frac{19879}{2285 \times 10^3} = 8.7 \times 10^{-3}$$

(ii) Since the path of condensate flow is shorter and the film is thinner, the heat transfer coefficient for tubes placed in horizontal position will be much higher. Obviously then the mass of steam condensed will increase.

The heat transfer coefficient for a horizontal tube is given by :

$$\bar{h}_h = 0.725 \left[\frac{k^3 \rho^2 g h_{f_g}}{\mu D (t_{sat} - t_s)}\right]$$

$$\therefore \quad \frac{\bar{h}_h}{\bar{h}_v} = \frac{0.725}{0.943} \left(\frac{l}{D}\right)^{0.25}$$

$$\bar{h}_h = \frac{0.725}{0.943} \left(\frac{1.25}{0.05}\right)^{0.25} \times 5064.8$$

$$= 8707 \text{ W/m}^2\text{K}$$

$$Q_h = 8707 \times (\pi \times 0.05 \times 1.25) \times (100 - 80)$$

$$= 34175 \text{ W}$$

$$m_h = \frac{34175}{2285 \times 10^3} \; 14.95 \times 10^{-3} \text{ kg/s}$$

Note : The Mc Adam's correction factor has not been applied while working out this problem.

EXAMPLE 13.9

A condenser is to be designed to condense 2250 kg of steam per hour at a pressure of 0.15 bar. A square array of 400 tubes, each of 6 mm in diameter, is available for the task. If the tube surface temperature is to be maintained at 26°C, make calculations for the length of each tube.

Solution : For the saturated water at 0.15 bar pressure

$$t_{sat} = 54°C$$

and $h_{f_g} = 2373 \times 10^3$ J/kg

For saturated water at the mean film tempeature,

$$t_f = \frac{t_{sat} + t_s}{2}$$

$$= \frac{54 + 26}{2} = 40°C$$

The relevant fluid properties are

$$\rho = 922.2 \text{ kg/m}^3$$

$$k = 63.34 \times 10^{-2} \text{ W/mK}$$

$$\mu = 235.602 \times 10^{-2} \text{ kg/hr-m}$$

$$= 6.54 \times 10^{-6} \text{ kg/sm}$$

The number of tubes n in a vertical column of the square array of 400 tubes is,

$$n = \sqrt{400} = 20$$

and for these 20-tubes in a vertical column the equivalent tube diameter is ;

$$D_e = nD = 20 \times 0.06 = 0.12 \text{ m}$$

The average heat transfer coefficient for steam condensing on horizontal tubes is given by :

$$\bar{h} = 0.725 \left[\frac{k^3 \rho^2 g h_{fg}}{D_e \mu (t_{sat} - t_s)}\right]^{0.25}$$

$$\bar{h} = 0.725 \left[\frac{(63.34 \times 10^{-2})^3 \times 993.3^2 \times 9.81 \times 2373 \times 10^3}{0.12 \times 6.54 \times 10^{-6} \times (53.95 - 26)}\right]^{0.25}$$

$$= 5205.34 \text{ W/m}^2\text{K}$$

Total surface area of 400 tubes

$$= 400 \times \pi d l$$

$$= 400 \times \pi \times 0.006 \times l = 7.536\, l$$

Heat flow rate,

$$Q = \bar{h} A (t_{sat} - t_s) = m\, h_{fg}$$

Thus,

$$5205.34 \times 7.536\, l \times (53.95 - 26)$$

$$= \frac{2250}{3600} \times 2373 \times 10^3$$

$$1096407\, l = 1483125$$

$\therefore$ length of each tube, l

$$= \frac{1483125}{1096407} = 1.353 \text{ m}$$

13.5. BOILING

Boiling constitutes the convective heat transfer process that involves a phase change from liquid to vapour state. This is achieved through heat supplied to the liquid and occurs when a heated surface is exposed to the liquid and is maintained at a temperature above the saturation temperature of the liquid. The boiling process has wide-spread applications, *e.g.*, in

(i) production of steam in nuclear and steam power plants for generation and for industrial processes and space heating

(ii) absorption of heat in refrigeration and air-conditioning systems

(iii) concentration, dehydration and drying of foods and materials

(iv) distillation and refining of liquids

Greater importance has recently been given to the boiling heat transfer because of developments in nuclear reactors, space-crafts and rockets where large quantities of heat are produced in a limited space and are to be dissipated at rates as high as 10^8 W/m². Such high heat transfer rates can be well appreciated by comparing it with the maximum heat transfer rate in a modern boiler which is about 2×10^5 W/m².

The boiling phenomenon is known to occur in the following forms :

(i) *Pool boiling* : The liquid above the hot surface is essentially stagnant, and the only motion near the surface is because of free convection and mixing induced by the bubble growth and detachment. The pool boiling occurs in steam boilers employing natural convection.

(ii) *Forced convection boiling* : The fluid motion is induced by external means. The liquid is pumped and forced across the surface in a controlled manner. The free convection and the bubble-induced-mixing also contribute towards fluid motion.

(iii) *Subcooled or local boiling* : The temperature of the liquid is below the saturation temperature and boiling takes place only in vicinty of the heated surface. The vapour bubbles travel a short path and then vanish; apparently they condense in the bulk of the liquid which is at a temperature less than the boiling point.

(iv) *Saturated boiling* : The temperature of the liquid exceeds the saturation temperature. The vapour bubbles generated at the solid surface (liquid-solid interface) are transported through the liquid by buoyancy effects and eventually escape from the surface (liquid-vapour interface). The actual evaporation process then sets in.

13.6. BOILING REGIMES

Whether the boiling phenomenon corresponds to pool boiling or forced circulation boiling, there are three definite regimes of boiling associated with progressively increasing heat flux. These regimes have been identified in Fig. 13.8 which shows the relationship between heat flux (or film coefficient) and the temperature excess $(t_s - t_{sat})$; t_s is the temperature of the hot surface and t_{sat} is the saturation temperature corresponding to the pressure at which the liquid is being evaporated.

A rise in the slope of the curve indicates an increasing heat flux with increasing temperature excess and the boiling regime is stable. When the slope decreases, the boiling regime is unstable and must be avoided. These different regimes can be illustrated by considering an electrically-heated horizontal wire submerged in a pool of liquid at saturation temperature. The heat flux is easily controlled by voltage drop across a wire of fixed resistance.

1. *Evaporation process with no bubble formation (interface evaporation)* : The boiling takes place in a thin layer of liquid which adjoins the heated surface. The liquid in the immediate vicinity of the wall becomes superheated, *i.e.*, temperature of the liquid exceeds the saturation temperature at the given pressure. The superheated liquid rises to the liquid-vapour interface where evaporation takes place. The fluid motion is determined primarily by free convection effects. The heat transfer rate increases, but gradually, with growth in a temperature excess.

2. *Nucleate boiling* : When the liquid is overheated in relation to saturation temperature, vapour bubbles are formed at certain favourable spots called the nucleation or active sites; these may be the wall surface irregularities, air bubbles and the particles of dust. The bubbles grow to certain size influenced by pressure, temperature and surface tension at the liquid-vapour interface. Depending on temperature excess, the nucleate boiling essentially consists of the following stages :

(*a*) Bubbles form and collapse on the surface itself

(*b*) Bubbles form on the heated surface, but get condensed in the liquid after detaching from the surface

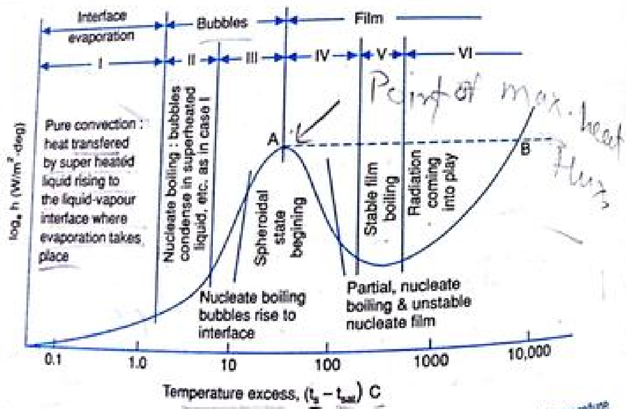


Fig. 13.8. Boiling curve for water : surface heat flux as a function of excess temperature

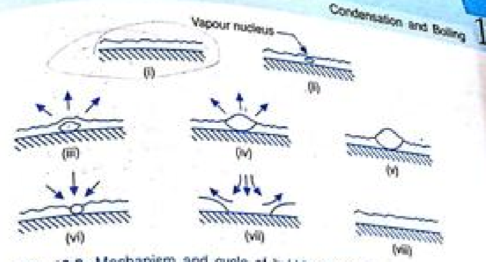


Fig. 13.9. Mechanism and cycle of bubble formation and collapse

(*c*) Bubbles form, break away from the heated surface with increasing frequency and intensity. The liquid is, however, quite hot and the bubbles do not condense in it. They rise to the liquid surface and are directly expelled to the vapour space and that helps rapid evaporation.

The mechanism and the cycle of bubble formation has been illustrated in Fig. 13.9.

(*i*) Liquid next to the heated surface becomes superheated

(*ii*) Vapour nucleus of sufficient size is created to initiate the formation of bubble

(*iii*) Bubble grows in size and pushes the layer of superheated liquid away from the heated surface

(*iv*) Top of the bubble comes into contact with the cooler liquid and that has a tendency to arrest the bubble growth

(*v*) Inertia of the liquid and the bubble has caused the bubble to grow to a size and position that it loses more heat to the cooler liquid than it gains by conduction from the heated surface

(*vi*) Bubble begins to collapse and the cooler liquid gains velocity to fill in the bubble volume

(*vii*) Bubble suffers a total collapse and inertia of the cooler liquid brings it into contact with the heating surface

(*viii*) Eventually the cooler liquid gets heated above the saturation temperature, and another cycle for bubble formation and its collapse begins.

The nucleation boiling is thus characterised by the formation of bubbles at the nucleation sites and the resulting liquid agitation. The bubble agitation induces considerable fluid mixing and that promotes substantial increase in the heat flux and the boiling heat transfer coefficient.

3. *Film boiling* : The bubble formation is very rapid; the bubbles blanket the heating surface and prevent the incoming fresh liquid from taking their place. Eventually the bubbles coalesce and form a vapour film which covers the surface completely. Insulating effect of the vapour film (its low thermal conductivity) overshadows the beneficial effect of liquid agitation and consequently the heat flux drops with growth in temperature excess. Within the temperature range $50 < \Delta t < 150$, conditions oscillate between nucleate and film boiling and this phase is referred to as *transition boiling, unstable film boiling* or *partial film boiling*. Eventually the temperature differences are so large that radiant heat-flux becomes significant, rather controlling factor and the heat flux curve begins to rise upward with increasing temperature excess. That marks the region of stable film boiling. The phenomenon of stable film boiling is referred to as *"Leidenfrost effect"*.

Critical heat flux - Burnout point : The boiling crisis or the burnout point A on the

boiling curve represents the point of maximum heat flux at which transition occurs from nucleate to film boiling. The maximum heat flux is called the critical heat flux and the corresponding temperature excess is termed as the critical temperature difference. For water evaporating at atmospheric pressure, the burnout occurs at temperature excess slightly above 55 K and has heat flux of the order of 1.58×10^6 W/m^2.

The boiling process remains in the unstable state beyond the burnout point A until situation corresponds to point B on the boiling curve. An increase in the temperature excess beyond burnout is accompanied by decrease in the heat transfer capability of the surface. The result is a continued increase in the surface temperature. At point B an equilibrium is established between the energy input into the surface and the heat flux away from the surface. The boiling conditions get stabilized but then the surface temperature has already become very large. And if the surface temperature exceeds the temperature limit of the wall material, burnout (structural damage and failure) of the wall occurs.

13.7. BUBBLE GROWTH AND NUCLEATE BOILING

The bubble formation in nucleate boiling is greatly influenced by the nature and condition of the heating surface and surface tension at the solid-liquid interface. The surface tension signifies wetting capability of the surface with the liquid and that influences the angle of contact between the bubble and the solid surface. Any contamination of the surface would affect its wetting characteristics and influence the size and shape of the vapour bubbles. Based upon different wetting characteristics, the typical vapour bubbles assume the shapes indicated in Fig. 13.10.

On an unwetted surface (high surface tension) the bubbles spread out and form a wedge between the heating surface and the liquid. For a partially wetted surface, the angle of contact is smaller than that for unwetted surface. The area occupied by the vapour bubbles is reduced and obviously the area of transfer of heat from the heated surface becomes larger. When surface roughness counteracts the surface tension forces, the surface becomes fully wetted. The vapour bubbles tend to become globular or oval in shape and they are more promptly disengaged from the surface. The situation is most favourable for efficient heat transfer because the area covered by the insulating vapour film is the smallest.

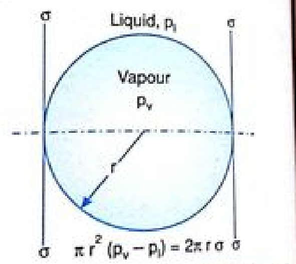


Fig. 13.11. Force balance for a spherical bubble

Experimental evidence does indicate that the vapour bubbles are not always in thermodynamic equilibrium with the surrounding liquid. The vapour inside the bubbles is not necessarily at the same temperature as the liquid and the vapour pressure p_v inside the bubble exceeds the liquid pressure p_l acting from outside of the bubble. Fig. 13.11 indicates one such spherical bubble with various forces acting on it.

(i) the resultant pressure $(p_v - p_l)$ acts on area πr^2 and the pressure force equals $\pi r^2 (p_v - p_l)$.

(ii) the surface tension σ of the vapour liquid interface acts on the interface length $2\pi r$ and the surface tension force equals $2\pi r \sigma$.

Under equilibrium conditions, the pressure force is balanced by the surface tension force. Thus

$$\pi r^2 (p_v - p_l) = 2\pi r \sigma$$

$$\text{or} \quad (p_v - p_l) = \frac{2\sigma}{r} \quad \text{...(13.27)}$$

The vapour may be approximated as a perfect gas for which the Clayperon equation is

$$\frac{dp}{dT} = \frac{p_v}{R_v} \frac{h_{fg}}{T_v^2}$$

$$\text{or} \quad \frac{p_v - p_l}{T_v - T_{sat}} = \frac{p_v}{R_v} \frac{h_{fg}}{T_v^2} \quad \text{...(13.28)}$$

where h_{fg} is the latent heat of vaporisation, T_v is the vapour temperature inside the bubble, T_{sat} is the saturation temperature of the vapour inside the bubble at pressure p_v and R_v is the effective gas constant for the vapour. Equivalence of pressure differential $(p_v - p_l)$ from 13.27 and 13.29 yields,

$$(T_v - T_{sat}) \frac{p}{R_v} \frac{h_{fg}}{T_{sat}} = \frac{2\sigma}{r}$$

$$\text{or} \quad (T_v - T_{sat}) = \frac{2\sigma}{r} \left(\frac{R_v}{p_v} \frac{T_v^2}{h_{fg}} \right) \quad \text{...(13.29)}$$

Considering a non-condensible gas inside the bubble exerting a pressure p_g, then

$$p_v + p_g - p_l = \frac{2\sigma}{r}$$

Equation 13.30 then takes the form

$$T_v - T_{sat} = \left(\frac{2\sigma}{r} - p_g \right) \frac{R_v T_v^2}{p_v h_{fg}} \quad \text{...(13.30)}$$

This is the equilibrium relationship between the bubble radius and the amount of superheat.

A bubble of radius r will grow if $(T_l - T_{sat}) > (T_v - T_{sat})$; otherwise it will collapse. Here T_l is the temperatre of the liqud surrounding the bubble.

The bubble diameter D_b at the time of detachment from the surface can be worked out from the relation proposed by Fritz :

$$D_b = C_d \beta \sqrt{\frac{2\sigma}{g(\rho_l - \rho_v)}} \quad \text{...(13.31)}$$

where β is the angle of contact and the empirical constant C_d has the value 0.0148 for water bubbles.

The nucleate pool boiling is influenced by the following factors :

(i) *Material, shape and condition of the surface* : Under identical conditons of pressure and temperature difference, the boiling heat transfer coefficient is different for different metals; copper has a high value compared to steel. Further a rough surface gives a better heat transmission than when the surface is either smooth or has been coated to weaken its tendency to get wetted.

(ii) *Pressure* : The temperature difference between the heating surface and the bulk and hence the rate of bubble growth is affected by pressure. The maximum allowable heat flux for a boiling liquid increases with pressure until critical pressure is reached and therafter it declines.

(iii) *Liquid properties* : Experiments have shown that the bubble size increases with the dynamic viscosity of the liquid. With increases in bubble size, the frequency of bubble formation decreases and that results in reduced rate of heat transfer.

High thermal conductivity of the liquid imporves the heat transfer rate.

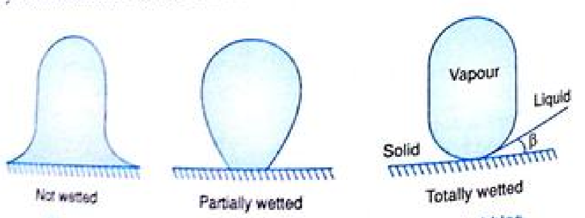


Fig. 13.10. Wetting characteristics for typical vapour bubbles

13.8. CORRELATION OF BOILING HEAT TRANSFER DATA

The heat transfer during a boiling process is governed by the convective heat transfer equation:

$$Q = h\,A\,\Delta t$$

where Q is the heat flow rate, h is the boiling film coefficient, A is the surface area and Δt is the temperature differential between the heating surface and the saturated liquid; $\Delta t = t_s - t_{sat}$.

The boiling is a phase change process; the latent heat of fluid is absorbed and it involves changes in density, viscosity, specific heat and thermal conductivity of the fluid. The fluid behaviour is very difficult to describe and as such there is no adequate analytical solution for boiling heat transfer. Because of this aspect, most of the engineering calculations involving boiling are made from empirical relations.

Some of the accepted and more widely used correlations for the boiling film coefficient are presented below for the different regimes of boiling.

13.8.1. Free Convection Boiling

When evaporation takes place at the liquid-vapour interface, the heat transfer is solely due to free convection and the film coefficient follows the relation

$$Nu = f_1(Gr)\,f_2(Pr)$$

The functions f_1 and f_2 depend upon the geometry of the heating surface. In terms of heat flux, this relation may be recast as :

$$\frac{Q}{A} = C\frac{k}{l}(Gr\,Pr)^m\,\Delta t \qquad ...(13.32)$$

The constant C and the exponent m are determined through experiments.

Since $Gr = l^3\,\beta g\,\Delta t/\nu^2$ and since the exponent m is usually 1/4 for laminar flow and 1/3 for turbulent flow, the heat transfer in this regime varies with $(\Delta t)^{5/4}$ for laminar and $(\Delta t)^{5/3}$ for turbulent flow conditions.

Fritz correlated the data of different investigators and formulated the following simplified correlation for water boiling at atmospheric pressure in free convection in a vertical tube heated from outside.

$$h = 1.973\left(\frac{Q}{A}\right)^{0.75}$$
$$= 15.11\,(\Delta t)^3\ \text{W/m}^2\text{-deg} \qquad ...(13.33)$$

This correlation considers the fact that the bubbles rising in a narrow space become more and more crowded and displace more liquid. Evidently the fluid changes in composition, density and velocity. With forced convection, the fraction of evaporation is small and the formation and motion of bubbles do not significantly disturb the turbulent flow pattern. The situation then corresponds to forced convection of a single-phase liquid. With water as the boiling liquid, an approximate correlation is of the form :

$$Nu = 0.028\,(Re)^{0.8}\,(Pr)^{0.4} \qquad ...(13.34)$$

13.8.2. Nucleate Boiling

The nucleate boiling regime is of great engineering importance because of the very high heat fluxes possible with moderate temperature differences.

The experimental data in the nucleate boiling regime is correlated by modifying the basic expression, $Nu = f_1(Gr)\,f_2(Pr)$, where f_1 and f_2 are the appropriate functions. The Reynolds number is replaced by a modulus significant of the agitation (mixing and turbulent motion) of the fluid particles in nucleate boiling. Such a dimensionless modulus is defined by the relation.

$$Re_b = \frac{D_b G_b}{\mu_f} \qquad ...(13.35)$$

where D_b is the average bubble diameter, G_b is the mass velocity of the bubble per unit area and μ_f is the fluid viscosity. The mass velocity is determined from,

$$G_b = \frac{Q/A}{h_{fg}}\left(\frac{\rho_f}{\rho_s}\right) \qquad ...(13.36)$$

where h_{fg} is the latent heat of vaporisation.

The bubble diameter has been expressed by Fritz as

$$D_b = C_d\,\beta\sqrt{\frac{2\sigma}{g(\rho_f - \rho_g)}} \qquad ...(13.37)$$

C_d is constant which has been evaluated as 0.0148 for hydrogen and water bubbles, σ is surface tension of the liquid and β is the bubble contact angle measured through liquid in degrees.

The Prandtl number is also modified by using the bubble diameter and thermal conductivity of the fluid as significant variables.

(*i*) Based on extensive experimental data, Rosenhow has developed the following empirical expression for nucleate pool boiling

$$\frac{Q}{A} = \mu_f h_{fg}\left[\frac{g(\rho_f - \rho_g)}{\sigma}\right]^{0.5}\left[\frac{c_{pf}\,\Delta t}{C_{sf}\,h_{fg}\,Pr_f^{1.7}}\right]^3 \qquad ...(13.38)$$

where μ_f = liquid viscosity in kg/m-s
h_{fg} = enthalpy of vaporisation in J/kg
ρ_g = density of dry saturated fluid in kg/m^3
ρ_f = density of saturated fluid in kg/m^3
σ = surface tension in N/m
c_{pf} = specific heat of saturated fluid in J/kgK
Pr = Prandtl number of saturated fluid
C_{sf} = surface fluid constant
Δt = $(t_s - t_{sat})$ excess temperature in °C or K
Q/A = heat flux per unit area, W/m^2

The subscripts f and g refer to fluid (liquid) and vapour (gas) states respectively. The surface-fluid constant C_{sf} depends upon particular combination of the fluid and heating surface involved in the boiling situation and is a function of the surface roughness (number of nucleating sites) and of the angle of contact between the heating surface. For example, C_{sf} = 0.013 for water-copper combination and C_{sf} = 0.006 for water-brass combination.

(*ii*) Jakob has suggested the following correlation for nucleate boiling at atmospheric pressure on a flat plate and with low heat fluxes

$$Nu = 0.16\,(Gr\,Pr)^{0.33}$$

and $$h = 0.16\left[\frac{k^2\rho^2\,\beta g\,\Delta t\,c_p}{\mu}\right]^{0.33} \qquad ...(13.39)$$

A radical departure from the Jakob correlation is encountered when the boiling heat flux increases. Fritz has recommended the following equation when heat flux lies in the range 15000-250000 W/m^2

$$h = 1.54\left(\frac{Q}{A}\right)^{0.75}$$
$$= 5.58\,(\Delta t)^3\ \text{W/m}^2\text{-deg} \qquad ...(13.40)$$

(*iii*) For the nucleate boiling on a vertical flat plate, the Jakob correlation is of the form

$$Nu = 0.61\,(Gr\,Pr)^{0.25}$$

and $$h = 0.61\left[\frac{k^3\rho^2\,\beta g\,\Delta t}{l\mu}\right]^{0.25} \qquad ...(13.41)$$

The characteristic length l is the vertical height of the plate.

(*iv*) Under the peak heat flux conditions, the vapour coming out of the vapour film is thought to be in the form of pulsating jet or vortex sheet. The relative velocity at the interface produces an instability that causes the jet to break into spheres. Taking this stipulation into account, Zuber has proposed the following relation for the maximum heat flux in nucleate boiling

$$\left(\frac{Q}{A}\right)_{max} = 0.18\rho_g h_{fg}\left[\frac{\sigma(\rho_f - \rho_g)g}{\rho_g^2}\right]^{0.25}\left[\frac{\rho_f}{\rho_g + \rho_f}\right]^{0.5} \qquad ...(13.42)$$

It may be observed that the peak heat flux is independent of the heating element.

Another correlation for maximum heat flux, as suggested by Rosenhow and Griffth is of the form :

$$\left(\frac{Q}{A}\right)_{max} = 0.00374\rho_g h_{fg} g^{0.25}\left(\frac{\rho_f - \rho_g}{\rho_g}\right)^{0.6} \quad ...(13.43)$$

Equation 13.43 is applicable for water and a variety of organic liquids.

13.8.3. Film Boiling

The minimum heat flux during film boiling can be calculated from the following expression suggested by Zuber

$$\left(\frac{Q}{A}\right)_{min} = 0.09\rho_g h_{fg}\left[\frac{g(\rho_f - \rho_g)}{\rho_f + \rho_g}\right]^{0.5}\left[\frac{\sigma}{g(\rho_f - \rho_g)}\right]^{0.25} \quad ...(13.44)$$

Heat transfer during stable film boiling is due to both heat conduction and radiation from the heating surface through the vapour film. Stable film boiling on the surface of horizontal tubes and vertical plates has been studied both analytically and experimentally by Bromley and his correlations for the convective and radiative film coefficients are :

(i) $$h_c = 0.62\left[\frac{k_g^3\rho_g(\rho_f - \rho_g)g(h_{fg} + 0.4c_g\,\Delta t)}{D\mu_g\,\Delta t}\right]^{0.25} \quad ...(13.45)$$

D is the outside diameter of the tube, and the vapour properties are taken at the mean film temperature, $t_f = (t_s + t_{sat})/2$

(ii) $$h_r = \frac{\sigma_s \in (T_s^4 - T_{sat}^4)}{(T_s - T_{sat})} \quad ...(13.46)$$

$\in$ is the emissivity of the solid surface and σ_s is the Stefan Boltzman constant for a black body.

(iii) Total heat transfer coeffcient,

$$h = h_c\left(\frac{h_c}{h}\right)^{0.33} + h_r \quad ...(13.47)$$

Equation 13.47. involves h implicitly and as such is difficult to use. For most problems of engineering interest, the following explicit equation is used with an approximation error of ± 5 percent.

$$h = h_c + \frac{3}{4}h_r \quad ...(13.48)$$

Equations 13.47 to 13.48 are equally applicable for forced convection of a liquid flowing normally across horizontal tubes when $V < \sqrt{gD}$. If $V > 2\sqrt{gD}$, equation 13.45 is replaced by

$$h_c = 2.7\left[\frac{V k_g \rho_g (h_{fg} + 0.4c_g\,\Delta T)}{D\,\Delta T}\right]^{0.5}$$

13.9. SIMPLIFIED RELATIONS FOR BOILING WATER

Certain simplified relationships have been developed by various investigators to estimate the boiling heat transfer for boiling of water at atmospheric pressure.

Table 13.1. Simplified relations for pool boiling water at atmoshperic pressure

Surface orientation	Q/A (kW/m²)	h (W/m²-deg)
Horizontal	$Q/A < 16$	$1042\,(\Delta t)^{1/3}$
	$16 < Q/A < 240$	$5.56\,(\Delta t)^3$
Vertical	$Q/A < 3$	$537\,(\Delta t)^{1/7}$
	$3 < Q/A < 63$	$7.96\,(\Delta t)^3$

For increased pressure, the values listed in Table 13.1. are increased by the factor $(p/p_o)^{0.4}$ where p is the boiling pressure and p_o is the atmospheric pressure. Further, the constants in these formulae are not dimensionless. Hence care must be taken to ensure that the quantities are expressed in the appropriate units.

EXAMPLE 13.10

Spherical bubbles of 3 mm diameter are observed in the bulk fluid boiling of water at standard atmospheric pressure. Assuming pure water vapour in the bubble and vapour pressure equal to 101.325 kN/m², calculate the temperature of the vapour

Solution : Invoking the equilibrium relation between the bubble radius and amount of superheat, we have

$$T_v - T_{sat} = \left(\frac{2\sigma}{r} - p_g\right)\frac{R_v T_v^2}{p_v h_{fg}}$$

Since the bubble contains no non-condensable gas, $p_g = 0$ and therefore

$$T_v - T_{sat} = \frac{2\sigma}{r} \times \frac{R_v T_v^2}{p_v h_{fg}}$$

Substituting the data,

$r = \frac{3}{2}$ mm = 0.0015 m

σ = 0.058 N/m

R_v = Gas constant of vapour = 4615 J/kgK

p_v = 101.325 kN/m²

h_{fg} = 2255 kJ/kg

T_v = 373 K

We get :

$$T_v - T_{sat} = \frac{2 \times 0.058}{0.0015} \times \frac{4615 \times 373^2}{(101.325 \times 1000) \times (2255 \times 1000)} = 0.217$$

$\therefore \quad T_v = T_{sat} + 0.217 = 100 + 0.217 =$ **100.217° C**

EXAMPLE 13.11

An electric wire of 1.25 mm diameter and 250 mm long is laid horizontally and submerged in water at 7 bar. The wire has an applied voltage of 2.2 V and carries a current of 130 amperes. If the surface of the wire is maintained at 200°C, make calculations for the heat flux and boiling heat transfer coefficient.

Solution : Electrical energy input to wire,

$Q = VI = 2.2 \times 130 = 286$ W

Surface area of the wire,

$A = \pi dl = \pi \times 0.00125 \times 0.25 = 9.81 \times 10^{-4}$ m²

$\therefore$ Heat flux,

$$q = \frac{Q}{A} = \frac{286}{9.81 \times 10^{-4}} = \mathbf{0.2915 \times 10^6\ W/m^2}$$

(b) Corresponding to 7 bar pressure :

$t_{sat} = 165°C$

Then from energy balance

$Q = h\,A\,\Delta t$

$286 = h \times (9.81 \times 10^{-4}) \times (200 - 165)$

$\therefore$ Boiling heat transfer coefficient,

$$h = \frac{286}{(9.81 \times 10^{-4}) \times (200 - 165)} = \mathbf{8330\ W/m^2\text{-}deg}$$

EXAMPLE 13.12

A 0.10 cm diameter and 15 cm long wire has been laid horizontally and submerged in water at atmospheric pressure. The wire has a steady state voltage drop of 14.5 V and a current of 42.5 A. Determine the heat flux and the excess temperature of the wire.

The following equation applies for water boiling on a horizontal submerged surface :

$$h = 1.54\,(Q/A)^{0.75} = 5.58\,(\Delta t)^3\ \text{W/m}^2\text{-K}$$ *where*

Q/A is the heat flux rate in W/m² and Δt is the temperature difference between surface and saturation.

Solution : Electrical input to wire,

$Q = EI = 14.5 \times 42.5 = 616.25$ W

Wire surface area,

$A = \pi dl = \pi \times 0.001 \times 0.15 = 4.71 \times 10^{-4}$ m²

Boiling energy flux,

$$\frac{Q}{A} = \frac{616.25}{4.71 \times 10^{-4}} = 1.308 \times 10^6\ \text{W/m}^2$$

Inserting the appropriate values in the given equation,

$1.54 \times (1.308 \times 10^6)^2 = 5.58\,(\Delta t)^3$

$\therefore$ Temperature excess Δt = **19.01 °C**

At atmospheric pressure, the saturation temperature is 100°C and as such the approximate wire temperature would be :

$t_w = 100 + 19.01 =$ **119.01° C**

EXAMPLE 13.13

Estimate the peak heat flux for water boiling at normal atmospheric pressure.

The relevant thermo-physical properties are

ρ_f (liquid) = 958.45 kg/m^3
ρ_g (vapour) = 0.61 kg/m^3
h_{fg} = 2.25 × 10^6 J/kg
and σ = 0.0585 N/m

Solution : At the point of maximum heat flux, the recommended correlation is

$$\left(\frac{Q}{A}\right)_{max} = 0.18\rho_g h_{fg}\left[\frac{\sigma(\rho_f-\rho_g)g}{\rho_g^2}\right]^{0.25} \times \left(\frac{\rho_f}{\rho_f+\rho_g}\right)^{0.5}$$

Substituting the appropriate values,

$$\left(\frac{Q}{A}\right)_{max} = 0.18 \times 0.61 \times 2.25 \times 10^6 \times \left[\frac{0.0585(958.45-0.61)\times 9.81}{0.61^2}\right]^{0.25} \times \left(\frac{958.45}{958.45+0.61}\right)^{0.5}$$

$$= 0.247 \times 10^6 \times (6.199) \times (0.999)$$
$$= 1.53 \times 10^6 \text{ J/s m}^2$$
$$= \mathbf{1.53 \times 10^6\ W/m^2}$$

EXAMPLE 13.14

A 1.0 mm diameter and 300 mm long nickel wire is submerged horizontal in water at atmospheric pressure. At burnout, the wire has a current of 195 A. Calculate the voltage at burnout.

The relevant thermo-physical properties are :

ρ_f (fluid) = 959.52 kg/m^3
ρ_g (vapour) = 0.597 kg/m^3
h_{fg} = 2.257 × 10^6 J/kg
and σ = 0.0533 N/m

Solution : At burnout, i.e., the point of maximum heat flux, the recommended correlation is

$$\left(\frac{Q}{A}\right)_{max} = 0.18\rho_g h_{fg}\left[\frac{\sigma(\rho_f-\rho_g)g}{\rho_g^2}\right]^{0.25} \times \left(\frac{\rho_f}{\rho_f+\rho_g}\right)^{0.5}$$

Substituting the appropriate values in consistent units,

$$\left(\frac{Q}{A}\right)_{max} = 0.18 \times 0.597 \times 2.257 \times 10^6 \times \left[\frac{0.0533(959.52-0.597)\times 9.81}{(0.597)^2}\right]^{0.25} \times \left(\frac{959.52}{959.52+0.597}\right)^{0.5}$$

$$= 0.2425 \times 10^6\ (6.124) \times 0.9997$$
$$= 1.48 \times 10^6 \text{ W/m}^2$$

Let E_b be the voltage at burnout. Then electric energy input to wire is

$$E_b \times I = 195\ E_b \text{ W}$$

The surface area of wire is

$$A = \pi dl = \pi \times 0.001 \times 0.3 = 9.42 \times 10^{-4} \text{ m}^2$$

From energy balance,

$$\frac{195E_b}{9.42\times 10^{-4}} = 1.48 \times 10^6$$

∴ Burnt out voltage E_b is equal to

$$\frac{1.48\times 10^6 \times 9.42\times 10^{-4}}{195} = 7.15 \text{ V}$$

EXAMPLE 13.15

A copper pan of 35 cm diameter contains water and its bottom surface is maintained at 115°C by an electric heater. Calculate the power required to boil water in this pan and the rate at which water evaporates from the pan due to the boiling process. Also make calculations for the heat flux for these conditions.

Solution : Temperature excess

$$\Delta t = t_s - t_{sat} = 115 - 100 = 15°\text{C}$$

The situation then corresponds to nucleate boiling for which the following correlation applies;

$$\frac{Q}{A} = \mu_f h_{fg}\left[\frac{g(\rho_f-\rho_g)}{\sigma}\right]^{0.5}\left[\frac{c_{pf}\,\Delta t}{C_{sf} h_{fg} Pr_f^{1.7}}\right]^3$$

The relevant fluid and vapour properties are :

μ_f = 2.83 × 10^{-4} kg/m-s
h_{fg} = 2.256 × 10^6 J/kg
ρ_f = 958.4 kg/m^3
ρ_g = 0.598 kg/m^3
σ = 58.9 × 10^{-3} N/m
c_{pf} = 4217 J/kg K
and Pr_f = 1.75

The surface fluid constant C_{sf} for the water-copper combination is 0.013

$$\therefore \quad \frac{Q}{A} = 2.83 \times 10^{-4} \times 2.256 \times 10^6 \times \left[\frac{9.81(958.4-0.598)}{58.9\times 10^{-3}}\right]^{0.5} \times \left[\frac{4217\times 15}{0.013\times 2.256\times 10^6\times 1.75^{1.7}}\right]^3$$

$$= 638.45 \times 399.40 \times 0.578$$
$$= \mathbf{147 \times 10^3\ W/m^2}$$

$$\text{Pan surface area} = \frac{\pi}{4}(0.35)^2 = 0.096 \text{ m}^2$$

Therefore the boiling heat transfer rate for the given pan area is :

$$Q = 147 \times 10^3 \times 0.096$$
$$= \mathbf{14.11 \times 10^3\ W}$$

Under steady state conditions, the entire heat will be utilized in the evaporation of water from the pan. Hence

$$m = \frac{Q}{h_{fg}} = \frac{14.11\times 10^3}{2.256\times 10^6}$$
$$= \mathbf{6.25 \times 10^{-3}\ kg/s}$$

The critical heat flux for nucleate pool boiling can be estimated from the correlation;

$$\left(\frac{Q}{A}\right)_{max} = 0.18\rho_g h_{fg}\left[\frac{\sigma(\rho_f-\rho_g)}{\rho_g^2}\right]^{0.25} \times \left(\frac{\rho_f}{\rho_f+\rho_g}\right)^{0.5}$$

Substituting the appropriate values, we get

$$\left(\frac{Q}{A}\right)_{max} = 0.018 \times 0.598 \times 2.256 \times 10^6 \times \left[\frac{58.9\times 10^{-3}(958.4-0.598)\times 9.8}{(0.598)^2}\right]^{0.25} \times \left[\frac{958.4}{958.4+0.598}\right]^{0.5}$$

$$= \mathbf{1.507 \times 10^6\ W/m^2}$$

EXAMPLE 13.16

In a nucleate pool boiling of saturated water at 101.32 kN/m^2, the vapour bubbles rise at an average velocity of 3 m/s. If the average diameter of the bubbles is 3.5 mm, calculate the average heat transfer coefficient.

Use may be made of the following empirical correlation for liquid flow over a single sphere :

$$\frac{hD}{k_\infty} = (1.2 + 0.53Re^{0.54})\,Pr^{0.3}\left(\frac{\mu_\infty}{\mu_s}\right)^{0.25}$$

valid over the range $1 < Re < 2 \times 10^5$, and all properties except μ_s are evaluated at the free stream temperature.

(b) Proceed to determine the drop in vapour temperature as the bubble moves up through 0.3 m distance.

Solution : For saturated water at 101.32 kN/m^2, the free stream temperature is 100°C and the corresponmding thermo-physical properties are :

ρ_∞ = 958.4 kg/m^3
k_∞ = 0.682 W/m-deg
ν_∞ = 0.297 × 10^{-6} m^2/s
$\mu_\infty = \mu_s = \rho\nu$ = 2.78 × 10^{-4} Ns/m^2

$$Pr = 1.75$$

$$c_p = 4220 \text{ J/kg K}$$

$$Re = \frac{\rho_\infty D V_\infty}{\mu_\infty} = \frac{D V_\infty}{\nu_\infty}$$

$$= \frac{3.5\times10^{-3}\times3}{0.297\times10^{-6}}$$

$$= 35353.5$$

Since the Reynolds number is less than 2×10^5, the given correlation applies. Substituting the given data, we obtain

$$\frac{hD}{k_\infty} = \left[1.2 + 0.53\times(35353.5)^{0.54}\right] \times (1.75)^{0.3} \times 1$$

$$= 152.70 \times 1.183 \times 1 = 180.64$$

$$\therefore \quad h = 180.64 \times \frac{k_\infty}{D}$$

$$= 180.64 \times \frac{0.682}{3.5\times10^{-3}}$$

$$= \mathbf{35.2 \times 10^3\ W/m^2\text{-}deg}$$

(*b*) By an energy balance and assuming no phase change,

$$hA\,(T_v - T_{sat})\,\tau = m\,c_p\,\Delta T = (\rho\,V\,c_p)\,\Delta T$$

$$\text{where } \tau = \frac{\text{distance}}{\text{velocity}} = \frac{0.3}{3} = 0.1 \text{ s}$$

$$A = \pi D^2 = \pi \times (3.5 \times 10^{-3})$$

$$= 38.46 \times 10^{-6} \text{ m}^2$$

$$V = \frac{\pi}{6}D^3 = \frac{\pi}{6} \times (3.5 \times 10^{-3})^3$$

$$= 2.244 \times 10^{-8} \text{ m}^3$$

Assuming the bubbles contain no condensable gas

$$T_v - T_{sat} = \frac{2\sigma}{r}\frac{R_v T_g^2}{p_v h_{fg}}$$

$$= \frac{2\times0.058}{\frac{1}{2}(3.5\times10^{-3})} \times \frac{4615\times373^2}{(101.32\times10^3)\times(2255\times10^3)}$$

$$= 0.217$$

$\therefore$ Vapour bubble temperature rise,

$$\Delta t = \frac{hA\,(T_v - T_{sat})\,\tau}{\rho\,V\,c_p}$$

$$= \frac{35.2\times10^3\times38.46\times10^{-6}\times0.217\times0.1}{958.4\times2.244\times10^{-8}\times4220}$$

$$= \mathbf{0.3237°\ C}$$

EXAMPLE 13.17

Water flows normal to a polished 15 mm diameter copper tube at the rate of 3.5 m/s. The tube is maintained at 116°C and film boiling occurs. Workout the boiling heat transfer coefficient.

Solution : Comparison of the values of

$$V_\infty = 3.5 \text{ m/s and}$$

$$2\sqrt{gD} = 2\times\sqrt{9.81\times0.015}$$

$$= 0.767 \text{ clearly shows that}$$

$$V_\infty > 2\sqrt{gD}$$

The following correlations then apply for the boiling film coefficient.

$$\underset{\text{(convection)}}{h_c} = 2.7\left[\frac{V_\infty k_g \rho_g (h_{fg} + 0.4 c_{pf}\Delta t)}{D\,\Delta t}\right]^{0.5}$$

$$\underset{\text{(radiation)}}{h_r} = \frac{\sigma_b \in (T_s^4 - T_{sat}^4)}{(T_s - T_{sat})}$$

$$\underset{\text{(total)}}{h} = h_c + \frac{3}{4}h_r$$

The required fluid properties evaluated at the mean film temperature

$$t_f = \frac{116+100}{2} = 108°\text{ C}$$

are :

$$k_g = 2.49 \times 10^{-2} \text{ W/mK}$$

$$\rho_g = 0.586 \text{ kg/m}^3$$

$$c_{pf} = 2058 \text{ J/kg}$$

$$h_{fg} = 2.25 \times 10^6 \text{ J/kg}$$

$$\sigma_b = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}$$

$$\text{and} \quad \in = 0.22$$

Substituting the appropriate values, we obtain :

$$h_c = 2.7\left[\frac{3.5\times2.49\times10^{-2}\times0.586\,(2.25\times10^6 + 0.4\times2058\times16)}{0.015\times16}\right]^{0.5}$$

$$= 1868.24 \text{ W/m}^2\text{K}$$

$$\text{and} \quad h_r = \frac{5.67\times10^{-8}\times0.22\left[(389)^4-(373)^4\right]}{(389-373)}$$

$$= 0.276 \text{ W/m}^2\text{K}$$

Therefore the total boiling coefficient becomes :

$$h = 1868.24 + \frac{3}{4}(0.276)$$

$$= 1868.447 \text{ W/m}^2\text{K}$$

SALIENT POINTS

1. Boiling and condensation are convective heat transfer processes which are associated with change of phase
   Boiling : liquid to vapour phase
   Condensation : vapour to liquid phase
2. The heat transfer coefficients and rates of heat transfer during boiling and condensation are generally much higher compared to that with normal convective processes without phase change.
3. Boiling is essentially evaporation at solid-liquid surface when the temperature of the surface t_s exceeds the saturation temperature t_{sat} corresponding to the liquid pressure.
   The temperature driving force for the boiling liquid, called the temperature excess, is defined as
   Temperature excess $\Delta t_e = t_s - t_{sat}$
4. The boiling phenomenon is known to occur in the following forms:
   (*i*) **Pool boiling:** Boiling of quiescent liquid in which the motion is caused by free convection and by the formation, growth detachment and rise of the bubbles.
   (*ii*) **Forced convection boiling:** Motion in the boiling medium is caused by external means, and also by free convection and bubble induced mixing.
   (*iii*) **Subcooled or local boiling:** Temperature of the liquid is below the saturation temperature and boiling takes place only in the vicinity of the heated surface.
   (*iv*) **Saturated boiling:** The temperature of the liquid exceeds the saturation temperature. The vapour bubbles generated at the solid surface are transported through the liquid by buoyancy effects and eventually escape from the liquid vapour interface.
5. The boiling curve depicts the surface heat flux as a function of excess temperature. Depending on excess temperature, the different regions of boiling curve are:
   (*i*) **Interface evaporation:** Evaporation process with no bubble formation. The liquid near the surface is slightly superheated, and the heat transfer is through this superheated liquid rising to the liquid vapour interface where evaporation take place.
   (*ii*) **Nucleate boiling:** Bubbles form at the nucleations sites and the resulting liquid agitation induces considerable fluid mixing and that promotes substantial increase in boiling heat transfer coefficient and the heat flux.
   (*iii*) **Film boiling:** Buble formation is very rapid, these bubbles blanket the heating surfaces and eventually coalesce to form a vapour film which completely covers the heated surface. The vapour film has low thermal conductivity and its insulating effect results in drop in heat flux.
   (*iv*) With in the temperature range $50 < \Delta t < 150$, conditions oscillate between nucleate and film boiling, and this phase is referred to as transition boiling, unstable and partical film boiling.
   With further increase in excess temperature, a continuous vapour film is formal on the heating surface, and this regime is called the stable film boiling regime.
   Burnout point or boiling crisis refers to a point on the boiling curve when the heat flux is maximum. This maximum heat flux is called critical heat flux and occurs at critical excees temperature.
6. Condensation refers to the process when a fluid changes from vapour to liquid phase.
   Condensation may occur in two possible way
   (*i*) **Film condensation:** The condensate ten to wet the surface and thereby form liquid film.

(v) **Dropwise condensation:** The vapour condenses into small liquid droplets which fall down from the surface in random fashion.

7. Film condensation usually occurs when a vapour, relatively free from impurities, is allowed to condense on a clean surface. Dropwise condensation has been observed to occur on highly polished surfaces, or on surfaces contaminated with impurities like fatty acids and organic compounds.

8. Dropwise condensation gives coefficient of heat transfer generally five to ten times larger than with film condensation.

9. For laminar film condensation on a verticle plate

(i) The velocity distribution is parabolic and the velocity distribution at distance δ from the top edge is given by

$$u = \frac{\rho g}{\mu}\left(\delta y - \frac{y^2}{2}\right)$$

(ii) The film thickness increases as the fourth root of the distance down the surface; the increase is rather rapid at the upper end of the verticle surface and slow thereafter

$$\delta = \left[\frac{4 k \mu (t_{sat} - t_s) x}{\rho^2 g h_{fg}}\right]^{0.25}$$

(iii) The mass flow rate of the condensate per unit depth of the film at any position x is given by

$$m = \frac{\rho g \delta^3}{3\mu}$$

The change in condensate flow rate is due to the energy transferred from the condensing vapour to the wall. The heat transferred from the condensing vapour to the wall is equal to the increased mass flow rate times the latent heat of vaporisation.

(iv) The local heat transfer coefficient at the lower edge of the plate, *i.e.*, at $x = l$ is

$$h_l = \left[\frac{k^3 \rho^2 g h_{fg}}{4 \mu l (t_{sat} - t_s)}\right]^{0.25}$$

The rate of condensation heat transfer is higher at the upper end of the plate than at the lower end.

The average heat transfer coefficient is $\frac{4}{3}$ times the local heat transfer coefficient at the trailing edge of the plate.

(v) For an inclined plate, the gravitational acceleration g is replaced by $g \sin\theta$ where θ is the inclination angle with the horizontal.

10. For condensation on tubes and tube banks, the average heat transfer coefficient for laminar condensation is given by

$$\bar{h} = A\left[\frac{k^3 \rho^2 g h_{fg}}{\mu d (t_s - t_w)}\right]$$

where d is diameter and $A = 0.725$ for tube and $A = 0.815$ for sphere.

11. The motion of the condensate becomes turbulent when its Reynolds number exceeds a critical value of about 1800.

$$R_e = \frac{\rho V d_h}{\mu} = \frac{4 A \rho V}{P \mu} = \frac{4 m}{P \mu}$$

where d_h is hydraulic diameter and P is wetted perimeter.

The following correlation has been suggested for calculating the average heat transfer coefficient when $R_e > 1800$:

$$\bar{h} = 0.0077 (R_e)^{0.4}\left[\frac{k^3 \rho^2 g}{\mu^2}\right]^{1/3}$$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

A. Condensation

1. What is condensation and when does it occur?
2. The rate of heat transfer in drop-wise condensation is many times larger than that in film condensation. Why?
3. How does filmwise condensation differ from dropwise condensation ? Which type has a higher heat transfer film coefficient and point out the reason thereof.

4. In the design of condensers, which of the two types of condensation is usually selected and why?
5. State the assumptions of Nusselt theory of condensation and obtain an expression for local value of condensing heat transfer coefficient over a vertical flat plate of lenght l. Also show that its average value is equal to 4/3 times the local value of condensing heat transfer coefficient at $x = l$.
What change, if any, would result in the average heat transfer coefficient if the plate is inclined at an angle θ to the vertical plane?
6. For laminar film condensation on a vertical plate, develop an expression for the film thickness, heat transfer coefficient and steam condensation rate in terms of relevant fluid properties, temperature difference and the plate dimensions. Are the fluid properties involved evaluated for the vapour phase ? If not, how are they evaluated ?
7. Based on the Nusselt theory of laminar film condensation, prove that for steam condensing on a vertical plate ;
(a) the thickness of condensate film at a distance x below the top edge of the plate is given by :

$$\delta = \left[\frac{4 k \mu (t_{sat} - t_s) x}{\rho^2 g h_{fg}}\right]^{0.25}$$

(b) the local heat transfer coefficient at a distance x below the top edge of the plate is given by :

$$h_x = 0.943\left[\frac{k^3 \rho^2 g h_{fg}}{\mu l (t_{sat} - t_s)}\right]^{0.25}$$

8. How is Reynolds number defined for film condensation ?
Analyse film condensation on a flat vertical plate by considering shear, gravity and vapour forces acting on the condensate layer. Determine an expression for the condensate velocity and the mass flow rate. Show that if the vapour density is much less than the liquid density, the Reynolds number is given by

$$Re = \frac{4}{3}\frac{\delta^3 \rho^2 g}{\mu^2}$$

The symbols have their usual meanings.
9. Considering film condensation on a vertical plate,
(i) state how does the local heat transfer coeficient vary with the distance from the top of plate
(ii) state how is the mass flow rate of the condensate related to the average heat transfer coefficient.
10. For laminar condensation, what is the ratio of heat transfer of a horizontal tube of large diameter to that of a vertical tube of the same size for the same temperature difference.
(Ans. $h_h/h_v = 0.64\ (l/d)^{0.25}$)
11. What l/d ratio will produce the same condensation heat transfer rate to a tube in both the horizontal and vertical orientations. Assume tube diameter to be large compared with condensate thickness.
(Ans. $l/d = 5.96$)
12. Dry saturated steam at atmospheric pressure condenses on the surface of a horizontal tube of 35 mm diamter. What should be the surface temperature of the tube if the rate of heat flow is required to be 6×10^4 W/m^2 ? Also, determine the heat transfer coefficient under these conditions.
13. Dry saturated steam at atmospheric pressure condenses on the surface of a horizontal tube. If the steam pressure, the temperature differences and the length of tube are maintained constant, how the heat transfer coefficient and the steam condensation rate would change if the tube diameter is increased to four-times.
(The heat transfer coefficient shall diminish 1.41 times; the steam condensation rate shall increase 2.828 times)
14. If the outside surface of a tube were to be used for condensing a vapour, does it make any difference if the axis of tube were positioned vertically or horizontally ?
Steam condenses on a 50 cm long vertical condenser tube of 20 mm outside diameter. Calculate the heat transfer rate if condensatio[n] pressure is 0.2 atm absolute and the tub[e] surface temperature is 5°C less than that [of] the condensing steam. Calculate also the ma[ss] of steam condensed per hour assuming ste[am] dry and saturated.
15. Dry saturated steam at 120°C satura[tion] temperature condenses on a vertical p[late] 100 mm in height and 50 mm in width ha[ving] a uniform surface temperature of 1

Estimate the average condensing film coefficient, heat transfer rate to the plate and the steam condensation rate.

16. A plane surface oriented vertically is 30 cm high and has a surface temperature of 90°C. What will be the condensing film coefficient and the rate of steam condensation if dry saturated steam at atmospheric pressure surrounds the surface ? How these quantities would be affected if the surface were positioned 60°C from the horizontal ?

[Ans. (i) 1.04×10^4 W/m²K ; 0.0137 kg/s-m]
(ii) 0.40×10^4 W/m²K ; 0.01186 kg/s-m]

17. (a) For laminar film condensation, what is the ratio of heat transfer to a horizontal tube of large diameter to that of a vertical tube of the same size for the same temperature difference?

(b) What length/diameter ratio will produce the same laminar film-condensation controlled heat transfer rate to a tube both in vertical and horizontal orientations ? Assume that the tube diameter is large compared with the condensate thickness.

While making the calculations, account for the Mc Adams correction factor for steam condensing on vertical surfaces.

$$\left[\text{Ans. } 0.641\left(\frac{l}{d}\right)^{0.25}, 5.02\right]$$

18. Saturated steam at 100°C condenses on a 4 cm diameter tube with a surface temperature of 60°C. Estimate the value of condensing film coefficient if the tube is 100 cm long and is oriented : (a) horizontally (b) vertically. What will be the steam condensation rate in each case ?

B. Boiling

19. (a) What is boiling ? When does it occur ? List some of the factors that affect boiling heat transfer.

(b) Explain pool boiling. How does it differ from forced convection boiling ?

(c) What do you understand by nucleation in nucleate boiling ? Explain subsequent growth and motion of bubbles.

20. (a) Define critical heat flux and boiling crisis

(b) When does radiation play a role in boiling heat transfer ? What is Leidenfrost phenomenon?

(c) Which boiling region is preferred in industrial boilers and vaporisers ?

21. (a) State the criterion for the collapse of vapour bubbles in a liquid.

(b) How does liquid pressure affect the boiling heat transfer ?

(c) What kind of surface site promotes origination of nucleus of a bubble (in boiling) or a drop (in condenstion).

22. (i) State the two conditions that are necessary for the formation of bubbles

(ii) Why do bubbles form on a heating surface?

23. Discuss in detail the various regimes in boiling and explain the condition for the growth of bubbles. What is the effect of bubble size on boiling ?

24. (a) Explain the phenomenon of nucleate boiling. List the factors that affect nucleate boiling.

(b) List some empirical correlations that have been suggested for determining the boiling heat transfer in nucleate and film boiling.

25. (i) How does nucleate boiling differ from film boiling?

(ii) State the reason of heat flux in nucleate boiling being much higher than in film boiling.

26. Discuss the various regines of boiling heat transfer. Comment on critical heat flux in nucleate boiling.

27. What is meant by burnout point and why is it identified as boiling crisis?

28. Calculate the heat flux and boiling heat transfer coefficient for water boiling on a horizontal flat plate. The temperature excess between the surface and saturation temperature is anticipated to be 10°C.

Use the Fritz correlation

$$h = 1.54\left(\frac{Q}{A}\right)^{0.75}$$

$$= 5.58\ (\Delta t)^3\ \text{W/m}^2\text{-deg}$$

where Q/A is heat rate W/m² and Δt is the temperature difference between surface and saturation.

(Ans. 54152 W/m², 5580 W/m²-deg)

29. A 1.0 mm diameter and 150 mm long wire is submerged horizontally in water at 7 bar pressure. The wire has a steady state applied voltage drop of 2.15 V and a current of 131.5 A. Calculate the heat flux and boiling heat transfer coefficient if the surface of wire is to be maintained at 180°C.

(Ans. 0.6×10^{-6} W/m², 39884 W/m²-deg)

30. In pool boiling on a horizontal surface with water at atmospheric pressure, the heat flow is 60 kW/m². Make calculations for the surface temperature required. How this value compares if the boiling occurs on a vertical flat plate ? Use the following correlation :

$$Nu = 0.16\ (Gr\ Pr)^{0.33}$$

for boiling on a horizontal plane

$$Nu = 0.16\ (Gr\ Pr)^{0.25}$$

for boiling on a vertical plane

31. Predict the peak heat flux and the minimum heat flux for water boiling at normal atmospheric pressure.

32. For equilibrium of a spherical bubble in a liquid, a balance of force gives

$$\pi r^2\ (p_v - p_l) = 2\pi r\sigma$$

where r is the bubble radius, σ is the surface tension, and subscripts v and l represents the vapour and liquid, respectively. Combining this equilibrium condition with the Clapeyron's equation

$$\frac{dp}{dT} = \frac{p_v}{R_v}\frac{h_{fg}}{T_v^2}$$

which relates the pressure and temperature between a saturated liquid and its saturated vapour, and with the ideal gas law, determine the relationship between the degree of superheat and bubble radius.

$$\left[\text{Ans. } T_v - T_{sat} = \frac{2\sigma}{r} \times \frac{R_v T_v^2}{p_v h_{fg}}\right]$$

B. Fill in the blanks with appropriate word/words :

1. In .... condensation, the liquid droplets fall from the plate's surface and there is no wetting of the surface by the condensate.
2. The convection heat transfer coefficient in drop-wise condensation is .... than that in case of film condensation.
3. The heat transfer coefficient for film condensation .... with increasing distance from the top edge of plate.
4. The critical Reynolds number for transition from laminar to turbulent film condensation is ....
5. Drop-wise condensation occurs on ____________ surfaces.
6. Boiling is a convective heat transfer process that involves a phase changes from ____________ to ____________ phase.
7. In the regine representing transition from nucleate boiling to film boiling, the heat flux ____________ with increasing excess temperature.
8. The driving force in boiling heat transfer is the ____________ .
9. The plot showing different regimes of boiling heat transfer, has ____________ plotted against excess temperature.
10. In the nucleate boiling regions, the heat flux rapidly increases with excess temperature and reaches a maximum value called the ____________ .
11. The .... is the point of maximum heat flux on the boiling curve at which transition from nucleate to film boiling initiates.

Answers : 1. drop-wise; 2. higher, 4 to 8 times; 3. decreases; 4. 1800; 5. highly polished; 6. liquid to vapour; 7. excess temperature; 8. decreased; 9. heat flux; 10. critical heat flux; 11. burnout point.

C. Multiple choice questions :

1. The convective coefficients for boiling and condensation usually lie in the range
   (a) 30-300
   (b) 60-3000
   (c) 300-10,000
   (d) 2500-10,000 W/m² K
2. Dropwise condensation usually occurs on
   (a) galzed surface (b) smooth surface
   (c) oily surface (d) coated surface
3. Consider the following statements :
   1. If a condensing liquid does not wet a surface, then dropwise condensation will not take place on it
   2. Dropwise condensation gives a higher transfer rate than filmwise condensation
   3. Reynolds number of condensing liquid is based on its mass flow rate
   4. Suitable coating or vapour additive is used to promote filmwise condensation

   Which of these statements is/are correct?
   (a) 1 and 2 (b) 2, 3 and 4
   (c) 4 only (d) 1, 2 and 3

4. Saturated steam is allowed to condense over a vertical flat surface and the condensate film flows down the surface. The local coefficient of heat transfer for condensation
(a) remains constant at all heights of the surface
(b) decreases with increasing distance from the top of the surface
(c) increases with increasing thickness of condensate film
(d) increases with increasing temperature differential between the surface and vapour

5. In condensation over a vertical surface, the value of convection coefficient varies as
(a) $k^{0.25}$ (b) $k^{0.33}$
(c) $k^{0.75}$ (d) $k^{-0.5}$
where k is the thermal conductivity of the liquid

6. Which of the followings is a wrong statement in the context of convective heat transfer coefficient in laminar film condensation?
The heat transfer coefficient varies as
(a) 1/2 power of density of liquid
(b) - 1/2 power of dynamic viscosity
(c) 1/4 power of enthalpy of evaporation
(d) - 1/4 power of acceleration due to gravity

7. For filmwise condensation on a vertical plane, the film thickness δ and heat transfer coefficient h vary with distance x from the leading edge as
(a) δ decreases, h increases
(b) both δ and h increases
(c) δ increases, h decrease
(d) both δ and h decrease

8. Mark the wrong statement with respect to laminar film condensation on a vertical plate
(a) the rate of condensation heat transfer is maximum at the upper edge of the plate and progressively decreases as the lower edge is approached.
(b) at a definite point on the heat transfer surface, the film coefficient is directly proportional to thermal conductivity and inversely proportional to thickness of film at the point
(c) the average heat transfer coefficient is two-third of the local heat transfer coefficient at the lower edge of the plate
(d) the film thickness increases as the fourth root of the distance down the upper edge

9. Natural convection heat transfer coefficients over surface of a vertical pipe and a vertical flat plate for same height and fluid are equal. What is/are the possible reasons for this?
1. same height
2. both vertical
3. same fluid
4. same fluid flow pattern
Select the correct answer using the code given below:
(a) 1 (b) 1 and 2
(c) 3 and 4 (d) 4

10. Milk spills over when it is boiled in an open vessel. The boiling of milk at this instant is referred to as
(a) interface evaporation
(b) sub-cooled boiling
(c) film boiling
(d) saturated nucleate boiling

11. With increase in excess temperature, the heat flux in boiling
(a) increases continuously
(b) decreases and then increases
(c) decreases, then increases and again decreases
(d) increases, then decreases and again increases

12. The excess temperature range 50°C < ΔT < 200°C is indicative of the region of
(a) interface evaporation
(b) nuclear boiling
(c) partial film boiling
(d) stable film boiling

13. Leiden-frost effect is associated with
(a) evaporation of a solution
(b) boiling of liquid on a hot surface
(c) exchange of heat between two fluids
(d) condensation of vapour on a cold surface

14. Heat flux increases with temperature excess beyond the Leiden-frost point due to
(a) occurrence of subcooled boiling
(b) promotion of nucleate boiling
(c) radiation effect becomes predominent
(d) vapour space becomes large

15. The heat flux in nucleate boiling varies in accordance with

(a) h_{fg} (b) $(h_{fg})^{0.5}$
(c) $(h_{fg})^3$ (d) $1/(h_{fg})^2$
where h_{fg} is the enthalpy of evaporation.

16. In nucleate pool boiling, the heat flux depends on
(a) material of the surface only
(b) material and roughness of the surface
(c) liquid properties and material of the surface
(d) liquid properties, material and condition of the surface

17. Identify the wrong statement with respect to boiling heat transfer ?
(a) boiling occurs when a heated surface is exposed to a liquid and maintained at a temperature lower than the saturation temperature of the liquid
(b) the steam boilers employing natural convection have steam raised through pool boiling.
(c) the nucleation boiling is characterised by the formation of bubbles at the nucleation sites and the resulting liquid agitation
(d) Leiden-frost effect refers to the phenomenon of stable film boiling
(e) the boiling crisis or the burn-out point on the boiling curve (surface heat flux as a function of excess temperature) represents the maximum heat flux at which transition occurs from nucleate to film boiling

18. Which of the following parameters affect burntout heat flux in the nucleate boiling region
1. heat of evaporation
2. temperature difference
3. density of vapour
4. density of liquid
5. surface tension at the vapour-liquid interface
Mark the correct answer from the codes indicated below :
(a) 1, 2, 3 and 5 (b) 1, 3, 4 and 5
(c) 1, 2, 3 and 4 (d) 1, 3 and 5

19. Consider the following statements regrading nucleate boiling :
1. The temperature of the surface is greater than the saturation temperature of the liquid.

2. Bubbles are created by the expansion of entrapped gas oil vapour at small cavities in the surface.
3. The temperature is greater than that in film boling.
4. The heat transfer from the surface to the liquid is greater than that in the film boiling.
Which of the these statements are correct ?
(a) 1, 2 and 4 (b) 1 and 3
(c) 1, 2 and 3 (d) 2, 3 and 4

20. Inspite of large heat transfer coefficeints in boiling liquids, fins are used advantageoulsy when the entire surface is exposed to
(a) film boiling
(b) transition boiling
(c) nucleate boiling
(d) all modes of boiling

21. All of the following statements are correct, except
(a) In subcooled boiling, the temperature of the heating surface is more than the boiling point of the liquid
(b) nucleate boiling gets promoted on a smooth surface
(c) there occurs transition from nucleate to film boiling at burn-out point on the boiling curve
(d) film boiling region is usually avoided in commercial equipment

22. Consider the following phenomena:
1. Boiling
2. Free convection in air
3. Forced convection in air
4. Conduction in air
The correct sequence in increasing order of heat transfer coefficient is
(a) 4 - 2- 3 - 1 (b) 4 - 1 - 3 - 2
(c) 4 - 3 - 2 - 1 (d) 3 - 4 - 1 - 2

Answers :

1. (d)	2. (c)	3. (d)	4. (b)	5. (c)
6. (d)	7. (c)	8. (c)	9. (c)	10. (b)
11. (d)	12. (c)	13. (b)	14. (c)	15. (d)
16. (d)	17. (a)	18. (b)	19. (a)	20. (d)
21. (b)	22. (a)			

HINTS AND COMMENTS

2(c):

Dropwise condensation has been observed to occur either on highly polished surfaces or on surfaces contaminated with impurities like fatty acids and organic compound (oils).

5(c), 6(d):

The average heat transfer coefficient for condensation on a vertical surface is prescribed by the correlation:

$$\bar{h} = 0.943 \left[\frac{k^3 \rho^2 g\, h_{fg}}{\mu\, l\,(t_{sat} - t_s)}\right]^{0.25}$$

Obviously in condensation over a vertical surface, the value of convection coefficient varies as $k^{0.75}$. Further, the heat tramsfer coefficient varies as $\frac{1}{4}$ power of acceleration due to gravity.

8(c):

$\bar{h} = \frac{4}{3} h_l$, that is, the average heat transfer coefficient is four-third of the local heat transfer coefficient at the lower edge of the plate.

12(c):

The physical phenomenon of pool boiling is generally divided into the following four different regions based on the excees temperature.

(i) Purely convective region : $\Delta t < 5°C$

(ii) Nucleate boiling: $5 < \Delta t < 50°C$

(iii) Unstable film boiling : $50 < \Delta t < 200°C$

(iv) Stable film boiling : $\Delta t > 200°C$

15(b):

Based on extensive experimental data, the following empirical correlation has been developed for nucleate pool boiling

$$\frac{Q}{A} = \mu_f\, h_{fg} \times \left[\frac{g(\rho_f - \rho_g)}{\sigma}\right]^{0.5} \left[\frac{C_f\, \Delta T}{h_{fg}\, \rho_f^n\, C_{sf}}\right]^3$$

Obviously the heat flux in nucleate pool boiling is proportional to $\frac{1}{(h_{fg})^2}$.

17(a):

For boiling to occur, the heated surface must be exposed to a liquid and maintained at a temperature higher than the saturation temperature of the liquid.

18(b):

According to Zuber relation, burn out

$$\left(\frac{Q}{A}\right)_{max} = 0.18\, \rho g\, h_{fg} \times \left[\frac{\rho(\rho_f - \rho_g)}{\rho_g^2}\right]^{0.25} \left[\frac{\rho_f}{\rho_g + \rho_f}\right]^{0.5}$$

It is to be observed that the peak heat flux is independent of the heating element.

21(b):

A rough surface gives a better heat transmission than when the surface is either smooth or has been coated to weake its tendency to get wetted.

□□□

Chapter

14

Heat Exchangers

Learning objectives : A study of the subject matter included in this chapter will help the readers to understand and comment upon the

- *utility and schematics of different types of heat exchangers*
- *fouling factor and overall heat transfer coefficient for heat exchangers*
- *logarithmic mean temperature difference (LMTD) for parallel flow and counter flow heat exchangers*
- *correction factors to LMTD of heat exchangers with different configurations*
- *heat exchanger effectiveness and number of transfer units*

Heat exchanger is process equipment designed for the effective transfer of heat energy between two fluids ; a hot fluid and a coolant. The purpose may be either to remove heat from a fluid or to add heat to a fluid. Notable examples are :

(i) boilers (evaporators), superheaters and condensers of a power plant

(ii) automobile radiators and oil coolers of heat engines

(iii) evaporator of an ice plant and milk-chiller of a pasteurising plant

(iv) condensers and evaporators in refrigeration units

(v) water and air heaters or coolers

The heat transferred in the heat exchanger may be in the form of latent heat (*e.g.* in boilers and condensers) or sensible heat (*e.g.* in heaters and coolers). This chapter presents the basic principles of heat transfer needed to design and to evaluate the performance of a heat exchanger.

14.1. CLASSIFICATION OF HEAT EXCHANGERS

Many types of heat exchangers have been developed to meet the widely varying applications. Based upon their

— operating principle

— arrangement of flow path

— design and certain constructional features,

the heat exchangers can be classified into the following categories :

14.1.1. Nature of Heat Exchange Process

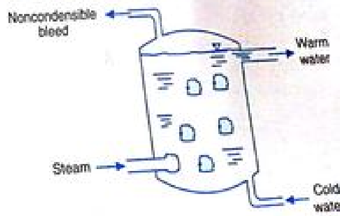


Fig. 14.1. Direct contact or open heat exchan

Based upon the nature of heat exchange process, the heat exchangers are classified into direct contact, regenerators and recuperators.

In **direct contact or open heat exchangers**, the energy transfer between the hot and cold fluids is brought about by their complete physical mixing ; there is simultaneous transfer of heat and mass. Use of such units is restricted to the situations where mixing between the fluids is either harmless or is desirable. Examples are water cooling towers and jet condensers in steam power plants. Figure 14.1. represents a direct contact heat exchanger. Steam is being bubbled into water; steam gets condensed and releases heat that warms up the water.

In a **regenerator**, the hot fluid is passed through a certain medium called matrix. The heat is transferred to the solid matrix and accumulates there; the operation is called heating period. The heat thus stored in the matrix is subsequently transferred to the cold fluid by allowing it to pass over the heated matrix. The regenerators are quite often used in connection with engines and gas turbines. Other applications are : regenerators of open hearth and glass melting furnaces and air heaters of blast furnaces. The operation of a regenerator is intermittent; the matrix alternately stores heat extracted from the hot fluid and then delivers it to the cold fluid. However in some of the regenerators the matrix is made to rotate through the fluid passages arranged side by side and that renders the heat exchange process continuous. The effectiveness of a regenerator depends upon the heat capacity of the regenerating material and the rate of absorption and release of heat.

Fig. 14.2. Heat exchange in a recuperator

In a **recuperator**, the fluids flow simultaneously on either side of a separating wall; the heat transfer occurs between the fluid streams without mixing or physical contact with each other. The wall provides an element of thermal resistance between the fluids and the heat transfer consists of:

- convection between the hot fluid and the wall
- conduction through the wall
- convection between wall and the cold fluid

Such exchangers are used when the two fluids cannot be allowed to mix, *i.e.*, when the mixing is undesirable. Majority of the industrial applications have exchangers of the recuperator type. Notable examples are :

(*i*) boilers, superheaters and condensers; economisers and the air preheaters in steam power plants

(*ii*) automobile radiators

(*iii*) condensers and evaporators in refrigeration units

(*iv*) oil heaters for an airplane

(*v*) heat exchanger inside a gas furnace etc.

The open-type (direct contact) heat exchangers and the recuperators operate under steady state conditions; and the transfer of heat inside a regenerator takes place essentially under transient conditions.

14.1.2. Relative Direction of Motion of Fluids

According to the direction of flow of fluids, the heat exchangers are classified into three categories: parallel flow, counter flow and the cross flow.

In the ***co-current*** or ***parallel*** flow arrangement, the fluids (hot and cold) enter the unit from the same side, flow in the same direction and subsequently leave from the same side (Fig. 14.3). Obviously the flow of fluids is unidirectional and parallel to each other.

In the ***counter-current*** or ***counter-flow*** arrangement, the fluids (hot and cold) enter the unit from opposite ends, travel in opposite

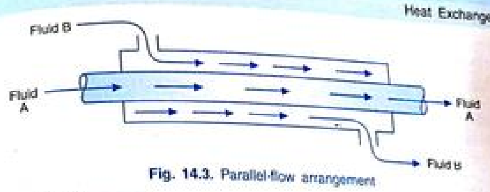


Fig. 14.3. Parallel-flow arrangement

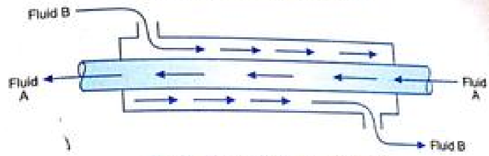


Fig. 14.4. Counter-flow arrangement

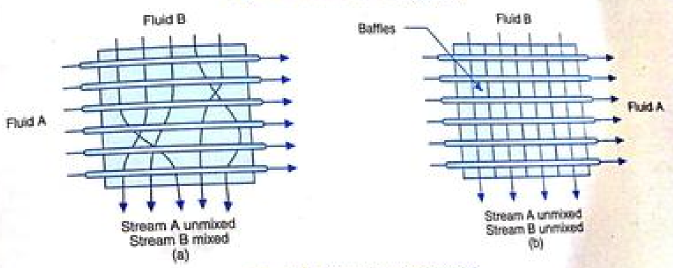


Fig. 14.5. Cross-flow arrangement

directions and subsequently leave from opposite ends. Obviously the flow of fluids is opposite in direction to each other. For a given surface area, the counter-flow arrangement gives the maximum heat transfer rate and is naturally preferred for the heating and cooling of fluids.

In the ***cross-flow*** arrangement, the two fluids (hot and cold) are directed at right angles to each other. Figure 14.5 shows two common arrangements of cross-flow heat exchangers. In Figure 14.5 (*a*) the fluid A flows inside the separate tubes and its different streams do not mix. The fluid B flows over the tube banks and gets perfectly mixed. In Figure 14.5 (*b*), each of the fluid stays in prescribed paths and are not allowed to mix as they flow through the heat exchanger. When mixing occurs, the temperature variations are primarily in the flow direction. When unmixed, there is temperature gradient along the stream as well as in the direction perpendicular to it. Apparently, temperatures of the fluids leaving the unit are not uniform for the unmixed streams. The cross flow heat exchangers are commonly employed in air or gas heating and cooling applications, *e.g.*, the automobile radiator and the cooling unit of air-conditioning duct.

14.1.3. Mechanical Design of Heat Exchange Surface

(*i*) *Concentric tubes* : Two concentric pipes are used, each carrying one of the fluids. The direction of flow may correspond to unidirectional or counter flow arrangement (Figure 14.3 and 14.4).

(*ii*) *Shell and tube* : One of the fluids is carried through a bundle of tubes enclosed by a shell. The other fluid is forced through the shell and flows over the outside surface of tubes. The direction of flow for either or both fluids may change during its passage through the heat exchanger.

(*iii*) *Multiple shell and tube passes* : The two fluids may flow through the exchanger only once (single pass), one or both fluids may traverse the exchanger more than once (multi-pass). By suitable header design, the fluid within the tubes (tube side fluid) can be made to traverse back and forth from one end of the shell to the other. Quite often longitudinal baffles are provided within the shell which cause the fluid surrounding the tubes (shell side fluid) to travel the length of shell a number of times. An exchanger having *n*-shell passes and *m*-tubes passes is designated as *n*-*m* exchanger.

A multiple shell and tube exchanger is preferred to ordinary counter-flow design due to its low cost of manufacture, easy dismantling for cleaning and repair and reduced thermal stresses due to expansion.

14.1.4. Physical State of Heat Exchanging Fluids (Condensation and Evaporation)

(*i*) *Condenser* : The hot fluid (condensing steam) remains at constant temperature all along its passage through the heat exchanger, whilst the temperature of the other fluid (coolant water) gradually increases from inlet to outlet. Obviously the hot fluid loses only part of its heat.

(*ii*) *Evaporator* : During heat exchange in the evaporation of water into steam, the cold-fluid (boiling water) evaporates at constant temperature whilst the temperature of hot gases continuously decreases from inlet to outlet.

The heat exchangers can be further classified on the basis of following design parameters:

(*i*) temperature and pressure levels of the fluid

(*ii*) corrosiveness, toxicity and scale forming tendency of the fluids.

(*iii*) economic considerations such as cost, ease of manufacture, necessary space and required life etc. The cost considerations may, however, be subordinate to weight and size

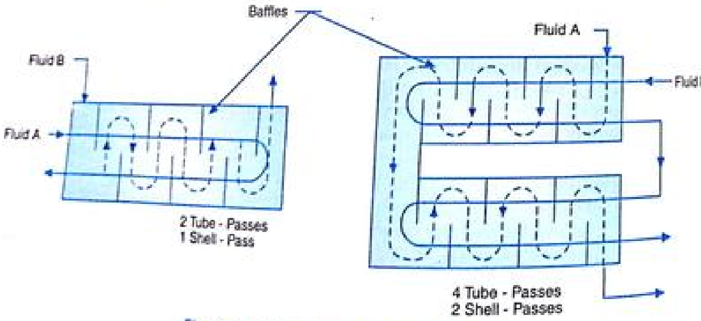


Fig. 14.6. Multiple shell and tube passes

limitation in space and aeronautical limitations. Exchangers of compact design are employed where weight, space and cost limitations are severe.

14.2. PERFORMANCE ANALYSIS

Figure 14.7 represents the block diagram of a heat exchanger. The indicated parameters are :

m = mass flow (kg/s)

c = specific heat (J/kg-deg)

t = fluid temperature (deg C)

Δt = temperature drop or rise of a fluid across the heat exchanger.

Subscripts h and c designate the hot and cold fluids respectively; subscripts 1 and 2 correspond to the inlet and outlet conditions of the fluid.

The following aspects are considered individually in the design and performance analysis of a heat exchanger.

(*i*) The hot fluid gives up heat

$$Q_h = m_h\, c_h\, (t_{h1} - t_{h2}) \qquad ...(14.1)$$

(*ii*) The coolant picks up heat

$$Q_c = m_c\, c_c\, (t_{c2} - t_{c1}) \qquad ...(14.2)$$

(*iii*) The structure of the heat exchanger transfers the heat from the hot fluid to the coolant.

$$Q_{ex} = U\, A\, \theta_m \qquad ...(14.3)$$

where U is the overall heat transfer coefficient between the two fluids, A is the effective heat transfer area and θ_m is the appropriate mean temperature difference across the heat exchanger structure.

From energy balance, the heat given up by the hot fluid is picked up by the coolant on being transferred through the heat exchanger.

$$Q_h = Q_c = Q_{ex} \qquad ...(14.4)$$

14.3. OVERALL HEAT TRANSFER COEFFICIENT

A heat exchanger is essentially a device in which energy is transferred from one fluid to another across a good conducting solid wall. Recapitulate that the rate of heat transfer between two fluids is given by :

$$Q = \frac{\Delta t}{\sum R_t} \quad \text{and} \quad Q = U\, A\, \Delta T$$

Apparently the overall coefficient of heat transfer (U) is defined in terms of the total thermal resistance (R_t).

$$U\, A = \frac{1}{\sum R_t} \qquad ...(14.5)$$

When the two fluids of the heat exchanger are separated by a plane wall (Figure 14.8), the thermal resistance comprises :

(*i*) convection resistance due to the fluid film at the inside surface, $\frac{1}{A_i h_i}$

(*ii*) wall conduction resistance, $\frac{\delta}{kA}$

(*iii*) convection resistance due to fluid film at the outside surface, $\frac{1}{A_o h_o}$

$$\therefore \quad U\, A = \frac{1}{\frac{1}{A_i h_i} + \frac{\delta}{kA} + \frac{1}{A_0 h_0}} \qquad ...(1$$

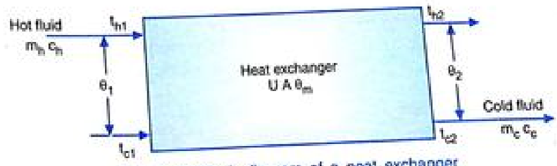


Fig. 14.7. Block diagram of a heat exchanger

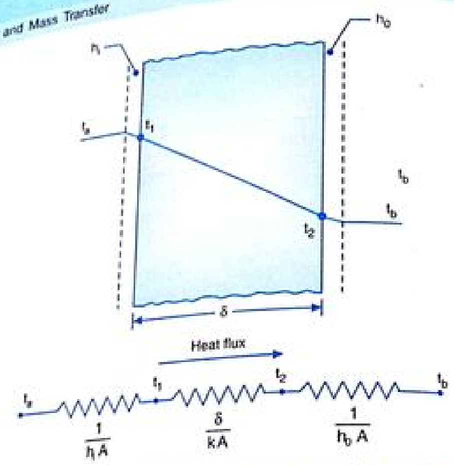


Fig. 14.8. Heat flux through a plate well separating two fluids

A plane wall has a constant cross-sectional area normal to the heat flow *i.e.*, $A_i = A = A_o$. That gives:

$$U = \frac{1}{\frac{1}{h_i} + \frac{\delta}{k} + \frac{1}{h_0}} \quad \text{...(14.7)}$$

For a cylindrical separating wall, the cross-sectional area of the heat flow path is not constant but varies with radius. It then becomes necessary to specify the area upon which the overall heat transfer coefficient is based. Thus depending upon whether the inner or outer area is specified, two different values are defined for U.

$$U_iA_i = \frac{1}{\frac{1}{A_ih_i} + \frac{1}{2\pi kl}\log_e\frac{r_0}{r_i} + \frac{1}{A_0h_0}} \quad \text{...(14.8)}$$

Since $A_i = 2\pi r_i l$ and $A_o = 2\pi r_o l$

$$U_i = \frac{1}{\frac{1}{h_i} + \frac{r_i}{k}\log_e\frac{r_0}{r_i} + \frac{r_i}{r_0h_0}} \quad \text{...(14.9)}$$

If resistance due to the material is neglected,

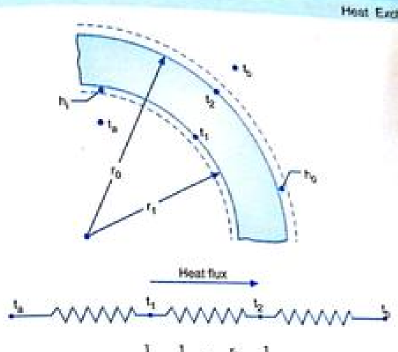
$$U_i = \frac{1}{\frac{1}{h_i} + \frac{r_i}{r_0h_0}} \quad \text{...(14.10)}$$

Further, if the wall thickness is small, i.e., $r_o = r_i$,

$$U_i = \frac{1}{\frac{1}{h_i} + \frac{1}{h_0}} = \frac{h_ih_o}{h_i + h_o} \quad \text{...(14.11)}$$

The corresponding relations for the outer surface would be :

$$U_o = \frac{1}{\frac{1}{h_o} + \frac{r_o}{k}\log_e\frac{r_o}{r_i} + \frac{r_o}{r_ih_o}}$$



$$\frac{1}{h_i A_i} \quad \frac{1}{2\pi kl}\log_e\frac{r_o}{r_i} \quad \frac{1}{h_o A_o}$$

Fig. 14.9. Heat flux through a cylindrical wall separating two fluids

$$\cong \frac{1}{\frac{1}{h_o} + \frac{r_o}{r_ih_i}} \cong \frac{1}{\frac{1}{h_o} + \frac{1}{h_i}}$$

$$= \frac{h_o h_i}{h_i + h_o} \quad \text{...(14.12)}$$

Equations 14.10 to 14.12 are essentially valid only for clean and uncorroded surface. However, during normal operation the tube surfaces get covered by deposits of ash, soot, dirt and scale etc.

This phenomenon of rust formation and deposition of fluid impurities is called ***fouling***. The surface deposits increase thermal resistance with a corresponding drop in the performance of the heat exchange equipment. Since the thickness and thermal conductivity of the scale deposits are difficult to ascertain, the effect of scale on heat flow is considered by specifying an equivalent ***scale heat transfer coefficient***, h_s. If h_{si} and h_{so} denote the heat transfer coefficients for the scale formed on the inside and outside surfaces respectively, then the thermal resistance due to scale formation on the inside surface is

$$R_{si} = \frac{1}{A_i h_{si}}$$

and the thermal resistance due to scale formation on the outside surface is

$$R_{so} = \frac{1}{A_o h_{so}} \quad \text{...(14.13)}$$

With the inclusion of these resistances at the inner and outer surfaces,

$$UA = \frac{1}{\frac{1}{A_ih_i} + \frac{1}{A_ih_{si}} + \frac{1}{2\pi kl}\log_e\frac{r_o}{r_i} + \frac{1}{A_oh_{so}} + \frac{1}{A_oh_o}} \quad \text{...(14.14)}$$

Therefore for the inner surface,

$$U_i = \frac{1}{\frac{1}{h_i} + \frac{1}{h_{si}} + \frac{r_i}{k}\log_e\frac{r_o}{r_i} + \frac{r_i}{r_oh_{so}} + \frac{r_i}{r_oh_o}} \quad \text{...(14.15)}$$

and for the outer surface,

$$U_o = \frac{1}{\frac{1}{h_o} + \frac{1}{h_{so}} + \frac{r_o}{k}\log_e\frac{r_o}{r_i} + \frac{r_o}{r_ih_{si}} + \frac{r_o}{r_ih_i}} \quad \text{...(14.16)}$$

The reciprocal of scale heat transfer coefficient is called the *fouling factor* R_f. Some typical values of R_f are given in Table 14.1. The values are only approximate and given to only one significant figure.

Table 14.1. Representative values of fouling factor

Fluid and situation	Fouling factor, $R_f = 1/h_s$	
	m^2 K/W	m^2-hr-K/kcal
(i) Distilled water	0.0001	1.163×10^{-4}
(ii) Sea water	0.0001 - 0.0002	$1.163 \times 10^{-4} - 2.326 \times 10^{-4}$
(iii) Clean river and lake water	0.0002 - 0.0006	$2.326 \times 10^{-4} - 6.978 \times 10^{-4}$
(iv) Muddy waters used in heat exchangers	< 0.0020	$< 2.326 \times 10^{-4}$
(v) Treated boiler feed water	0.0001 - 0.0002	$1.163 \times 10^{-4} - 2.326 \times 10^{-4}$
(vi) Transformer or lubricating oil	0.0002	2.326×10^{-4}
(vii) Fuel oil	0.0010	1.163×10^{-3}
(viii) Industrial liquids0.002	2.326×10^{-4}	
(ix) Non-oil bearing steam	0.0001	1.163×10^{-4}
(x) Oil bearing steam (turbine exhaust)	0.0002	2.326×10^{-4}
(xi) Liquid gasoline	0.0008	9.304×10^{-4}
(xii) Engine exhaust and flue gases	0.002	2.326×10^{-3}

These values are based on reasonable maintenance and the use of conventional heat exchangers. The resistance generally drops with increased flow velocity and increases with temperature and age.

Fouling factors are determined experimentally by testing the heat exchanger in both the clean and dirty conditions :

$$R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$$

A heat exchanger might be designed either to restrict or to enhance the heat exchange rate. If the heat exchanger is intended to improve heat exchange, U will generally be much greater than 40 W/m^2 K. If it is intended to impede heat flow, U will be much less than 0 W/m^2 K.

The following points are worth bearing mind :

(i) The over all heat transfer coefficient ends upon the flow rate and properties he fluid, the material thickness and surface dition of tubes and the geometrical guration of the heat exchanger.

(ii) The fluids with low thermal conductivities such as tars, oils or any of the gases usually give low values of heat transfer coefficient, h. The overall coefficient U will generally decrease when such a fluid flows on one side of the exchanger.

(iii) The highly conducting liquids such as water and liquid metals give higher values of heat transfer coefficient h and overall heat transfer coefficient U. Condensation and boiling process also have high values of U.

(iv) For an efficient and effective design, there should be no high thermal resistance in the heat flow path; all the resistance in the heat exchanger must be low.

Table 14.2 gives approximate values of U for some commonly fluid combinations. The wide range of the cited values results from a diversity of heat exchange materials (having different thermal conductivities) and flow conditions (influencing the film coefficient h) as well as geometric configuration.

EXAMPLE 14.1.

The heat loss form unpainted aluminium side of a house has been calculated on the presumption

Table 14.2. Representative values of the overall heat transfer coefficient

Fluid combination	Overall heat transfer coefficient U	
	W/m^2K	kcal/m^2-hr-K
(i) Air to heavy tars and liquids	As low as 45	As low as 39
(ii) Air to low viscosity liquids	As high as 600	As high as 516
(iii) Air to various gases	60 - 550	52 - 475
(iv) Air condensers	350 - 780	300 - 675
(v) Steam or water to oil	60 - 340	52 - 290
(vi) Ammonia condensers	800 - 1400	690 - 1200
(vii) Alcohol condensers	250 - 700	215 - 600
(viii) Steam condensers	1500 - 5000	1290 - 4300
(ix) Feed water heaters	110 - 8500	95 - 7300

that overall coefficient of heat transfer is 5 W/m² K. Later, it was discovered that the air pollution levels are such that fouling factor on this side is of the order of 0.0005 m² K/W. Should the side be redesigned ?

Solution : Substituting the appropriate values in the relation,

$$R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$$

we obtain :

$$0.0005 = \frac{1}{U_{dirty}} - \frac{1}{5}$$

or $U_{dirty} = 4.9875$ W/m^2 K

Due to pollution, the overall heat transfer coefficient is reduced from 5 to 4.9875 W/m^2 K; an insignificant change. Apparently, the fouling is not of much importance in the calculation of domestic heat loss. Its inclusion will make only insignificant contribution to the heat loading.

EXAMPLE 14.2.

The design of a water cooled steam condenser has been made by presuming that the overall heat transfer coefficient U = 5000 W/m² K. While deciding this value, the engineer presumed that the flowing water is very clean and accordingly he neglects the fouling resistance. Later, it is discovered that the cooling water is not clean at all and that it has a fouling resistance of the order of 0.0006 to 0.002 m² K/W. Should the design calculations be remade ? Comment.

Solution : Substituting the appropriate values in the relation :

$$R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$$

we obtain :

$$(0.0006 \text{ to } 0.002) = \frac{1}{U_{dirty}} - \frac{1}{5000}$$

$$U_{dirty} = 1250 \text{ to } 454.54 \text{ W/m}^2\text{K}$$

Thus the over all heat transfer coefficient is reduced from 5000 to between 454.5 and 1250 W/m^2K. Fouling is crucial and the condenser needs to be redesigned.

EXAMPLE 14.3.

After being in service for a period of six months, a heat exchanger transfers 10% less heat than it does when new. Determine the effective fouling factor in terms of its clean (new) overall heat transfer coefficient. It may be presumed that the heat exchanger operates between the same temperature differentials and that there is no change in the effective surface area due to scale build up.

Solution : For the heat exchange ratio, we may write

$$\frac{Q_{clean}}{Q_{dirty}} = \frac{U_{clean}\, A\, \Delta T}{U_{dirty}\, A\, \Delta T} = \frac{U_{clean}}{U_{dirty}}$$

Further,

$$Q_{dirty} = 0.9\, Q_{clean}$$

$\therefore \quad U_{dirty} = 0.9\, U_{clean}$

Substituting this in the relation,

$$R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$$

$$= \frac{1}{0.9U_{clean}} - \frac{1}{U_{clean}} = \frac{0.11}{U_{clean}}$$

EXAMPLE 14.4.

In a counter flow heat exchanger, water flowing through a tube of 10 cm inner diameter is heated by steam condensing on the outside of the tube. The convective film coefficient on the water and steam side are estimated to be 12000 and

20000 kJ/m²-hr-deg

Neglecting tube thickness and its resistance to heat flow, work out the overall heat transfer coefficient for the heat exchanger.

(b) A heat exchanger to preheat oil for a furnace was designed without considering the possibility of scale formation, and the overall heat transfer coefficient based on the fuel oil side was

3200 kJ/m²-hr-deg

What would be the corrected coefficient of heat transfer if a fouling factor of

0.00025 m²-hr-deg/kJ

for the fuel oil is taken into account ?

Solution : Neglecting tube thickness, $A_i = A_o$ where suffixes i and o denote the inside and outside surfaces of the tube. Then, the overall coefficient of heat transfer is given by

$$U_i = U_o = U = \frac{1}{\dfrac{1}{h_i} + \dfrac{1}{h_o}}$$

$$= \frac{h_i\, h_o}{h_i + h_o}$$

$$= \frac{12000 \times 20000}{12000 + 20000}$$

$$= 7500 \text{ kJ/m}^2\text{-hr-deg}$$

(b) Corrected heat transfer coefficient,

$$U = \frac{1}{R + \dfrac{1}{h_s}}$$

where R represents the total thermal resistance per unit heat transfer surface of the heat exchanger without fouling, and $1/h_s$ denotes the fouling or scale resistance

$$\therefore \quad U = \frac{1}{1/3200 + 0.00025} = \frac{1}{0.0005625}$$

$$= \mathbf{1777.78 \text{ kJ/m}^2\text{-hr-deg}}$$

EXAMPLE 14.5.

A copper pipe (k = 350 W/mK) of 1.75 cm inner diameter and 2.0 cm outside diameter conveys water and the oil flows through the annular passage between this pipe and a steel pipe. On the water side, the film coefficient is 4600 W/m² K and the fouling factor is 0.00034 m² K/W. The corresponding values for the oil side are 1200 W/m² K and 0.00086 m² K/W. Workout the overall heat transfer coefficient between the water and oil.

Solution : The overall coefficient, based on the outer surface of the inner pipe is prescribed by the relation

$$U_o = \frac{1}{\dfrac{1}{h_o} + \dfrac{1}{h_{so}} + \dfrac{r_o}{k}\log_e\dfrac{r_o}{r_i} + \dfrac{r_o}{r_i\, h_{si}} + \dfrac{r_o}{r_i\, h_i}}$$

A fouling factor represents the reciprocal of the scale heat transfer coefficient. Therefore, in terms of the fouling factors,

$$U_o = \frac{1}{\dfrac{1}{h_o} + R_{fo} + \dfrac{r_o}{k}\log_e\dfrac{r_o}{r_i} + \dfrac{r_o}{r_i}R_{fi} + \dfrac{r_o}{r_i\, h_i}}$$

Given: $r_o = 0.001$ m

and $r_o/r_i = 2.00/1.75 = 1.143$

$$\therefore \; U_o = \frac{1}{\dfrac{1}{1200} + 0.00086 + \dfrac{0.001}{350}\log_e 1.143 + 1.143 \times 0.00034 + \dfrac{1.143}{4600}}$$

$$= \frac{1}{2.298 \times 10^{-3}} = 435.16 \text{ W/m}^2\text{K}$$

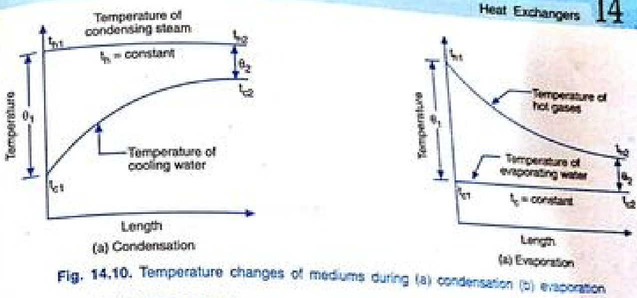


Fig. 14.10. Temperature changes of mediums during (a) condensation (b) evaporation

14.4. LOGARITHMIC MEAN TEMPERATURE DIFFERENCE

During heat exchange between two fluids, the temperature of the fluids change in the direction of flow and consequently there occurs a change in the thermal head causing the flow of heat. Fig. 14.10(*a*) represents the temperature conditions existing in surface condenser or feed water heater. The hot fluid is steam and the cold fluid is water. Here the temperature of steam remains constant but the temperature of water is progressively rising. During heat exchange in evaporation of water into steam (Fig. 14.10 b), the water evaporates at constant temperature and the temperature of hot gases continuously decreases in flowing from inlet to outlet.

In many instances, both the fluids experience a change in temperature while flowing through the heat exchange equipment. In a parallel flow system, the thermal head (temperature potential) causing the flow of heat is maximum at the inlet. However, the thermal head goes on diminishing along the flow path and is minimum at the outlet. In a counter flow system, both the fluids are in their coldest state at the exit.

To calculate the rate of heat transfer by the expression, $Q = U\,A\,\Delta t$, an average value of the temperature difference between the

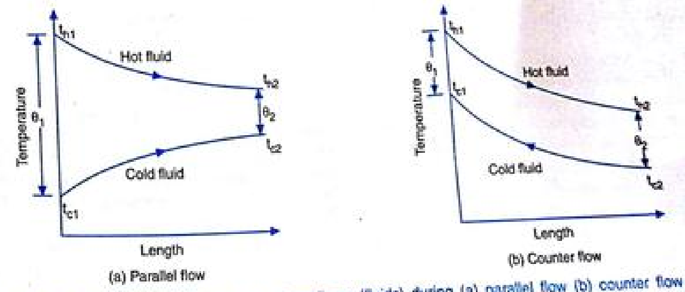


Fig. 14.11. Temperature changes of mediums (fluids) during (a) parallel flow (b) counter flow

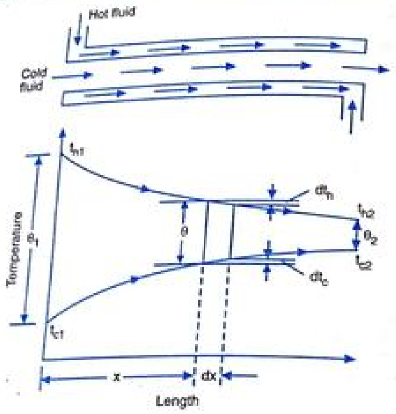


Fig. 14.12. Temperature changes of mediums (fluids) during parallel flow arrangement

fluids has to be determined. The calculations become simplified if it is assumed that :

(i) the overall heat transfer coefficient U is constant throughout the heat exchanger

(ii) the specific heats and mass flow rates of both the fluids are constant. This also implies that heat capacities (a product of mass and specific heat) of the fluids are constant over the entire length of flow path

(iii) the exchanger is perfectly insulated and so the heat loss to the surroundings is negligible

(iv) the temperature at any cross-section of the stream is uniform, *i.e.*, at any cross-section of heat exchanger, each of the fluid can be characterised by a single temperature

(v) there is no conduction of heat along he tubes of heat exchanger

(vi) the kinetic and potential energy hanges are negligible

(vii) there is no partial phase change in e systems. The analysis would thus be plicable for sensible heat changes and for the cases when condensation or vaporisation is isothermal over the entire flow path.

Consider heat transfer across an element of length dx at a distance x from the entrance side of the heat exchanger. Let at this section, the temperature of the hot fluid be t_h and that of cold fluid be t_c. Within the limits of this elementary length, the temperature t_h and t_c of the heat exchanging fluids can be considered to remain constant. Then heat flow dQ through this elementary length is given by :

$$\begin{aligned} dQ &= U\, dA\, (t_h - t_c) \\ &= U\, dA\, \theta \qquad \text{...(14.17)} \end{aligned}$$

where $\theta = (t_h - t_c)$, the temperature difference between the fluids. Due to heat exchange, the temperature of the hot fluid decreases by dt_h and the temperature of cold fluid increases by dt_c. Then for heat exchange between the fluids, we can write

$$\begin{aligned} dQ &= -\, m_h\, c_h\, dt_h = m_c\, c_c\, dt_c \\ &= -\, C_h\, dt_h = C_c\, dt_c \qquad \text{...(14.18)} \end{aligned}$$

where $C_h = m_h\, c_h$

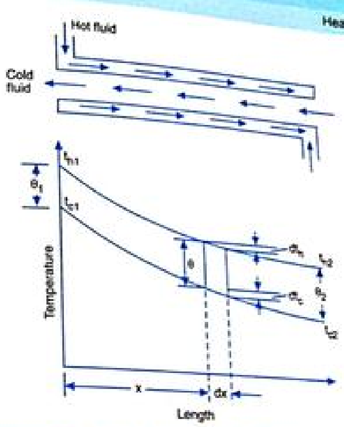


Fig. 14.13. Temperature changes of mediums (fluids) during counterflow arrangements

= heat capacity or water equivalent of the hot fluid

and $C_c = m_c\, c_c$

= heat capacity or water equivalent of the cold fluid.

m_h and m_c are the mass flow rates of fluids and c_h and c_c are the respective specific heats.

From equation 14.18,

$$dt_h = \frac{-dQ}{C_h} \quad \text{and} \quad dt_c = \frac{dQ}{C_c}$$

$$\therefore\ dt_h - dt_c = -dQ\left[\frac{1}{C_h} + \frac{1}{C_c}\right]$$

$$d\theta = -dQ\left[\frac{1}{C_h} + \frac{1}{C_c}\right] \qquad \text{...(14.19)}$$

In a counter flow system, the temperatures of both the fluids decrease in the direction of heat exchanger length. In that case

$$dQ = -\, C_h\, dt_h = -\, C_c\, dt_c$$

$$\therefore \quad dt_h = -\frac{dQ}{C_h} \quad \text{and} \quad dt_c = -\frac{dQ}{C_c}$$

$$\text{or } (dt_h - dt_c) = d\theta = -dQ\left[\frac{1}{C_h} - \frac{1}{C_c}\right] \qquad \text{...(14.20)}$$

From equations 14.19 and 14.20, we can write

$$d\theta = -dQ\left[\frac{1}{C_h} \pm \frac{1}{C_c}\right] \qquad \text{...(14.21)}$$

The +ve sign refers to the parallel flow heat exchanger and –ve sign refers to the counter flow heat exchanger.

Integration of equation 14.21 between th inlet 1 and outlet 2 gives :

$$\theta_1 - \theta_2 = Q\left[\frac{1}{C_h} \pm \frac{1}{C_c}\right] \qquad \text{...(14.2}$$

Further, from equations 14.17 and 14.

$$U\, dA\, \theta = \frac{-d\theta}{1/C_h \pm 1/C_c}$$

$$\text{or } \frac{d\theta}{\theta} = -U\, dA\left(\frac{1}{C_h} \pm \frac{1}{C_c}\right)$$

Integrating between the inlet and outlet sections, we get :

$$\log_e \frac{\theta_1}{\theta_2} = U A\left(\frac{1}{C_h} \pm \frac{1}{C_c}\right) \quad ...(14.23)$$

Substituting for $1/C_h \pm 1/C_c$ for equation 14.22, we get :

$$Q = U A \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = U A \theta_m \quad ...(14.24)$$

where $\theta_m = (\theta_1 - \theta_2)/\log_e(\theta_1/\theta_2)$ is called the **logarithmic mean temperature difference** (LMTD)

If the heat capacity of the two fluids are equal in a counter flow arrangement, then it follows from equation 14.22, that $\theta_1 = \theta_2$

Then

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{0}{\log_e 1} = \frac{0}{0} = \text{indeterminate}$$

Applying L' Hospital's rule :

$$\lim_{\theta_1 \to \theta_2} = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{\partial}{\partial \theta_1}(\theta_1 - \theta_2)_{\theta_1 = \theta_2} \div \frac{\partial}{\partial \theta_1} \log_e \frac{\theta_1}{\theta_2}\bigg|_{\theta_1 = \theta_2}$$

$$= \frac{1}{\frac{1}{\theta_1}\big|_{\theta_1 = \theta_2}} = \theta_1 = \theta_2 \quad ...(14.25)$$

Clearly the effective temperature difference must equal θ_1 and θ_2 in case the temperature differences on either side of a heat exchanger are equal. Further, the logarithmic mean temperature difference for a counter flow unit is greater than that of a parallel flow system and accordingly the counter flow heat exchanger can transfer more heat than a similar parallel flow heat exchanger. Conversely a counter flow exchanger needs a smaller heating surface for the same rate of heat transfer.

If the variation in the temperature of the fluids is relatively small, then temperature variation curves are approximately straight lines and sufficiently accurate results are obtained by taking the arithmetic mean temperature difference (AMTD).

$$AMTD = \frac{t_{h1} + t_{h2}}{2} - \frac{t_{c1} + t_{c2}}{2}$$

$$= \frac{(t_{h1} - t_{c1}) + (t_{h2} - t_{c1})}{2} = \frac{\theta_1 + \theta_2}{2} \quad ...(14.26)$$

Practical consideration, however, suggest that at ratios $\theta_1/\theta_2 > 1.7$, the logarithmic mean temperature difference should be invariably used.

EXAMPLE 14.6.

Explain the difference between parallel flow and counter flow heat exchanger and show how the temperature of the two fluids vary along the path of flow.

A cold liquid (sp. heat 2.95 kJ/kg K) at 10 kg/min is to be heated from 25°C to 55°C in a heat exchanger. The task is accomplished by extracting heat from hot water (sp heat 4.186 kJ/kg K) available at mass flow rate 5 kg/min and inlet temperature 85°C. Should the thermal engineer make design calculations based on parallel flow or counter flow configuration ? State the reason thereof.

Solution : Making the energy balance between the hot (suffix h) and cold (suffix c) fluids,

$$m_h\, c_h\, (t_{h1} - t_{h2}) = m_c\, c_c\, (t_{c2} - t_{c1})$$

or $5 \times 4.186 \times (85 - t_{h2})$

$$= 10 \times 2.95 \times (55 - 25) = 855$$

$$\therefore \quad t_{h2} = 85 - \frac{855}{5 \times 4.186} = \mathbf{42.72°C}$$

The hot water exit temperature (t_{h2} = 42.72°C), is less than the cold fluid exit temperature (t_{c2} = 55°C). The parallel flow configuration is not possible with such temperature distribution. Hence the design calculations need to be made for counter flow configuration of the heat exchanger.

EXAMPLE 14.7.

In a food processing plant, a brine solution is heated from – 12°C to – 65°C in a double pipe parallel flow heat exchanger by water entering at 35°C and leaving at 20.5°C at the rate of 9 kg/min. Determine the heat exchanger area for an overall heat transfer coefficient of 860 W/m² K. For water c_p = 4.186 × 10³ J/kgK.

Solution : Heat transfer from the water,

$$Q = m\, c_p\, \Delta t$$

$$= 9 \times 4.186 \times 10^3 (35 - 20.5)$$

$$= 546.27 \times 10^3 \text{ J/min}$$

$$= 9104.5 \text{ J/s}$$

Log-mean temperature difference, θ_m

$$= \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

For parallel flow arrangement (Figure 14.12) :

$$\theta_1 = t_{h1} - t_{c1} = 35 - (-12) = 47°C$$

$$\theta_2 = t_{h2} - t_{c2} = 20.5 - (-6.5) = 27°C$$

The subscripts h and c represent the hot and cold fluids respectively :

$$\therefore \quad \theta_m = \frac{47 - 27}{\log_e \frac{47}{27}} = 36.10°C$$

Heat exchange, $Q = U A \theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U \theta_m}$$

$$= \frac{9104.5}{860 \times 36.10} = \mathbf{0.293\ m^2}$$

EXAMPLE 14.8.

A tubular heat exchanger is to be designed for cooling oil from a temperature of 80°C to 30°C by a large of stagnant water which may be assumed to remain constant at a temperature of 20°C. The heat transfer surface consists of 30 m long straight tube of 20 mm inside diameter. The oil (specific heat = 2.5 kJ/kg K and specific gravity = 0.8) flows through the cylindrical tube with an average velocity of 50 cm/s. Calculate the overall heat transfer coefficient for the oil cooler.

Solution : The mass rate of flow of oil (hot fluid) through the heat transfer tube is

$$m_h = V A \rho$$

$$= 0.5 \times \frac{\pi}{4}(0.02)^2 \times (0.8 \times 1000)$$

$$= 0.1256 \text{ kg/s}$$

An energy balance on the hot fluid then gives the total energy transferred

$$Q = m_h\, c_h\, (t_{h1} - t_{h2})$$

$$= 0.1256 \times 2.5 \times (80 - 30)$$

$$= 15.7 \text{ kJ/s}$$

The energy transferred is also given by

$$Q = U A \theta_m = U A \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

Since temperature of one of the fluid remains constant during the flow passage, it is immaterial whether the calculations are made for parallel flow or counter flow arrangement. Both arrangements would give the same values for log mean temperature difference and heating surface area for a specified load.

For parallel flow arrangement,

$$\theta_1 = t_{h1} - t_{c1} = 80 - 20 = 60°C$$

$$\theta_2 = t_{h2} - t_{c2} = 30 - 20 = 10°C$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{60 - 10}{\log_e \frac{60}{10}}$$

$$= 27.90°C$$

$$A = \pi d l$$

$$= \pi \times 0.02 \times 30 = 1.884 \text{ m}^2$$

∴ Overall heat transfer coefficient for the oil cooler,

$$U = \frac{Q}{A \theta_m} = \frac{15.7}{1.884 \times 27.90}$$

$$= \mathbf{0.2987\ kJ/m^2\text{-}s\text{-}deg}$$

EXAMPLE 14.9.

Exhaust gases (c_p = 1.12 kJ/kg-deg) flowing through a tubular heat exchanger at the rate of 1200 kg/hr are cooled from 400°C to 120°C. The cooling is affected by water (c_p = 4.18 kJ/kg K) that enters the system at 10°C at the rate of 1500 kg/hr. If the overall heat transfer coefficient is 500 kJ/m²-hr-deg, what heat exchanger area is required to handle the load for (a) parallel flow and (b) counter flow arrangement ?

Solution : The unknown exit temperature of the cooling water may be found from an energy balance on the two fluids, *i.e.*,

heat gained by water = heat lost by exhaust gases

$$m_c\, c_c\, (t_{c2} - t_{c1}) = m_h\, c_h\, (t_{h1} - t_{h2})$$

$$t_{c2} = t_{c1} + \frac{m_h c_h}{m_c c_c}(t_{h1} - t_{h2})$$

$$= 10 + \frac{1200 \times 1.12}{1500 \times 4.18}(400 - 120)$$

$$\cong 70°C$$

From an energy balance on the hot fluid, the heat transfer rate is,

$$Q = m_h\, c_h\, (t_{h1} - t_{h2})$$
$$= 1200 \times 1.12\, (400 - 120)$$
$$= 376320 \text{ kJ/hr}$$

(*a*) **Parallel flow arrangement** (Figure 14.12). The log mean temperature difference is,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

where $\theta_1 = t_{h1} - t_{c1} = 400 - 10 = 390°C$

and $\theta_2 = t_{h2} - t_{c2} = 120 - 70 = 50°C$

$$\therefore \quad \theta_m = \frac{390 - 50}{\log_e \frac{390}{50}} = 165.53°C$$

Now, heat exchange $Q = U\, A\, \theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U\theta_m} = \frac{376320}{500 \times 165.53}$$
$$= 4.547 \text{ m}^2$$

(*b*) **Counter flow arrangement** (Figure 14.13). The log mean temperature difference is,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$\theta_1 = t_{h1} - t_{c2} = 400 - 70 = 330°C$$
$$\theta_2 = t_{h2} - t_{c1} = 120 - 10 = 110°C$$

$$\theta_m = \frac{320 - 110}{\log_e \frac{330}{110}} = 200.25°C$$

Heat exchange, $Q = U\, A\, \theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U\theta_m} = \frac{376320}{500 \times 200.25} = 3.758 \text{ m}^2$$

EXAMPLE 14.10.

A steam condenser is transferring 250 kW of thermal energy at a condensing temperature of 65°C. The cooling water enters the condenser at 20°C with a flow rate of 7500 kg/hr. Calculate the log mean temperature difference. If overall heat transfer coefficient for the condenser surface is 1250 W/m²-deg, what surface area is required to handle this load ? What error would be introduced if the arithmetic mean temperature difference is used rather than the log-mean temperature difference ?

Solution : For the condensing steam, the temperature remains constant throughout the flow passage, *i.e.*,

$$t_{h1} = t_{h2} = 65°C$$

The unknown exit temperature of the cooling water may be found from an energy balance on the cooling water

$$Q = m_c\, c_c\, (t_{c2} - t_{c1})$$

$$250 = \frac{7500}{3600} \times 4.186\,(t_{c2} - 20)$$

$$t_{c2} = 20 + \frac{250 \times 3600}{7500 \times 4.186} = 48.67°C$$

Log mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

where ; $\theta_1 = t_{h1} - t_{c2} = 65 - 20 = 45°C$

$\theta_2 = t_{h2} - t_{c1} = 65 - 48.67 = 16.33°C$

$$\therefore \quad \theta_m = \frac{45 - 16.33}{\log_e \frac{45}{16.33}} = 28.25°C$$

Now, heat exchange $Q = U\, A\, \theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U\theta_m} = \frac{250 \times 10^3}{1250 \times 28.25} = 7.08 \text{ m}^2$$

It may be pointed out that when one of the fluid remains at constant temperature during the flow passage, it is immaterial whether the calculations are made for parallel or counter flow arrangement. Both arrangements would give the same values for log mean temperature difference and heating surface area for a specified load.

(*b*) The arithmetic mean temperature difference is

$$\bar{\theta} = \frac{\theta_1 + \theta_2}{2} = \frac{45 + 16.33}{2} = 30.66°C$$

This gives a heat exchange area of,

$$\bar{A} = \frac{250 \times 10^3}{1250 \times 30.66} = 6.52 \text{ m}^2$$

which under specifies the area by

$$\text{error} = \frac{7.08 - 6.52}{7.08} = 7.91\ \%$$

EXAMPLE 14.11.

Establish the expression for log-mean temperature difference for a (i) parallel flow and a (ii) counter flow heat exchanger.

For what value of end temperature difference ratio θ_1/θ_2, is the arithmetic mean temperature difference 5 % higher than the log-mean temperature difference ?

Solution : The arithmetic mean temperature difference ($\bar{\theta}$) and the log-mean temperature difference (θ_m) ratio may be written as,

$$\frac{\bar{\theta}}{\theta_m} = \left[\frac{\frac{1}{2}(\theta_1 + \theta_2)}{\dfrac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}}\right]$$

$$= \frac{1}{2}\,\frac{\frac{\theta_1}{\theta_2} + 1}{\frac{\theta_1}{\theta_2} - 1}\log_e \frac{\theta_1}{\theta_2}$$

It is stated that $\bar{\theta}$ is to be 5% higher than θ_m

$$\therefore \quad \frac{\bar{\theta}}{\theta_m} = 1.05 = \frac{1}{2}\,\frac{\frac{\theta_1}{\theta_2} + 1}{\frac{\theta_1}{\theta_2} - 1}\log_e \frac{\theta_1}{\theta_2}$$

or $$\frac{1}{2}\,\frac{\frac{\theta_1}{\theta_2} + 1}{\frac{\theta_1}{\theta_2} - 1}\log_e \frac{\theta_1}{\theta_2} = 2.10$$

Solution through hit and trial gives :

$$\frac{\theta_1}{\theta_2} = 2.2$$

Thus the simple arithmetic mean temperature difference gives results to within 5% when the end temperature differences vary by no more than a factor of 2.2

EXAMPLE 14.12.

A company is heating a gas by passing it through a pipe with steam condensing on the outside. What percentage change in length would be needed if it is proposed to triple the heater capacity. Presume that the same terminal conditions as regards temperature are maintained.

(b) A properly designed steam-heated tubular preheater is heating 20,000 kg per hour of air from 20°C to 76°C when using steam at 1.3 bar pressure. It is proposed to double the rate of air flow through the heater and yet heat the air from 20°C to 76°C; this is to be accomplished by increasing the steam pressure. Calculate the required steam pressure to meet the changed conditions. For the changed conditions, calculations may be made by using arithmetic mean temperature difference rather than logarithmic mean temperature difference.

Solution : Present capacity

$$Q_1 = U_1\, A_1\, \theta_{m1}$$

New capacity

$$Q_2 = U_2 A_2 \theta_{m2}$$

According to the given condition :

$$U_2 A_2 \theta_{m2} = 3 U_1 A_1 \theta_{m1}$$

Since the same terminal conditions as regards to temperature are maintained,

$$U_1 = U_2 \quad \text{and} \quad \theta_{m1} = \theta_{m2}$$

$$\therefore \quad A_2 = 3 A_1$$

$$\text{or} \quad \pi d l_2 = 3 \pi d l_1 \qquad (d_1 = d_2 = d)$$

$$\text{or} \quad l_2 = 3 l_1$$

Obviously the required increase in length is 200 percent

(b) The heat transfer under the present conditions is,

$$Q_1 = m\, c_p\, \Delta t$$
$$= 20000 \times 1.05 \times (76 - 20)$$
$$= 1176000 \text{ kJ/hr}$$

The heat transfer under the revised operating conditions would be

$$Q_2 = 2Q_1$$
$$= 2 \times 1176000 = 2352000 \text{ kJ/hr}$$

With surface area A and the overall heat transfer coefficient U remaining constant, the only change is in the logarithmic mean temperature difference θ_m.

Log mean temperature difference θ_m

$$= \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

For the condensing steam, the temperature remains constant throughout the flow passage, i.e., $t_{h1} = t_{h2} = 107°C$ (saturation temperature at 1.3 bar pressure of steam). Then

$$\theta_1 = t_{h1} - t_{c1} = 107 - 20 = 87°C$$
$$\theta_2 = t_{h2} - t_{c2} = 107 - 76 = 31°C$$

$$\therefore \quad \theta_m = \frac{87 - 31}{\log_e \frac{87}{31}} = 54.26°C$$

Under the revised conditions

$$\theta_m = 2 \times 54.26 = 108.52°C$$

The net effect will be a rise in the horizontal steam line at temperature x°C. Then

$$\theta_1 = x - 20 \quad \text{and} \quad \theta_2 = x - 70$$

$$\text{AMTD} = \frac{\theta_1 + \theta_2}{2} = \frac{(x - 20) + (x - 76)}{2}$$
$$= (x - 48)°C$$

As per the given conditions,

$$x - 48 = 108.52 \text{ or } x = 156.52°C$$

Corresponding to 156.52°C, the saturated steam pressure as obtained from the steam tables is 5.6 bar.

EXAMPLE 14.13.

For a double-pipe heat exchanger in which the overall heat transfer coefficient varies linearly with the temperature difference, i.e., $U = a + b\theta$, determine an expression for the heat transfer using subscripts 1 and 2 to represent values at each end of the exchanger.

Solution : For a parallel flow heat exchanger, (Figure. 14.12), the heat flow dQ through an elementary strip of area dA is given by :

$$dQ = - m_h\, c_h\, dt_h = m_c\, c_c\, dt_c$$
$$= - C_h\, dt_h = C_c\, dt_c$$

$$dt_h = - \frac{dQ}{C_h} \quad \text{and} \quad dt_c = \frac{dQ}{C_c}$$

$$d(t_h - t_c) = - dQ \left[\frac{1}{C_h} + \frac{1}{C_c}\right]$$

$$dQ = \frac{-d(t_h - t_c)}{\frac{1}{C_h} + \frac{1}{C_c}} = \frac{-d\theta}{\frac{1}{C_h} + \frac{1}{Cc}} \qquad ...(i)$$

Also, $dQ = U\, dA\, \theta = (a + b\theta)\, dA\, \theta$ ...(ii)

Through integration of expression (i)

$$Q = \frac{\theta_1 - \theta_2}{\frac{1}{C_h} + \frac{1}{C_c}}$$

$$\text{or} \quad \left(\frac{1}{C_h} + \frac{1}{C_c}\right) = \frac{\theta_1 - \theta_2}{Q} \qquad ...(iii)$$

From expression (i) and (ii)

$(a + b\theta)\, dA\, \theta$

$$= \frac{-d\theta}{\frac{1}{C_h} + \frac{1}{C_c}} = \frac{-d\theta}{\frac{\theta_1 - \theta_2}{Q}} = \frac{Q\, d\theta}{\theta_1 - \theta_2}$$

Separating the variable and integrating between the inlet and outlet sections of the heat exchanger :

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\theta(a + b\theta)} = \frac{\theta_1 - \theta_2}{Q} \int_0^A dA$$

$$\frac{\theta_1 - \theta_2}{Q} A = \frac{1}{a} \log_e \left[\frac{\theta}{a + b\theta}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{a} \log_e \left[\frac{\theta_2}{a + b\theta_2} - \frac{\theta_1}{a + b\theta_1}\right]$$

$$= \frac{1}{a} \log_e \left[\frac{\theta_2 (a + b\theta_1)}{\theta_1 (a + b\theta_2)}\right]$$

$$= \frac{1}{a} \log_e \left(\frac{\theta_2 U_1}{\theta_1 U_2}\right) \qquad ...(iv)$$

From the identities

$$U_1 = a + b\theta_1$$

$$\text{and} \quad U_2 = a + b\theta_2$$

the constant a is found to be

$$a = \frac{U_1 \theta_2 - U_2 \theta_1}{\theta_2 - \theta_1}$$

Upon substitution of this value in expression (iv)

$$\frac{\theta_1 - \theta_2}{Q} A = \frac{\theta_2 - \theta_1}{U_1 \theta_2 - U_2 \theta_1}$$

$$= \frac{1}{a} \log_e \left(\frac{\theta_2 U_1}{\theta_1 U_2}\right)$$

Therefore, the heat exchange rate is :

$$Q = \frac{U_1 \theta_2 - U_2 \theta_1}{\log_e (\theta_2 U_1 / \theta_1 U_2)} A$$

EXAMPLE 14.14.

Is it better to arrange for the flow in a heat exchanger to be parallel or counter flow ?

In a counter flow heat exchanger, oil ($c_p = 3$ kJ/kg K) at the rate of 1400 kg/hr is cooled from 100°C to 30°C by water that enters the exchanger at 20°C at the rate of 1300 kg/hr. Determine the heat exchanger area for an overall heat transfer coefficient of 3975 kJ/m²-hr-K. Also derive a relationship between oil and water temperatures at any section of the heat exchanger.

Solution : When there is a choice between the counter-flow and parallel flow arrangements, the counter-flow design is usually preferred for the following reasons :

(i) The exchange of heat may raise the temperature of the cold fluid to more nearly the initial temperature of the hot fluid.

(ii) Log-mean temperature difference is higher and accordingly more heat can be transferred. Conversely, a smaller surface area is required for the same rate of heat transfer.

(b) The unknown exit temperature of the cooling water may be found from an energy balance on the two fluids, i.e.,

$$m_w\, c_w\, (t_{w2} - t_{w1}) = m_o\, c_o\, (t_{o1} - t_{o2})$$

The subscripts w and o refer to water and oil respectively. Inserting the appropriate values :

$$1300 \times 4.186 \times (t_{w2} - 20) = 1400 \times 3 \times (100 - 30)$$

Therefore water outlet temperature is

$$t_{w2} = \frac{1400 \times 3 (100 - 30)}{1300 \times 4.186} + 20$$
$$= 74°C$$

From an energy balance on the oil, the heat transfer is

$$Q = m_o\, c_o\, (t_{o1} - t_{o2})$$
$$1400 \times 3 \times (100 - 30) = 294000 \text{ kJ/hr}$$

Referring to Fig. 14.13, the log-mean temperature difference is

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$\text{where :} \quad \theta_1 = t_{o1} - t_{w2} = 100 - 74 = 26°C$$
$$\theta_2 = t_{o2} - t_{w1} = 30 - 20 = 10°C$$

$$\therefore \quad \theta_m = \frac{26 - 10}{\log_e \frac{26}{10}} = 16.745°C$$

Heat exchange, $Q = U\, A\, \theta_m$

$\therefore$ Heating surface area, A

$$= \frac{Q}{U\,\theta_m} = \frac{294000}{3975 \times 16.745} = 4.417\ m^2$$

(c) At any section of the heat exchanger, we have

$$m_s\, c_s\, (100 - t_s) = m_w\, c_w\, (74 - t_w)$$

$$1400 \times 3 \times (100 - t_s) = 1300 \times 4.186 \times (74 - t_w)$$

$$(100 - t_s) = 1.296\,(74 - t_w)$$

$$\therefore \quad t_s = 100 - 1.296\,(74 - tw) = 4.096 + 1.295 t_w$$

EXAMPLE 14.15.

A feed water heater with steam outside and water inside the tubes is required to heat water from a temperature t_1 to temperature t_2. The overall heat transfer coefficient is anticipated to be [m/(m + Ba)] kJ/m²-s-deg, where a is the total cross-sectional area of water flow in m², m is the mass flow rate of water per tube in kg/s and B is a dimensional constant. Establish the following relation between length l and diameter d of the tube

$$4\frac{l}{d} = c\left(B + \frac{m}{a}\right)\log_e \frac{t_s - t_1}{t_s - t_2}$$

where c is the specific heat of water and t_s is the temperature of condensing steam.

Hence proceed to determine the number and length of tube in each pass of a feed water heater having 6 passes. The relevant data is :

$$t_1, t_2, t_s = 150°C,\ 200°C \text{ and } 205°C \text{ respectively}$$

$$m = 9.5 \text{ kg/s (total)};$$

$$B = 500;$$

$$d = 2.5 \text{ cm}$$

and speed of water through tubes,

$$V = 0.9 \text{ m/s}$$

Solution : An energy balance on the coolant (water) gives the heat exchange rate as

$$Q = n\, m\, c\, (t_2 - t_1)$$

where m is the mass flow rate of water per tube and n is the number of tubes.

The overall heat exchange rate is also given by :

$$Q = U\,A\,\theta_m = U\,A\,\frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= U\,A\,\frac{(t_s - t_1) - (t_s - t_2)}{\log_e \frac{t_s - t_1}{t_s - t_2}}$$

The heating surface area A equals $n\pi d l$

$$\therefore\ n\,m\,c\,(t_2 - t_1) = \left(\frac{m}{m + Ba}\right) \times n\pi d l \times \frac{(t_2 - t_1)}{\log_e\left[\frac{t_s - t_1}{t_s - t_2}\right]}$$

$$\pi\, d\, l = c\,(m + B\,a)\log_e \frac{t_s - t_1}{t_s - t_2}$$

Dividing both sides by the flow area a

$$= \frac{\pi}{4} \times d^2,$$

$$\frac{\pi\, d\, l}{\frac{\pi}{4} \times d^2} = c\left(\frac{m}{a} + B\right)\log_e \frac{t_s - t_1}{t_s - t_2}$$

$$\therefore \quad \frac{4\,l}{d} = c\left(B + \frac{m}{a}\right)\log_e \frac{t_s - t_1}{t_s - t_2}$$

which is the required expression

(b) Mass flow of water per tube

$$= \frac{\pi}{4} d^2 \times V \times \rho$$

$$= \frac{\pi}{4} \times (0.225)^2 \times 0.9 \times 1000$$

$$= 0.441 \text{ kg/s}$$

$\therefore$ Number of tubes $n = 9.5/0.441 = 21.54$

From the relation established above

$$\frac{4\,l}{d} = 1.0\left(500 + \frac{0.441}{\frac{\pi}{4}(0.025)^2}\right)\log_e \frac{205 - 105}{205 - 200}$$

$$= 3354.29$$

Length of each tube l

$$= 354.29 \times 0.025/4 = 20.96 \text{ m}$$

Length of tube per pass

$$= 20.96/6 = \mathbf{3.49\ m}$$

EXAMPLE 14.16.

An existing heat exchanger of 20 m² surface area is to be used to condense low pressure steam. The cooling medium will be feed water available at 40°C; its flow rate being 0.9 kg/s. From previous experience, the overall heat transfer coefficient is estimated at 120 W/m² K. Calculate the quantity of steam condensed and the exit temperature of the feed water. At the condensing pressure steam has saturation temperature t_s = 100°C and latent heat of vaporisation h_{fg} = 2257 kJ/kg. Presume that the steam is initially just saturated and that the condensate leaves the exchanger without sub-cooling, i.e., only the latent heat of condensing steam is transferred to water.

(b) How would the performance of the exchanger be affected if the overall heat transfer coefficient can be doubled by a modification of feed water flow through the exchanger ?

Solution : From energy balance,

$$Q = U\,A\,\theta_m = m_c\, c_c\,(t_{c2} - t_{c1})$$

where,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$$

$$= \frac{(t_s - t_{c1}) - (t_s - t_{c2})}{\log_e\left[(t_s - t_{c1})/(t_s - t_{c2})\right]}$$

$$= \frac{(100 - 40) - (100 - t_{c2})}{\log_e\left[(100 - 40)/(100 - t_{c2})\right]}$$

$$= \frac{t_{c2} - 40}{\log_e\left[(60)/(100 - t_{c2})\right]}$$

$$\therefore\ 120 \times 20 \times \frac{t_{c2} - 40}{\log_e\left[(60)/(100 - t_{c2})\right]} = 0.9 \times (4.18 \times 10^3)\,(t_{c2} - 40)$$

$$\log_e \frac{60}{100 - t_{c2}} = \frac{120 \times 20}{0.9 \times (4.18 \times 10^3)} = 0.6379 \text{ or } \frac{60}{100 - t_{c2}} = 1.892$$

Therefore, the exit temperature of the feed water is :

$$t_{c2} = 100 - \frac{60}{1.892} = \mathbf{68.29°C}$$

Again, the energy balance between the condensing steam and the feed water gives :

$$m_s\, h_{fg} = m_c\, c_c\,(t_{c2} - t_{c1})$$

$$m_s \times 2257 = 0.9 \times 4.18 \times (68.29 - 40)$$

$\therefore$ Steam condensation rate m_s

$$= \frac{0.9 \times 4.18\,(68.29 - 40)}{2257}$$

$$= 0.0471 \text{ kg/s}$$

(b) With overall heat transfer coefficient

$$U = 2 \times 120 = 240 \text{ W/m}^2\text{ K},$$

a similar calculation procedure would give

$$\log_e \frac{60}{100 - t_{c2}} = \frac{240 \times 20}{0.9 \times 4.18 \times 10^3} = 1.2749$$

from which t_{c2} = 83.24°C

Then rate of steam condensation is

$$m_s = \frac{0.9 \times 4.18 \times (83.24 - 40)}{2257}$$

$$= 0.0721 \text{ kg/s}$$

Increase in steam condensation

$$= \frac{0.0721 - 0.0471}{0.0471}$$

$$= 0.5307 = \mathbf{53.07\%}$$

Thus a 100% increase in the overall heat transfer coefficient produces only 50% increase in the rate of steam condensation.

EXAMPLE 14.17.

A counter-flow concentric tube heat exchanger is used to cool the lubricating oil of a large industrial gas turbine engine. The oil flows through the tube at 0.19 kg/s (c_p = 2.18 kJ/kgK), and the coolant water flows in the annulus in the opposite direction at a rate of 0.15 kg/s (c_p = 4.18 kJ/kg K). The oil enters the coolant at 425 K and leaves at 345 K while the coolant enters at 285 K. How long must the tube be made to perform this duty if the heat transfer coefficient from oil to tube surface is 2250 W/m² K and from tube surface to water is 5650 W/m² K ? The tube has a mean diameter of 12.5

mm and its wall presents negligible resistance to heat transfer.

Solution : A heat balance over the exchanger gives :

heat lost by oil

= heat gained by water

$$m_o\, c_o\, (t_{o1} - t_{o2}) = m_w\, c_w\, (t_{w2} - t_{w1})$$

The subscripts o and w refer to oil and water respectively. Inserting the appropriate values :

$$0.19 \times 2.18\ (425 - 345) = 0.15 \times 4.18\ (t_{w2} - 285)$$

∴ Outlet temperature of water ;

$$t_w = 285 + \frac{0.19 \times 2.18 \times 80}{0.15 \times 4.18} = 337.85 \text{ K}$$

From an energy balance on the oil, the heat transfer is given by

$$Q = m_o\, c_o\, (t_{o1} - t_{o2}) = 0.19 \times 2.18\ (425 - 345) = 33.136 \text{ kJ/s} = 33.136 \times 10^3 \text{ J/s}$$

For counter current operation of the exchanger (Fig. 14.13).

$$\theta_1 = t_{o1} - t_{w2} = 425 - 337.85 = 87.15 \text{ K}$$

$$\theta_2 = t_{o2} - t_{w1} = 345 - 285 = 60 \text{ K}$$

∴ log-mean temperature difference is,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}} = \frac{87.15 - 60}{\log_e \dfrac{87.15}{60}} = 72.79 \text{ K}$$

Let U be overall heat transfer coefficient between oil and water;

$$\frac{1}{U} = \frac{1}{h_o} + \frac{1}{h_w}$$

$$U = \frac{h_o h_w}{h_o + h_w} = \frac{2250 \times 5650}{2250 + 5650} = 1609.18 \text{ W/m}^2 \text{ K}$$

Heat exchanger, $Q = U A \theta_m$

∴ Heat surface area,

$$A = \frac{Q}{U \theta_m} = \frac{33.136 \times 10^3}{1609.18 \times 7279} = 0.2829 \text{ m}^2$$

The surface area also equals $\pi\, d\, l$ where d and l represent the tube diameter and length respectively.

∴ Required length of the tube is

$$l = \frac{0.2829}{\pi \times 12.5 \times 10^{-3}} = \mathbf{7.21\ m}$$

EXAMPLE 14.18.

A one-shell, two-tube pass heat exchanger having 3000 thin wall brass tubes of 20 mm diameter has been installed in a steam power plant with a heat load of 2.3×10^8 W. The steam condenses at 50°C and the cooling water enters the tubes at 20°C at the rate of 3000 kg/s. Calculate the overall heat transfer coefficient, the tube length per pass, and the rate of condensation of steam. Take the heat transfer coefficient for condensation on the outer surfaces of the tubes as 15500 W/m² K and the latent heat of steam as 2380 kJ/kg. Further presume the following fluid properties :

$c = 4180$ J/kgK,

$\mu = 855 \times 10^{-6}$ Ns/m²

$k = 0.613$ W/mK

and $Pr = 5.83$

Solution : For thin walled tubes, the overall heat transfer coefficient is given by :

$$\frac{1}{U} = \frac{1}{h_o} + \frac{1}{h_w} \quad \text{or} \quad U = \frac{h_o h_w}{h_o + h_w}$$

The heat transfer coefficient for condensation on the outer surfaces of the tubes is stated as $h_o = 15500$ W/m² K and the coefficient on the inner surface h_i can be calculated from the internal flow situation as follows :

$$Re = \frac{V d \rho}{\mu} = \frac{4m}{\pi d \mu}$$

where m is the mass flow rate through each tube = 1 kg/s

$$Re = \frac{4 \times 1}{\pi \times 0.02 \times (855 \times 10^{-6})} = 74496$$

Obviously the flow is turbulent and therefore the following correlation applies :

$$Nu = 0.023\ (Re)^{0.8}\ (Pr)^{0.4} = 0.023 \times (74496)^{0.8} \times (5.83)^{0.4} = 367.6$$

$$\therefore \quad h_i = \frac{Nu \times k}{d} = \frac{367.6 \times 0.613}{0.02} = 11267 \text{ W/m}^2 \text{ K}$$

and so, $$U = \frac{11267 \times 15500}{11267 + 15500} = 6524 \text{ W/m}^2 \text{ K}$$

The outlet temperature of cooling water can be obtained from the energy balance. That is

$$Q = m_c\, c_c\, (t_{c2} - t_{c1})$$

$$2.3 \times 10^8 = 3000 \times 4180 \times (t_{c2} - 20)$$

$$\therefore \quad t_{c2} = \frac{2.3 \times 10^8}{3000 \times 4180} + 20 = (t_{c2} - 20)$$

Log mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}}$$

where ; $\theta_1 = t_{h1} - t_{c1} = 50 - 20 = 30$°C

$\theta_2 = t_{h2} - t_{c2} = 50 - 38.34 = 11.66$°C

$$\therefore \quad \theta_m = \frac{30 - 11.66}{\log_e \dfrac{30}{11.66}} = \frac{18.34}{0.945} = 19.40°\text{C}$$

Heat exchange, $Q = U A \theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U \theta_m} = \frac{2.3 \times 10^8}{6524 \times 19.40} = 1817 \text{ m}^2$$

For a two pass heat exchanger, the surface area also equals, $N \times 2\pi\, d\, l$, where N is the number of tubes, d and l represent the tube diameter and length respectively,

∴ l (length of tube per pass)

$$= \frac{1817}{3000 \times 2 \times \pi \times 0.02} = 4.82 \text{ m}$$

The mass flow rate of condensation is obtained from, $Q = m \times h_{fg}$

∴ mass flow of condensation, m

$$= \frac{2.3 \times 10^8}{2380 \times 1000} = \mathbf{96.64\ kg/s}$$

EXAMPLE 14.19.

A heat exchanger is to be designed to condense 8 kg/s of an organic liquid (t_{sat} = 80°C ; h_{fg} = 600 kJ/kg) with cooling water available at 15°C and at a flow rate of 60 kg/s. The overall heat transfer coefficient is 480 W/m²-deg. Calculate :

(a) the number of tubes required. The tubes are to be of 25 mm outer diameter, 2 mm thickness and 4.85 m length.

(b) the number of tube passes. The velocity of the cooling water is not to exceed 2 m/s

Solution : From energy balance,

heat lost by vapour

= heat gained by water

$$m_h\, h_{fg} = m_c\, c_c\, (t_{c2} - t_{c1})$$

$$8 \times 600 = 60 \times 4.186\ (t_{c2} - 15)$$

∴ Outlet temperature of water,

$$t_{c2} = \frac{8 \times 600}{60 \times 4.186} + 15 = 34.11°\text{C}$$

Logarithmic mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}}$$

For a condensing vapour, the temperature remains constant through out the flow passage. That is $t_{h1} = t_{h2} = 80$°C.

$\theta_1 = t_{h1} - t_{c1} = 80 - 15 = 65$°C

$\theta_2 = t_{h2} - t_{c2} = 80 - 34.11 = 45.89$°C

$$\therefore \quad \theta_m = \frac{65 - 45.89}{\log_e \dfrac{65}{45.89}} = 54.91°\text{C}$$

Heat transfer rate is given by,

$$Q = m_h\, h_{fg}$$

$= U A \theta_m$

$= U (\pi d_o l N) \times \theta_m$

or $8 \times (600 \times 10^3)$

$= 480 \times (\pi \times 0.025 \times 4.85 \times N) \times 54.91$

$\therefore$ Number of tubes, $N = 478$

Let n be the number of tubes in each pass. Then mass of cold water passing through each pass,

$$m_c = \left(\frac{\pi}{4} d_i^2 \times V \times \rho\right) n$$

$$\therefore \quad 60 = \frac{\pi}{4} (0.021)^2 \times 2 \times 1000 \times n \; ;$$

$$n = 86.66$$

$$\text{Number of passes} = \frac{478}{86.66} = 5.51 = 6$$

EXAMPLE 14.20.

Dry saturated steam at 10 bar enters a counter-flow heat exchanger at the rate of 15 kg/s and leaves at 300°C. The entry of gas at 600°C is with mass flow rate of 25 kg/s. If the condenser tubes are 30 mm diameter and 3 m long, make calculations for the heating surface area and the number of tubes required. Neglect the resistance offered by the metallic tubes

Take the following properties for steam and gas :

For steam :

$t_{sat} = 180°C$ (at 10 bar)

$c_{ps} = 2.7$ kJ/kgK

$h_s = 600$ W/m²-deg

For gas :

$c_{pg} = 1$ kJ/kgK

$h_g = 250$ W/m²-deg

Solution : From energy balance, rate of heat transfer is

$$Q = m_h c_{ph} (t_{h1} - t_{h2})$$

$$= m_c c_{pc} (t_{c2} - t_{c1})$$

where subscripts h and c refer to hot and cold fluids respectively.

$$25 \times 1 \times (600 - t_{h2})$$

$$= 15 \times 2.7 \times (300 - 180)$$

$$= 4860 \text{ kJ}$$

$$\therefore \quad t_{h2} = 600 - \frac{4860}{25} = 405.6°C$$

The overall heat transfer coefficient is given by

$$\frac{1}{U} = \frac{1}{h_g} + \frac{d_o}{d_i}\frac{1}{h_s} = \frac{1}{h_g} + \frac{1}{h_s} \text{ as } d_i = d_o$$

$$U = \frac{h_g \times h_s}{h_g + h_s} = \frac{250 \times 600}{250 + 600}$$

$$= 176.47 \text{ W/m}^2\text{-deg}$$

The log mean temperature difference for counter flow arrangement,

$$\theta_1 = t_{h1} - t_{c1} = 600 - 300 = 300°C$$

$$\theta_2 = t_{h2} - t_{c2} = 405.6 - 180 = 225.6°C$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{300 - 225.6}{\log_e \frac{300}{225.6}} = 261.05°C.$$

Heat transfer rate, $Q = U A \theta_m$

$4860 \times 10^3 = 176.47 \times A \times 261.05$

$\therefore$ Heating surface area, A

$$= \frac{4860 \times 10^3}{176.47 \times 261.05} = 105.5 \text{ m}^2$$

The heating surface area also equals ($N \times \pi d l$) where d and l represent the tube diameter and length respectively, and N is the number of tubes.

$$\therefore \quad N = \frac{105.5}{\pi \times (30 \times 10^{-3}) \times 3} = 374$$

EXAMPLE 14.21.

Steam at 0.5 bar pressure is condensed in a single-pass steam condenser consisting of 100 thin-walled tubes of 25 mm nominal diameter and 2 m length. The cooling water enters at a temperature of 10°C, leaves at 50°C and flows through the tubes with a mean velocity of 2 m/s. The condensing heat transfer coefficient is 5 kW/m² K. You are required to determine :

(i) overall heat transfer coefficient for heat exchange

(ii) condensation rate of steam

(iii) mean temperature of the tube metal at the centre of the condenser length

Thermal resistance of the tube metal may be neglected.

Solution : The heat transfer coefficient on the cooling (water) side is determined from the correlation,

$$Nu = 0.023 \, (Re)^{0.8} \, (Pr)^{0.4}$$

At the average bulk temperature,

$$t_b = \frac{t_{w1} + t_{w2}}{2} = \frac{10 + 50}{2} = 30°C$$

the thermo physical properties of water are

$\rho = 995.7$ kg/m³

$\mu = 2.88$ kg/hr-m

$c_p = 4.174$ kJ/kgK

$k = 0.617$ W/mK

$\therefore$ Reynolds number,

$$Re = \frac{V d \rho}{\mu} = \frac{2 \times 0.025 \times 995.7}{(2.88 / 3600)}$$

$$= 62331$$

Prandtl number,

$$Pr = \frac{\mu c_p}{k}$$

$$= \frac{(2.88 / 3600) \times 4.174 \times 10^3}{0.617}$$

$$= 5.41$$

Nusselt number,

$$Nu = 0.023 \, (62231)^{0.8} \, (5.41)^{0.4}$$

$$= 309.18$$

From $Nu = hd/k$ the heat transfer coefficient on the cooling water side is

$$h = \frac{Nu \times k}{d} = \frac{309.18 \times 0.617}{0.025}$$

$$= 7630 \text{ W/m}^2\text{K} = 6.63 \text{ kW/m}^2\text{K}$$

Neglecting thermal resistance of the metal tube, the overall heat transfer coefficient is given by the expression :

$$\frac{1}{U} = \frac{1}{h_o} + \frac{1}{h_i}$$

where the subscripts o and i indicate outer and inner sides respectively.

$$U = \frac{h_o \times h_i}{h_o + h_i} = \frac{5 \times 7.63}{5 + 7.63}$$

$$= 3.02 \text{ kW/m}^2\text{K}$$

(b) At a pressure of 0.5 bar the steam condensing temperature is 81°C.

Log-mean temperature difference, θ_m

$$= \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

where $\theta_1 = 81 - 10 = 71°C$

and $\theta_2 = 81 - 50 = 31°C$

$$\therefore \quad \theta_m = \frac{71 - 31}{\log_e \frac{71}{31}} = 48.27°C$$

Heat exchange per tube

$= U A \theta_m$

$= 3.02 \times (\pi \times 0.025 \times 2) \times 48.27$

$= 23.09$ kW

Total heat exchange rate Q

$= 23.09 \times 100 = 2309$ kW

The mass flow rate of condensation is obtained from Q

$= m \times h_{fg}$

where h_{fg} is the enthalpy of vaporisation or latent heat at 0.5 bar pressure ;

$h_{fg} = 2305$ kJ/kg K

$\therefore$ Mass flow of condensate, m

$= 2309/2305 = 1$ kg/s

(c) Let the temperature difference between the two fluids be expressed by the following general equation,

$$\log_{10} \theta = Ax + B$$

where x is the distance along the exchanger length. With the conditions

$\theta = 81 - 10 = 71°C$ at $x = 20$

$\theta = 81 - 50 = 31°C$ at $x = 2$

the constants are obtained as :

$A = -0.175$ and $B = 1.854$

and the equation becomes

$$\log_{10} \theta = -0.175 x + 1.854$$

At the mid-plane ($x = 1$),

$$\log_{10} \theta = 1.697 \; ; \; \theta = 47.64$$

Therefore, the temperature of water at the mid-plane is,

$$t_w = 81.0 - 47.64 = 33.36°C$$

EXAMPLE 14.22.

A surface condenser used in a steam power plant deals with 27000 kg of steam per hour at a pressure of 4.15 kN/m² and 0.9 dryness fraction. The cooling medium will be water that enters the condenser at 15°C and leaves at 25°C. From previous experience, a water velocity of 1.5 m/s is maintained through the tubes and the overall coefficient of heat transfer is estimated at 3500 W/m² K. Calculate (i) mass flow rate of water, (ii) surface area required for the given duty and (iii) passes and number of tubes.

The tubes used in condenser are 20 mm outside diameter, 1.5 mm thick and the space limitation restricts the condenser length to 4 metres. At the condensing pressure, steam has saturation temperature t_s = 29.5°C and laten heat of vaporisation h_{fg} = 2435 kJ/kg. Presume that the condensate coming out of the condenser is saturated water at the condenser pressure, i.e., there is no under cooling and the steam loses only latent part of its heat.

Solution : The exchanger would extract $x\,h_{fg}$ amount of heat to condense 1 kg of wet vapour (steam) into 1 kg of saturated liquid ; x is the dryness fraction of steam. The overall energy balance gives :

heat lost by wet steam

= heat gained by cooling medium

$$m_s\,(x\,h_{fg}) = m_w\,c_w\,(t_{w2} - t_{w1})$$

$$27000 \times (0.9 \times 2345)$$

$$= m_w \times 4.18\,(25 - 15)$$

∴ Mass flow rate of water, m_w

$$= \frac{27000 \times (0.9 \times 2345)}{4.18 \times (25 - 15)}$$

$$= 1.416 \times 10^6 \text{ kg/hr}$$

Heat transfer , Q

$$= 27000 \times (0.9 \times 2345)$$

$$= 5.917 \times 10^7 \text{ kJ/hr}$$

$$= 16.436 \times 10^3 \text{ kJ/s}$$

(*i*) For the condenser,

$$\theta_1 = t_s - t_{w1} = 29.5 - 15 = 14.5°C$$

$$\theta_2 = t_s - t_{w2} = 29.5 - 25 = 4.5°C$$

∴ log-mean temperature difference is

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{14.5 - 4.5}{\log_e \frac{14.5}{4.5}} = 8.547°C$$

(*ii*) Now ; mass flow = velocity × cross-sectional area × density

$$m = V\,A_c\,\rho\,; \quad A_c = \frac{m}{V\rho}$$

Therefore, the total cross-sectional area for water flow is

$$A_c = \frac{1.416 \times 10^6}{(1.5 \times 3600) \times 998} = 0.2627 \text{ m}^2$$

Cross sectional area per tube

$$= \pi/4 \times (0.017)^2$$

$$= 2.268 \times 10^{-4} \text{ m}^2$$

∴ Number of tubes, n

$$= 0.02627/(2.268 \times 10^{-4}) \approx 1158$$

Surface area of each tube

$$= \pi\,d_o\,l$$

$$= \pi \times 0.02 \times 4 = 0.2512 \text{ m}^2$$

If the condenser has p-passes, then number of tubes = $n\,p$

$$\therefore 0.2512\,n\,p = 549.43$$

Number of passes, p

$$= 549.43/(0.2512 \times 1158)$$

$$= 1.89 \approx 2$$

EXAMPLE 14.23.

A two pass surface condenser is required to handle the exhaust from a turbine developing 15 MW with specific steam consumption of 5 kg/kWh. The quality of exhaust steam is 0.9, the condenser vacuum is 66 cm of mercury while the bar meter reads 76 cm of mercury. The condenser tubes are 28 mm inside diameter, 4 mm thick and water flows through tubes with a speed of 3 m/s and inlet temperature 20°C. All the steam is condensed, the condensate is saturated water and temperature of cooling water at exit is 5°C less than the condensate temperature.

Assuming that overall coefficient of heat transmission is 4 kW/m²-deg, determine : (a) mass of cooling water circulated ; (b) surface area required ; (c) length and number of tubes in each pass.

Solution : Refer Figure 14.14 for the arrangement of a two-pass surface condenser and temperature profiles for the hot (steam) and cold (cooling water) fluids along the length of heat exchanger.

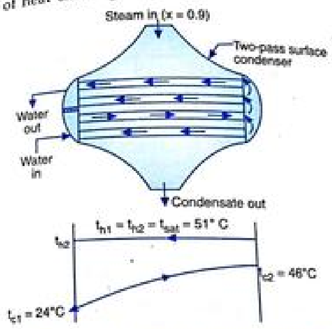


Fig. 14.14. A two-pass surface condenser

The pressure of steam in the condenser,

$$P_s = \frac{76 - 66}{76} \times 1.0133 = 0.133 \text{ bar}$$

At 0.133 bar, the properties of steam as read from steam tables are :

saturation temperature t_{sat}

= 51°C, and

specific enthalpy h_{fg} = 2590 kJ/kg

∴ Temperature of cooling water at exit,

$$t_{c2} = 51 - 5 = 46°C$$

Steam condensed

$$= \frac{(15 \times 1000)}{3600} = 20.83 \text{ kg/s}$$

(*a*) From energy balance,

heat lost by steam

= heat gained by cooling water

$$m_h \times (x\,h_{fg}) = m_c\,c_c\,(t_{c2} - t_{c1})$$

$$20.83 \times (0.9 \times 2590)$$

$$= m_c \times 4.186 \times (46 - 20)$$

∴ Mass of cooling water, m_c

$$= \frac{20.83 \times (0.9 \times 2590)}{4.186 \times 26}$$

$$= 446.13 \text{ kg/s}$$

(*b*) Assuming counter flow arrangement,

$$\theta_1 = t_{h2} - t_{c1} = 51 - 20 = 31°C$$

$$\theta_2 = t_{h1} - t_{c2} = 51 - 46 = 5°C$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{31 - 5}{\log_e \frac{31}{15}}$$

$$= 14.25°C$$

Heat exchange,

$$Q = m_h\,(x\,h_{fg}) = U\,A\,\theta_m$$

or $20.83 \times (0.9 \times 2590) = 1 \times A \times 14.25$

∴ Condenser surface area, A

$$= \frac{20.83 \times (0.9 \times 2590)}{4 \times 14.25}$$

$$= 851.84 \text{ m}^2$$

(*c*) Mass flow rate of water through each tube,

$$m = \rho\,A\,V$$

$$= 1000 \times \frac{\pi}{4}(0.028)^2 \times 3$$

$$= 1.846 \text{ kg/s}$$

∴ Number of tubes per pass

$$= \frac{446.13}{1.846} = 241.67 \text{ say } 242$$

∴ Total number of tubes required

$$= 2 \times 142 = 484$$

The condenser surface area A also equals $(N \times \pi\,d_o\,l)$ where d_o and l represent the outside tube diameter and length respectively, and N is the total number of tubes.

$$\therefore 851.84 = 484 \times (\pi \times 0.032 \times l)$$

or Length of tube, l

$$= \frac{851.84}{484 \times \pi \times 0.032} = 17.51 \text{ m}$$

EXAMPLE 14.24.

In a petrol engine arranged for evaporative cooling, the steam formed at 100°C is condensed in the radiator where it surrounds the tubes through which the cooling air at 17°C is made to flow with a mean air velocity of 5.5 m/s. The radiator is made up of parallel 12.5 cm long and 0.8 cm diameter tubes, and is designed to suit an engine which develop 8.8 kW of power with brake thermal efficiency of 22 percent. If 35 percent of heat supplied to the engine is dissipated at the radiator, workout the number of tubes and the temperature of air at exit from the radiator. For air :

$$c_p = 1005 \text{ J/kg-deg}$$
$$\rho = 1.4 \text{ kg/m}^3$$

and overall coefficient of heat transfer through the tubes $U = 19.08$ W/m²-deg

Solution : Heat supplied to the engine

$$= \frac{8.8}{0.22} = 40 \text{ kJ/s}$$

Heat dissipated to the radiator Q

$$= 40 \times 0.35$$
$$= 14 \text{ kJ/s} = 14000 \text{ J/s}$$

From energy balance,

$$Q = n\, m_a\, c_p\, (t_{a2} - t_{a1})$$

The subscript a refers to the coolant which is air for the given arrangement and n represents the number of tubes number of tubes.

Mass flow rate of air per tube

$$m_a = \rho\left(\frac{\pi}{4} d^2 V\right)$$

$$= 1.4 \times \frac{\pi}{4} \times 0.008^2 \times 5.5$$

$$= 3.87 \times 10^{-4} \text{ kg}$$

$$\therefore\ 14000 = n \times (3.87 \times 10^{-4}) \times 1005 \times (t_{a2} - 17)$$

$$\text{or } n\,(t_{a2} - 17) = 35995 \quad \text{...}(i)$$

The heat exchange can also be expressed in terms of overall heat transfer coefficient and the corresponding mean temperature difference θ_m.

$$\theta_m = \frac{\theta_i - \theta_e}{\log_e(\theta_i/\theta_e)}$$

$$= \frac{(100-17)-(100-t_{a2})}{\log_e\left[(100-17)/(100-t_{a2})\right]}$$

$$= \frac{t_{a2}-17}{\log_e\left[83/(100-t_{a2})\right]}$$

$$\therefore\ 1400 = U A \theta_m = U \times (n \pi d l)\theta_m$$

$$= 19.08 \times (n \times \pi \times 0.008 \times 0.125) \times \frac{t_{a2}-17}{\log_e\left[83/(100-t_{a2})\right]}$$

$$\text{or } \frac{n\,(t_{a2}-17)}{\log_e\left[83/(100-t_{a2})\right]} = 233679 \quad \text{...}(ii)$$

From expression (i) and (ii)

$$\log_e\left(\frac{83}{100-t_{a2}}\right) = \frac{35995}{233679} = 0.154$$

$$\text{or } \frac{83}{100-t_{a2}} = 1.68$$

Therefore, temperature of air at exit from the radiator

$$t_{a2} = 100 - \frac{83}{1.166} = 28.8°\text{C}$$

Then using expression (i)

$$n\,(28.8 - 17) = 35995$$

$$\therefore \text{ Number of tubes, } n = \frac{35995}{18.6-17} = 3050$$

EXAMPLE 14.25.

An intercooler of air compressor takes in air at 6 bar and 150°C and passes it to the next stage at 30°C and at the equivalent rate of 6 m³ of free air (15°C and 1 bar) per minute. The cooling water passes in parallel flow over the tubes which are 100 mm outside diameter and 1.2 mm thick. The inlet and outlet water temperatures are 10°C and 20°C respectively and the air velocity at entry into the tubes is 6 m/s. Take the following data :

h_i (inside heat transfer coefficient, air side)
= 360 kJ/m²-hr-deg

h_o (outlet heat transfer coefficient, water side)
= 7200 kJ/m²-hr-deg

c_p (air) = 1.00 kJ/kgK

and R = 287 J/kgK

Find the number tubes in the intercooler and the length of each tube. What will be the saving in the total tube length if the cooler is made counter-flow with the inlet and outlet temperatures being maintained same as before ?

Solution : The log-mean temperature difference is given by

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

Reference to Figure 14.12 for the parallel flow arrangement

$$\theta_1 = t_{a1} - t_{w1} = 150 - 10 = 140°\text{C}$$
$$\theta_2 = t_{a2} - t_{w2} = 30 - 20 = 10°\text{C}$$

$$\therefore\ \theta_m = \frac{140-10}{\log_e \frac{140}{10}} = 49.26°\text{C}$$

The overall heat transfer coefficient referred to the outside surface of the tube is

$$\frac{1}{U} = \frac{1}{h_o} + \left(\frac{r_o}{r_i}\right)\frac{1}{h_i}$$

$$r_o = 5 \text{ mm}$$

and $r_i = 5 - 1.2 = 3.8$ mm

$$\frac{1}{U} = \frac{1}{7200} + \left(\frac{5}{3.8}\right) \times \frac{1}{360};$$

$$U = 263.66 \text{ kJ/m}^2\text{-hr-deg}$$

The mass of air passing through the tube is

$$m_a = \frac{pV}{RT} = \frac{1 \times 10^5 \times 6}{287 \times (273+15)}$$

$$= 7.26 \text{ kg/min}$$

The heat exchange rate is worked out form energy balance on air,

$$Q = m_a\, c_a\, (t_{a1} - t_{a2})$$
$$= 7.26 \times 1.00 \times (150 - 30)$$
$$= 871.2 \text{ kJ/min} = 52272 \text{ kJ/hr}$$

Further, $Q = U A \theta_m$

$\therefore$ Heat transfer area, A

$$= \frac{Q}{U\theta_m} = \frac{52272}{263.66 \times 49.26}$$

$$= 4.025 \text{ m}^2$$

(a) The density of air flowing through the intercooler tubes at 6 bar pressure and an average temperature of 90°C is

$$\rho = \frac{p}{RT} = \frac{6 \times 10^5}{287 \times (273+90)}$$

$$= 5.76 \text{ kg/m}^3$$

Now, mas flow

= velocity × cross-sectional area × density

$$m = V A_c \rho\ ;\ A_c = \frac{m}{V\rho}$$

Therefore total cross-sectional area for the air flow is

$$A_c = \frac{7.26}{(60 \times 6) \times 5.76} = 3.5 \times 10^{-3} \text{ m}^2$$

Cross-sectional area per tube

$$= \pi \times (0.0038)^2 = 4.53 \times 10^{-5} \text{ m}^2$$

$\therefore$ Number of tubes, n

$$= \frac{3.53 \times 10^{-3}}{4.53 \times 10^{-5}} = 77.29 \approx 78$$

The heating surface area equals $n \pi d l$ where n is the number of tubes, d and l are the tube outside diameter and length respectively

$\therefore$ Required length of tube,

$$l = \frac{4.025}{78 \times \pi \times 0.01}$$

$$= 1.643 \text{ m per tube}$$

(b) For the counter-flow arrangement (Figure 14.13)

$$\theta_1 = t_{a1} - t_{w2} = 150 - 20 = 130°\text{C}$$
$$\theta_2 = t_{a2} - t_{w1} = 30 - 10 = 20°\text{C}$$

$$\therefore\ \theta_m = \frac{130-20}{\log_e \frac{130}{20}} = 58.77°\text{C}$$

For the same heat exchange rate, the area required is inversely proportional to θ_m

obviously the length of tube required is inversely proportional to θ_m. Therefore the length required for the counter-flow arrangement is

$$l_c = 1.643 \times \frac{49.26}{58.77}$$

$$= 1.377 \text{ m per tube}$$

Saving in length

$$= \frac{1.643 - 1.377}{1.643}$$

$$= 0.162 \text{ or } 16.2\%$$

It should be noted that the above calculation is valid on the assumption that the modified temperature distribution throughout the exchanger does not affect the value of heat transfer coefficient.

EXAMPLE 14.26.

The heat energy for space heating is extracted from the exhaust gases of an I.C. engine. The single pass multi tube heat exchanger employed for this task operates in counter flow and has a normal tube diameter of 20 mm. The exhaust gases with a mean specific heat of 1.088 kJ/kgK and a mean gas constant of 300 J/kgK flow through the tubes with a velocity of 25 m/s and are cooled from 300°C to 100°C. Water flowing through the tubes is heated from 20°C to 85°C and the overall coefficient of heat transfer is estimated to be 720 kJ/m²-hr-deg.

These conditions apply when the engine produces 100 kW brake power, has brake specific fuel consumption 0.25 kg/kWh, air fuel ratio 15 : 1 and the exhaust gas pressure equivalent to 100 mm of water. You are required to make an estimate of the mass flow rate of water, the number and length of the exchanger tubes.

Solution : The mass flow rate of the exhaust gases m_g may be found from the engine data :

fuel flow = 0.25 × 100 = 25 kg/hr

air flow = 15 × fuel flow

= 15 × 25 = 375 kg/hr

total exhaust gas flow m_g

= 25 + 375 = 400 kg/hr

(a) A heat balance over the exchanger gives,

heat lost by flue gases

= heat gained by water

$$m_g\, c_g\, (t_{g1} - t_{g2}) = m_w\, c_w\, (t_{w2} - t_{w1})$$

The subscripts g and w refer to gas and water respectively. Inserting the appropriate values :

$$400 \times 1.088 \times (300 - 100) = m_w \times 4.186 \times (85 - 20)$$

Therefore, the mass flow rate of water

$$m_w = \frac{400 \times 1.088 \times (300 - 100)}{4.186 \times (85 - 20)}$$

$$= 320 \text{ kg/hr}$$

(b) The exhaust gas pressure of 100 mm of water is equivalent to

$$\frac{100}{13.6} \text{ mm of Hg} = \frac{100}{13.6} \times \frac{1.0132}{760}$$

$$= 0.0098 \text{ bar}$$

The corresponding absolute pressure is obtained by adding atmospheric pressure (1.0132 bar) to this gauge pressure.

∴ absolute gas pressure p_g

= 1.0132 + 0.0098 = 1.023 bar

From the characteristic gas equation,

$$\rho = \frac{P}{RT} = \frac{1.023 \times 10^5}{300 \times (300 + 273)}$$

$$= 0.595 \text{ kg/m}^3$$

Now, mass flow

= velocity × cross-sectional area × density

$$m = V A_c \rho\ ;\ A_c = \frac{m}{V\rho}$$

Therefore total cross-sectional area for the air flow is

$$A_c = \frac{400}{(25 \times 3600) \times 0.595}$$

$$= 7.49 \times 10^{-3} \text{ m}^2$$

Cross-sectional area per tube

$$= \frac{\pi}{4}(0.02)^2 = 3.14 \times 10–4 \text{ m}^2$$

∴ Number of tube, n

$$= \frac{7.49 \times 10^{-3}}{3.14 \times 10^{-4}} = 23.85 = 24$$

(c) From an energy balance on the gases, the heat transfer is given by

$$Q = m_g\, c_g\, (t_{g1} - t_{g2})$$

$$= 400 \times 1.088\ (300 - 100)$$

$$= 87040 \text{ kJ/hr}$$

For counter current operation of the exchanger (Figure. 14.13)

$$\theta_1 = t_{g1} - t_{w2} = 300 - 85 = 215°C$$

$$\theta_2 = t_{g2} - t_{w1} = 100 - 20 = 80°C$$

∴ Log mean temperature difference

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{215 - 80}{\log_e \frac{215}{80}}$$

$$= 136.55°C$$

Heat exchange, $Q = U A \theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U\theta_m} = \frac{87040}{720 \times 136.55}$$

$$= 0.885 \text{ m}^2$$

The heating surface area also equals $n \pi d l$, where n is the number of tubes, d and l denote the tube diameter and length respectively.

∴ Required length of tube, l

$$= \frac{0.885}{24 \times \pi \times 0.02} = 0.587$$

EXAMPLE 14.27.

A single pass shell and tube heat exchanger, consisting of a bundle of 100 tubes (inner diameter 25 mm and thickness 2 mm) is used for heating 8 kg/s of water from 25°C to 75°C with the help of steam condensing at atmospheric pressure on the shell side with condensing heat transfer coefficient 5000 W/m²-deg. Make calculations for the overall heat transfer coefficient based on the inner area and length of the tube bundle

Take fouling factor on the water side to be 0.0002 m²-deg/W per tube and neglect effect of fouling on the shell side and thermal resistance of the tube wall

The thermo-physical properties of water at the mean bulk temperature of 50°C are :

ρ = 998 kg/m³

c_p = 4175 J/kg-deg

k = 0.65 W/m-deg

μ = 55 × 10⁻⁵ kg/m-s

Solution : Refer Figure 14.15 for the temperature variation in counter/parallel arrangement of the heat exchanger.

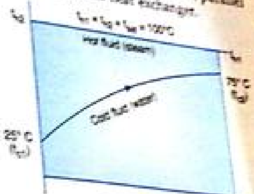


Fig. 14.15.

Heat transfer rate from steam to water is given by

$$Q = m_c\, c_c\, (t_{c2} - t_{c1})$$

$$= 8 \times 4175\ (75 - 25)$$

$$= 1.67 \times 10^6 \text{ W}$$

(a) Flow Reynolds number, $Re = \frac{\rho V d_i}{\mu}$

The average flow velocity V of water can be worked out from the following identity

$$m_c = (\rho A_i V) N = \rho \times \frac{\pi}{4} d_i^2 V \times N$$

$$8 = 998 \times \frac{\pi}{4}(0.025)^2 V \times 100$$

$$= 48.96 V$$

or $V = \frac{8}{48.96} = 0.163 \text{ m/s}$

$$\therefore\ Re = \frac{998 \times 0.163 \times 0.025}{55 \times 10^{-5}} = 7394$$

Prandtl number, Pr

$$= \frac{\mu c_p}{k} = \frac{55 \times 10^{-5} \times 4175}{0.65} = 3.53$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies

for calculating the convective heat transfer coefficient on the water side

$$Nu = \frac{h_i d_i}{k}$$
$$= 0.023\,(Re)^{0.8} \times (Pr)^{0.4}$$
$$= 0.023\,(7394)0.8 \times (3.53)0.4$$
$$= 47.41$$

$$\therefore \quad h_i = Nu \times \frac{k}{d_i} = 47.41 \times \frac{0.65}{0.025}$$
$$= 1232.7 \text{ W/m}^2\text{-deg}$$

The overall heat transfer coefficient based on inside area is

$$\frac{1}{U_i} = \frac{1}{h_i} + R_{fi} + \frac{r_i}{r_o} \times \frac{1}{h_o}$$
$$= \frac{1}{1232.7} + 0.0002 + \frac{0.0125}{0.0145} \times \frac{1}{5000}$$
$$= 0.00081 + 0.0002 + 0.00017$$
$$= 0.00118$$

$$\therefore \quad U_i = \frac{1}{0.00118} = 847.46 \text{ W/m}^2\text{-deg}$$

(b) The logarithmic mean temperature difference for counter flow arrangement

$$\theta_1 = t_{h2} - t_{c1} = 100 - 75 = 25°\text{C}$$
$$\theta_2 = t_{h1} - t_{c2} = 100 - 25 = 75°\text{C}$$
$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{25 - 75}{\log_e \frac{25}{75}}$$
$$= 45.51°\text{C}$$

Heat exchange, $Q = U_i A_i \theta_m$

$\therefore$ Heating surface area, A_i

$$= \frac{Q}{U_i \theta_m} = \frac{1.67 \times 10^6}{847.46 \times 45.51}$$
$$= 43.30 \text{ m}^2$$

The heating surface area A_i also equals $(N \times \pi\, d_i\, l)$ where d_i and l represent the inside tube diameter and length l respectively, and N is the number of tubes

$\therefore$ Length of tube

$$= \frac{A_i}{n \times \pi d_i} = \frac{43.30}{100 \times \pi \times 0.025}$$
$$= \mathbf{5.52 \text{ m}}$$

EXAMPLE 14.28.

A water preheater consists of an iron pipe (k = 60 W/m-deg) with inner and outer diameters as 3.5 cm and 4 cm. The water enters the pipe at 30°C, flows with a mean velocity of 1.25 m/s and leaves at 90°C temperature. The heating fluid is steam at 180°C and the heat transfer coefficient on the steam side is 12 kW/m²-deg. Determine the heat transfer rate, overall coefficient of heat transfer and the length of the pipe required to achieve the desired objective

The thermo-physical properties of water at the bulk temperature of 60°C are stated as

$$\mu = 4.6 \times 10^{-6} \text{ kg/m-s};$$
$$k = 0.65 \text{ W/m-deg};$$
$$c_p = 4200 \text{ J/kgK};$$
$$\rho = 1000 \text{ kg/m}^3$$

Solution : The heat transfer rate from steam to water is given by

$$Q = m_c\, c_c\, (t_{c2} - t_{c1})$$

where $\quad m_c = \rho\, A\, V = \rho \times \frac{\pi}{4} d_i^2 \times V$

$$= 1000 \times \left(\frac{\pi}{4} \times 0.035^2\right) \times 1.25$$
$$= 1.202 \text{ kg/s}$$
$$\therefore \quad Q = 1.202 \times 4200 \times (90 - 300)$$
$$= \mathbf{302904 \text{ W}}$$

(b) Flow Reynolds number Re

$$= \frac{\rho V d_i}{\mu} = \frac{1000 \times 1.25 \times 0.035}{4.6 \times 10^{-4}}$$
$$= 95109$$

Prandtl number Pr

$$= \frac{\mu c_p}{k} = \frac{4.6 \times 10^{-4} \times 4200}{0.65}$$
$$= 2.972$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies for calculating the convective heat transfer coefficient on the water side

$$Nu = \frac{h_i d_i}{k}$$
$$= 0.023\,(Re)^{0.8} \times (Pr)^{0.4}$$
$$= 0.023\,(95109)^{0.8} \times (2.972)^{0.4}$$
$$= 341.6$$

$$h_i = Nu \times \frac{k}{d_i} = 341.6 \times \frac{0.65}{0.035}$$
$$= 6344 \text{ W/m}^2\text{-deg}$$

The overall heat transfer coefficient based on inner surface is

$$\frac{1}{U_i} = \frac{1}{h_i} + \frac{r_i}{r_o}\frac{1}{h_o} + \frac{r_i}{k_{pipe}} \log_e \frac{r_o}{r_i}$$
$$= \frac{1}{6344} + \frac{0.0175}{0.02} \times \frac{1}{10 \times 10^3}$$
$$+ \frac{0.0175}{60} \log_e \frac{0.02}{0.0175}$$
$$= 0.000158 + 0.0000875 + 0.000039$$
$$= 0.0002845$$

$\therefore$ Overall heat transfer coefficient, U_i

$$= \frac{1}{0.0002845} = 3515 \text{ W/m}^2\text{-deg}$$

(c) The logarithmic mean temperature difference for counter flow arrangement

$$\theta_1 = t_{h2} - t_{c1} = 180 - 30 = 150°\text{ C}$$
$$\theta_2 = t_{h1} - t_{c2} = 180 - 90 = 90°\text{ C}$$
$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$
$$= \frac{150 - 90}{\log_e \frac{150}{90}} = 117.46°\text{ C}$$

Heat exchange, $Q = U_i A_i \theta_m$

$\therefore$ Heating surface area, A_i

$$= \frac{Q}{U_i \theta_m} = \frac{302904}{35.15 \times 117.46}$$
$$= 0.7336 \text{ m}^2$$

The heating surface area for a pipe also equals $(\pi\, d_i\, l)$ where d_i and l represent the inside take diameter and length respectively.

$\therefore$ Length of pipe

$$= \frac{A_i}{U_i \theta_m} = \frac{0.7336}{\pi \times 0.035} = \mathbf{6.675 \text{ m}}$$

EXAMPLE 14.29.

A feed water heater which supplies hot water to a boiler comprises a one-shell pass and two-tube pass heat exchanger. The unit has 100 thin walled tubes of 20 mm diameter and space limitations restrict the tube length of 2 m per pass. Saturated steam condenses at 1 atm (t_{sat} = 100°C) on surface of the tubes with convective coefficient 10 kW/m²-deg. If water enters the tube at 10 kg/s and 15°C, determine overall heat transfer coefficient and temperature of water at exit.

Use the following fluid properties and make calculations on arithmetic mean temperature difference.

$$c_p = 4180 \text{ J/kgK}$$
$$\mu = 0.6 \times 10^{-3} \text{ Ns/m}^2$$
$$k = 0.64 \text{ W/m-deg}$$
$$\rho = 1000 \text{ kg/m}^3$$

Solution : Refer Fig. 14.16. for the feed water heater and corresponding temperature profile for the hot fluid (steam) and cold fluid (water).

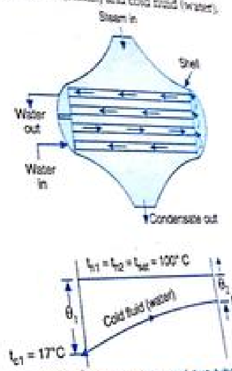


Fig. 14.16. One-shell pass and two-tube passes condenser

Mass flow of water/sec

$$= \rho\, AV \times \text{number of tubes per pass}$$
$$10 = 1000 \times \frac{\pi}{4}(0.02)^2 \times V \times 100$$
$$= 31.4\, V$$

$\therefore$ Velocity of water, V

$$= \frac{10}{31.4} = 0.318 \text{ m/s}$$

Flow Reynolds number Re

$$= \frac{V d \rho}{\mu} = \frac{0.318 \times 0.02 \times 1000}{0.6 \times 10^{-3}}$$
$$= 10600$$

Prandtl number Pr

$$= \frac{\mu c_p}{k}$$
$$= \frac{0.6 \times 10^{-3} \times 4180}{0.64} = 3.918$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies for calculating the heat transfer coefficient on the water side.

$$Nu = \frac{h_i d}{k}$$
$$= 0.023 (Re)^{0.8} \times (Pr)^{0.4}$$
$$= 0.023 \, (10600)^{0.8} \times (3.918)^{0.4}$$
$$= 65.95$$

$$\therefore \quad h_i = Nu \times \frac{k}{d}$$
$$= 65.95 \times \frac{0.64}{0.02}$$
$$= 2110 \text{ W/m}^2\text{-deg}$$

The overall heat transfer coefficient,

$$\frac{1}{U} = \frac{1}{h_o} + \frac{d_o}{d_i} \times \frac{1}{h_i}$$
$$= \frac{1}{h_o} + \frac{1}{h_i} \text{ as } d_o = d_i$$
$$U = \frac{h_o h_i}{h_o + h_i}$$
$$= \frac{10000 \times 2110}{10000 + 2110}$$
$$= 1742.2 \text{ W/m}^2\text{-deg}$$

(b) From temperature profiles,

$$\theta_1 = t_{h1} - t_{c1} = 100 - 15 = 85$$
$$\theta_2 = t_{h2} - t_{c2} = 100 - t_{c2}$$
$$AMTD = \frac{\theta_1 + \theta_2}{2}$$
$$= \frac{(100 - t_{c2}) + 85}{2} = 92.5 - 0.5 \, t_{c2}$$

Heat exchange, Q

$$= m_c \, c_c \, (t_{c2} - t_{c1}) = UA \times AMTD$$

The heating surface area A equals $(N \times \pi \, d \, l)$ where d and l represent the tube diameter and length respectively, and N is the total number of tubes in the two passes. As such

$$10 \times 4180 \times (t_{c2} - 15)$$
$$= 1742.4 \times (200 \times \pi \times 0.02 \times 2) \times (92.5 - 0.5 \, t_{c2})$$
$$\text{or } t_{c2} - 15 = 1.047 \, (92.5 - 0.5 \, t_{c2})$$
$$= 96.847 - 0.5235 \, t_{c2}$$

$\therefore$ Temperature of water at outlet from the feed water heater

$$t_{c2} = \frac{96.847 + 15}{1.5235} = 73.41° \text{ C}$$

EXAMPLE 14.30.

The hot air at 135°C needed for a drying plant is obtained by passing 2.5 m³/s of atmospheric air at 1 bar pressure and 27°C temperature over tubes through which hot oil is circulated. The tubes have 2 cm bore, 1.5 mm thickness and are made of material having thermal conductivity 52.5 W/m-deg. The oil enters these tubes at 305°C and leaves at 210°C. Assuming that air is flowing in opposite direction to oil, calculate :

(a) overall heat transfer coefficient

(b) total heating surface, and

(c) number of tubes and number of passes if the overall length of the heater is restricted to 3.2 metres.

For air :

$$c_p = 1005 \text{ J/kg-deg}$$
$$R = 287 \text{ J/kg-deg}$$

and convective heat transfer coefficient from air to metal

$$h = 172.42 \text{ W/m}^2\text{-deg}$$

For oil :

$$c_p = 1885 \text{ J/kg-deg}$$
$$k = 0.129 \text{ W/m-deg}$$
$$\mu = 2.07 \times 10^{-3} \text{ kg/m-s}$$

and flow rate = 500 kg/s/m²

The heat transfer coefficient for oil to metal is governed by the relation

$$\frac{h \, d}{k} = 0.023 \, (Re)^{0.8} \, (Pr)^{0.3}$$

Solution : Flow velocity of oil

$$V = m/\rho A = 500/\rho$$
$$Re = \frac{V d \rho}{\mu} = \frac{(500/\rho) \times 0.02 \times \rho}{2.07 \times 10^{-3}}$$
$$= 4831$$
$$Pr = \frac{\mu c_p}{k} = \frac{2.07 \times 10^{-3} \times 1885}{0.129}$$
$$= 30.25$$
$$\therefore h = \frac{k}{d} \times 0.023 \times (Re)^{0.8} \, (Pr)^{0.3}$$
$$= \frac{0.129}{0.02} \times 0.023 \times (4831)^{0.8} \times (30.25)^{0.3}$$
$$= 365.4 \text{ W/m}^2\text{-deg}$$

For the tube :

$$r_1 = 1.0 \text{ cm ;}$$
$$r_2 = 1.0 + 0.15 = 1.15 \text{ cm}$$

Considering 1 m length of tube, the various thermal resistances to flow of heat are offered by

(i) inner oil film

$$= \frac{1}{h_i A_i} = \frac{1}{365.4 \times (2\pi \times 0.01 \times 1)}$$
$$= 0.04358$$

(ii) pipe material

$$= \frac{1}{2\pi k l} \log_e \frac{r_2}{r_1}$$
$$= \frac{1}{2\pi \times 0.115 \times 1} \log_e \frac{1.15}{1}$$
$$= 0.00042$$

(iii) outside air film

$$= \frac{1}{h_o A_o}$$
$$= \frac{1}{172.42 \times (2\pi \times 0.115 \times 1)}$$
$$= 0.0803$$

Total thermal resistance of the arrangement

$$\Sigma R_t = 0.04358 + 0.00042 + 0.083$$
$$= 0.1243 \text{ deg/W}$$

$$\text{Heat loss} = \frac{\Delta t}{\Sigma R_t}$$

The heat loss can also be expressed as

$$Q = U_o \, A_o \, \Delta t = U_i \, A_i \, \Delta t$$

where U_o and U_i are the overall coefficients of heat transfer based on the outside area A_o and inside are A_i respectively

$$\therefore \quad U_o = \frac{1}{A_o \Sigma R_t}$$
$$= \frac{1}{(2\pi \times 0.0115 \times 1) \times 0.1243}$$
$$= 111.40 \text{ W/m}^2\text{-deg}$$
$$\text{and} \quad U_i = \frac{1}{A_i \Sigma R_t}$$
$$= \frac{1}{(2\pi \times 0.01 \times 1) \times 0.1243}$$
$$= 128.1 \text{ W/m}^2\text{-deg}$$

(b) Mass flow rate of air m_a

$$= \frac{pV}{RT} = \frac{1 \times 10^5 \times 2.5}{287 \times (273 + 27)}$$
$$= 2.9 \text{ kg/s}$$

A heat balance over the exchanger gives :

heat lost by oil = heat gained by air

$$m_o \, c_o \, (t_{o1} - t_{o2}) = m_a \, c_a \, (t_{a2} - t_{a1})$$

The subscripts o and a refer to oil and air respectively. Inserting the appropriate values,

$$m_o \times 1885 \, (305 - 210)$$
$$= 2.9 \times 1005 \, (135 - 27) = 314766$$

$\therefore$ Total oil flow

$$= \frac{314766}{1885 \, (305 - 210)} = 1.758 \text{ kg/s}$$

Mass of oil flowing through each tube

$$= \frac{\pi}{4} (0.02)^2 \times 500 = 0.157 \text{ kg/s}$$

$\therefore$ Number of tubes, n

$$= \frac{1.758}{0.157} = 11.2 \text{ say 12 tubes}$$

The overall heat exchange is also given by

$$Q = U\,A\,\theta_m = U_i\,A_i\,\frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

The subscript i refers to conditions based on inside diameter. Reference Fig. 14.13. for the counter flow arrangement.

$$\theta_2 = 210 - 27 = 183°C$$
$$\theta_1 = 305 - 135 = 170°C$$

$$\therefore\ 314766 = 128.1\,A_i\,\frac{170 - 183}{\log_e \frac{170}{183}}$$

$$= 22596\,A_i$$

∴ Heating surface area, A

$$= \frac{314766}{22596} = 13.93\ m^2$$

The heating surface area also equals $n\,(\pi\,d_i\,l)$ where n is the number of tubes, d_i and l denote inside tube diameter and length respectively.

∴ Required length of tube

$$= \frac{13.93}{12 \times \pi \times 0.02} = 18.48\ m$$

Since the overall length of the heat is restricted to 3.2 metres,

Number of passes

$$= 18.48/3.2 = 5.77, \text{ say } 6$$

EXAMPLE 14.31.

In a single-pass counter flow heat exchanger, 1800 kg/hr of ethylene glycol is cooled from 100°C to 60°C by cooling water that enters the annular space of the heat exchanger at 15°C and has a mass flow rate of 1200 kg/hr Calculate (a) overall heat transfer coefficient if convective heat transfer coefficient of water side is 31400 kJ/m²-hr-deg, (b) necessary length of copper tubing if it has an internal diameter of 1.25 cm. Also calculate the length of the tube required if water flows in the same direction as ethylene glycol

For turbulent flow of fluid inside tube, the Dittus-Boelter relationship,

$$Nu = 0.023\,(Re)^{0.8}\,(Pr)^{0.4}$$

is to be used. The relevant physical properties of ethylene glycol at its bulk temperature, t_b

$$= (100 + 60)/2 = 80°C \text{ are :}$$
$$c_p = 2.65\ kJ/kgK$$
$$\mu = 11.72\ kg/hr\text{-}m$$
$$k = 0.942\ kJ/m\text{-}hr\text{-}deg$$

Solution : The mass flow rate of ethylene glycol is

$$m_g = \frac{\pi}{4}\,d^2\,V\rho$$

or $1800 = \frac{\pi}{4}\,(0.0125)^2 \times V\rho$

or $V\rho = 14.675 \times 10^6$

Reynold number,

$$Re = \frac{V\,d\,\rho}{\mu} = \frac{14.675 \times 10^6 \times 0.0125}{11.72}$$
$$= 15652$$

Prandtl number,

$$Pr = \frac{\mu\,c_p}{k} = \frac{11.72 \times 2.65}{0.942} = 32.97$$

From the given Dittus-Boelter relationship,

$$Nu = 0.023\,(15652)^{0.8}\,(32.97)^{0.4}$$
$$= 211.04$$

Since $Nu = hd/k$, the heat transfer coefficient on the ethylene glycol side is :

$$h_g = \frac{Nu \times k}{d} = \frac{211.04 \times 0.942}{0.0125}$$
$$= 15904\ kJ/m^2\text{-hr-deg}$$

If thermal resistance of the tube is negligible, then overall heat transfer coefficient is given by the following equation

$$\frac{1}{U} = \frac{1}{h_g} + \frac{1}{h_w}$$

where subscripts w and g refer to water and glycol respectively.

$$U = \frac{h_w \times h_g}{h_w + h_g} = \frac{31400 \times 15904}{31400 + 15904}$$
$$= 10557\ kJ/m^2\text{-hr-deg}$$

(*b*) The unknown exit temperature of the cooling water may be found from an energy balance on the two fluids, *i.e.*,

$$m_w\,c_w\,(t_{w2} - t_{w1}) = m_g\,c_g\,(t_{g1} - t_{g2})$$
$$1200 \times 4.186\,(t_{w2} - 15)$$
$$= 1800 \times 2.65\,(100 - 60)$$

∴ Water outlet temperature,

$$t_{w2} = \frac{1800 \times 2.65\,(100 - 60)}{1200 \times 4.186} + 15$$
$$= 52.98°C$$

From an energy balance on ethylene glycol, the heat transfer is given by ;

$$Q = m_g\,c_g\,(t_{g1} - t_{g2})$$
$$= 1800 \times 2.65\,(100 - 60)$$
$$= 190800\ kJ/hr$$

Reference Figure 14.13 for the counter flow arrangement, the log mean temperature difference is given by

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

where ; $\theta_1 = t_{g1} - t_{w2} = 100 - 52.98 = 47.02°C$

$\theta_2 = t_{g2} - t_{w1} = 60 - 15 = 45°C$

$$\theta_m = \frac{47.02 - 45.00}{\log_e \frac{47.02}{45.00}} = 46.22°C$$

Heat exchanger, $Q = U\,A\,\theta_m$

∴ Heating surface area, A

$$= \frac{Q}{U\,\theta_m} = \frac{190800}{10557 \times 46.22}$$
$$= 0.391\ m^2$$

The surface area also equals $\pi\,d\,l$ where d and l represent the tube diameter and length respectively.

∴ Required length of the tube

$$= \frac{0.391}{\pi \times 0.0125} = 9.936\ m$$

(*c*) For parallel flow arrangement (Figure 14.12)

$$\theta_1 = t_{g1} - t_{w1} = 100 - 15 = 85°C$$
$$\theta_2 = t_{g2} - t_{w2} = 60 - 52.98 = 7.02°C$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{85 - 7.02}{\log_e \frac{85}{7.02}} = 31.36°C$$

For the same heat exchange rate, the area required is inversely proportional to θ_m; obviously the length of tube required is inversely proportional to θ_m. Therefore the length required for parallel flow arrangement is

$$l_p = 9.936 \times \frac{46.22}{31.36} = 14.64\ m$$

EXAMPLE 14.32.

The lubricating oil for a large industrial gas turbine engine is cooled in a counter flow, concentric tube heat exchanger. The cooling water flows through the inner tube (d_i = 25 mm) with inlet temperature 25°C and mass flow rate 0.2 kg/s. The oil flows through the annulus (d_o = 50 mm) with mass flow rate 0.125 kg/s and its temperature at entry and exit are 90°C and 60° respectively. Neglecting tube wall thermal resistance, fouling factors and heat loss to surroundings, make calculations for outlet temperature of cooling water, overall heat transfer coefficient and length of the tube.

The relevant thermo-physical properties of engine oil and water are as given below :

For oil :

$$c = 2135\ J/kgK$$
$$\mu = 0.03\ Ns/m^2$$
and $$k = 0.14\ W/m\text{-deg}$$

For water :

$$c = 4180\ J/kgK$$
$$\mu = 725 \times 10^{-6}\ Ns/m^2$$
and $$k = 0.6\ W/m\text{-deg}$$

Solution : Refer Fig. 14.17. for the flow arrangement and temperature distribution of hot fluid (engine oil) and cold fluid (water) in a counter flow, concentric tube heat exchanger.

(*a*) From energy balance,

$$m_h\,c_h\,(t_{h1} - t_{h2}) = m_c\,c_c\,(t_{c2} - t_{c1})$$

The subscripts h and c refer to hot (coil) and cold (water) fluids respectively.

$$0.125 \times 2135 \times (90 - 60)$$
$$= 0.2 \times 4180\,(t_{c2} - 25)$$

∴ Output temperature of cooling water,

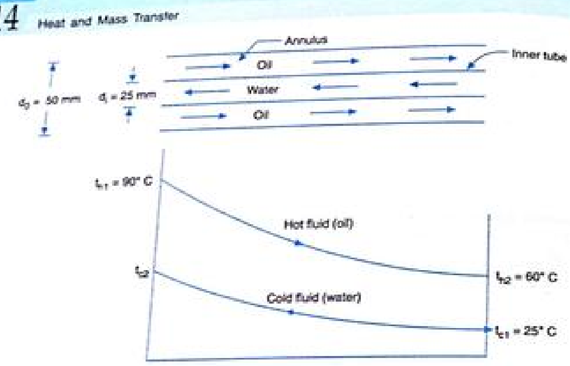


Fig. 14.17.

$$t_{c2} = \frac{0.125 \times 2135 \times 30}{0.2 \times 4180} + 25$$
$$= 34.58° \text{ C}$$

(b) Reynolds number for flow of water through the tube is

$$Re = \frac{4m}{\pi d \mu}$$
$$= \frac{4 \times 0.2}{\pi \times 0.025 \times 725 \times 10^{-6}} = 14057$$

$$Pr = \frac{\mu c_p}{k} = \frac{725 \times 10^{-6} \times 4180}{0.6} = 5.05$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies for calculating the heat transfer coefficient on the inside (water side).

$$Nu = \frac{h d_i}{K}$$
$$= 0.023\ (Re)^{0.8} \times (Pr)^{0.4}$$
$$= 0.023 \times (14057)^{0.8} \times (5.05)^{0.4}$$
$$= 91.475$$

$$\therefore \quad h_i = Nu \times \frac{k}{d_i}$$
$$= 91.475 \times \frac{0.6}{0.025}$$
$$= 2195 \text{ W/m}^2\text{-deg}$$

The oil flows through the annulus with hydraulic diameter,

$$d_h = 4\frac{A}{P}$$
$$= 4 \times \frac{\frac{\pi}{4}(d_0^2 - d_i^2)}{p(d_0 + d_i)}$$
$$= d_0 - d_i = 0.025 \text{ m}$$
$$Re = \frac{\rho V d_h}{\mu}$$
$$= \frac{\rho(d_o - d_i)}{\mu} \times \frac{m}{\rho \frac{\pi}{4}(d_0^2 - d_i^2)}$$
$$= \frac{4m}{\pi(d_0 + d_i)\mu}$$
$$= \frac{4 \times 0.125}{\pi(0.05 + 0.025) \times 0.03} = 7.77$$

Since $Re < 2300$, the flow is laminar. Further assuming constant wall temperature along the inner surface of the annulus, the

heat transfer coefficient on the outside (inner surface of the annulus) is

$$Nu = \frac{h_0 d_h}{k}$$
$$= 3.66 \text{ (fully developed flow)}$$
$$h_0 = 3.66 \times \frac{k}{d_h}$$
$$= 3.66 \times \frac{0.14}{0.025} = 20.5 \text{ W/m}^2\text{-deg}$$

Neglecting fouling effects and thermal resistance of the tube material, the overall heat transfer coefficient is

$$\frac{1}{U} = \frac{1}{h_i} + \frac{1}{h_0}$$
$$U = \frac{h_0 \times h_i}{h_0 + h_i}$$
$$= \frac{20.5 \times 2195}{20.5 + 2195} = 20.31 \text{ W/m}^2\text{-deg}$$

(c) The logarithmic mean temperature difference for counter flow arrangement

$$\theta_1 = t_{h2} - t_{c1} = 60 - 25 = 35° \text{ C}$$
$$\theta_2 = t_{h1} - t_{c2} = 90 - 34.58 = 55.42° \text{ C}$$
$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$
$$= \frac{35 - 55.42}{\log_e \frac{35}{55.42}} = 44.43° \text{ C}$$

Heat exchange, $Q = U\ A\ \theta_m$

$\therefore$ Heating surface area A

$$= \frac{Q}{U\theta_m} = \frac{m_h c_h (t_{h2} - t_{h1})}{U\theta_m}$$
$$= \frac{0.125 \times 2135(90 - 60)}{20.31 \times 44.43}$$
$$= 8.872 \text{ m}^2$$

The heating surface area also equal ($\pi\ d\ l$) where d and l represent the pipe diameter and length respectively.

$\therefore$ Length of pipe

$$= \frac{A}{\pi d} = \frac{8.872}{\pi \times 0.025} = 113 \text{ m}$$

EXAMPLE 14.35

0.5 kg/s of ethylene glycol flows through a thin walled copper tube of 1.25 cm diameter, and 0.35 kg/s of water flows in the opposite direction through the annular space formed by this tube and a tube of diameter 2 cm. The ethylene glycol, which enters at 100°C is required to leave at 60°C, while the water enters at 10°C. Calculate :

(a) heat transfer coefficient on ethylene glycol side,

(b) heat transfer coefficient on water side,

(c) overall heat transfer coefficient, and

(d) length of the tube required.

Use the correlation $N_u = 0.023\ R_e^{0.8}\ P_r^{0.3}$ and take the properties of water and ethylene glycol for the bulk temperature as listed below :

Property	Ethylene glycol at 80°C	Water at 27°C
ρ (kg/m³)	1075	995
μ (kg/ms)	3200×10^{-6}	850×10^{-6}
c_p (J/kgK)	2650	4180
k (W/mK)	0.26	0.615

Solution : Flow velocity for ethylene glycol,

$$V = \frac{m}{\rho A} = \frac{0.5}{1075 \times \frac{\pi}{4} \times (0.0125)^2}$$
$$= 3.792 \text{ m/s}$$

Reynolds number,

$$R_e = \frac{\rho V d}{\mu} = \frac{1075 \times 3.792 \times 0.0125}{3200 \times 10^{-6}}$$
$$= 15923$$

Prandtl number,

$$P_r = \frac{\mu c_p}{k} = \frac{3200 \times 10^{-6} \times 2650}{0.26}$$
$$= 32.16$$

From the given correlation,

$$N_u = \frac{hd}{k}$$
$$= 0.023 \times (15923)^{0.8} \times (32.16)^{0.3}$$
$$= 149.81$$

Heat transfer coefficient on ethylene glycol side (*i.e.*, inside the tubes)

$$h_1 = \frac{149.81 \times k}{d}$$

$$= \frac{149.81 \times 0.26}{0.0125} = 3116 \text{ W/m}^2 \text{ K}$$

(b) Flow velocity for water,

$$V = \frac{m}{\rho A}$$

$$= \frac{0.35}{995 \times \frac{\pi}{4} \times [0.02^2 - 0.0125]^2}$$

$$= 1.837 \text{ m/s}$$

Hydraulic diameter d

$$= \frac{4A}{\text{wetted perimeter}}$$

$$= \frac{4 \times \frac{\pi}{4}(d_o^2 - d_i^2)}{\pi(d_o + d_i)}$$

$$= d_o - d_i = 2 - 1.25 = 0.75 \text{ cm}$$

Reynold number :

$$R_e = \frac{\rho V d}{\mu}$$

$$= \frac{995 \times 1.837 \times 0.0075}{850 \times 10^{-6}} = 16128$$

Prandtl number :

$$P_r = \frac{\mu c_p}{k} = \frac{850 \times 10^{-6} \times 4180}{0.615}$$

$$= 5.78$$

From the given correlation,

$$N_u = \frac{hd}{k}$$

$$= 0.023 \times (16128)^{0.8} \times (5.78)^{0.3}$$

$$= 90.44$$

Heat transfer coefficient on water side (i.e., for annular space)

$$h_2 = \frac{90.44 \times k}{d}$$

$$= \frac{90.44 \times 0.615}{0.0075} = 7416 \text{ W/m}^2 \text{ K}$$

(c) Overall heat transfer coefficient

$$UA = \frac{1}{\frac{1}{h_1 A_1} + \frac{1}{h_2 A_2}}$$

$$= \frac{1}{\frac{1}{3116 \times \pi \times 0.0125 l} + \frac{1}{7416 \times \pi \times 0.0125 l}}$$

$$= \frac{1}{\frac{1}{122.31 l} + \frac{1}{291 l}}$$

Solution gives :

$$UA = 86.35\, l$$

(d) Heat transfer rate between hot fluid (ethylene glycol) and cold fluid (water) is

$$Q = m_h c_{ph} (T_{h1} - T_{h2})$$
$$= m_c c_{pc} (T_{c2} - T_{c1})$$
$$= 0.5 \times 2650 (100 - 6)$$
$$= 0.35 \times 4180 (T_{c2} - 10)$$
$$53000 = 1463 (T_{c2} - 10)$$

That gives :

$$T_{c2} = \frac{53000}{1463} + 10 = 46.23°\text{C}$$

Now, for a counter flow heat exchanger,

$$\theta_1 = T_{h1} - T_{c2} = 100 - 46.23 = 53.77$$
$$\theta_2 = T_{h2} - T_{c1} = 60 - 10 = 50$$

Then, logarithmic mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{53.7 - 50}{\log_e \frac{53.77}{50}} = 51.85$$

Heat exchange, $Q = UA\theta_m = 86.35\, l \times 51.85$

$\therefore$ Length of tube required, l

$$= \frac{53000}{86.35 \times 51.85} = 11.84 \text{ m}$$

EXAMPLE 14.34.

A chemical industry operates continuously and produces 2×10^5 kg of sulphuric acid per day which needs to be cooled from 60°C to 40°C in a counter flow double pipe heat exchanger. The acid flows through the inner pipe, while water employed as cooling medium flows through the annulus with temperature 15°C at entry and 20°C at exit. The inner diameters for the inner and outer pipe are 70 mm and 120 mm respectively, and each pipe is 5 mm thick. Presuming that thermal conductivity of inner pipe material is 48 W/m K, make calculations for the mass flow rate of water and length of the heat exchanger.

The thermo-physical properties of water and sulphuric acid at the mean bulk temperature are given below :

Property	Water	Acid
Density (kg/m³)	998	1800
Sp. heat (J/kgK)	4187	1465
Thermal conductivity (W/mK)	0.60	0.30
Kinematic viscosity (m²/s)	1.0×10^{-6}	6.8×10^{-6}
Fouling factor (m²K/W)	—	0.0002

Solution : The mass flow rate of water can be determined from an energy balance on the two fluids. That is

heat gained by water
= heat lost by acid

$$m_c c_c (t_{c2} - t_{c1}) = m_h c_h (t_{h1} - t_{h2})$$

$$m_c \times 4187 \times (20 - 15)$$

$$= \frac{2 \times 10^5}{24 \times 3600} \times 1465 \times (60 - 40)$$

$$= 67824 \text{ J/s}$$

$$\therefore \quad m_c = \frac{67824}{4187 \times 5}$$

$$= 3.24 \text{ kg/s} = 11664 \text{ kg/hr}$$

(b) For counter flow arrangement

$$\theta_1 = t_{h1} - t_{c2} = 60 - 20 = 40°\text{C}$$
$$\theta_2 = t_{h2} - t_{c1} = 40 - 15 = 25°\text{C}$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{40 - 25}{\log_e \frac{40}{25}} = 31.9° \text{ C}$$

For the given configuration, the overall heat transfer coefficient is prescribed by the relation

$$\frac{1}{U} = \frac{d_o}{d_i h_i} + \frac{d_o}{2k}\log_e\left(\frac{d_o}{d_i}\right) + F + \frac{1}{h_o} \quad \text{...(i)}$$

The heat transfer coefficients h_i for inner side (for acid side) and h_o for outer side of pipe (for water side) are worked out as follows

For inner side of pipe :

$$Re = \frac{4m_h}{\pi d_i \mu} = \frac{4m_h}{\pi d_i \rho v}$$

$$= \frac{4 \times (2 \times 10^5 / 24 \times 3600)}{\pi \times 0.07 \times (1800 \times 6.8 \times 10^{-6})}$$

$$= 3441$$

$$Pr = \frac{\mu c_p}{k} = \frac{\rho v c_p}{k}$$

$$= \frac{1800 \times 6.8 \times 10^{-6} \times 1465}{0.3}$$

$$= 59.77$$

Since $Re > 2300$, the flow is turbulent and the following correlation applies for cooling

$$Nu = \frac{h_i d_i}{k} = 0.023\, (Re)^{0.8}\, (Pr)^{0.3}$$

...(Dittus-Boelter equation)

$$= 0.023\, (3441)^{0.8}\, (59.77)^{0.3} = 52.97$$

$$\therefore \quad h_i = 79.34 \times \frac{k}{d_i}$$

$$= 52.97 \times \frac{0.3}{0.07} = 227 \text{ W/m}^2\text{-deg}$$

For outer side of pipe :

Hydraulic diameter for annulus, d_h

$$= \frac{4A_c}{P}$$

$$= \frac{4 \times \frac{\pi}{4}(D_i^2 - d_o^2)}{\pi(D_i - d_o)}$$

$$= D_i + d_o = 0.12 + 0.08 = 0.2 \text{ m}$$

$$Re = \frac{4 m_c}{\pi d_h \rho v}$$

$$= \frac{4 \times 3.24}{\pi \times 0.2 \times 998 \times 1 \times 10^{-6}}$$

$$= 20678$$

$$Pr = \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k}$$

$$= \frac{998 \times 1 \times 10^{-6} \times 4187}{0.6} = 6.96$$

Since $Re > 2300$, the flow is turbulent and accordingly Dittus Boeltier correlation for heating applies

$$Nu = \frac{h_0 d_h}{k}$$

$$= 0.023 (Re)^{0.8} (Pr)^{0.4}$$

$$= 0.023\ (20678)^{0.8} \times (6.96)^{0.4}$$

$$= 141.63$$

$$\therefore \quad h_0 = 141.63 \times \frac{k}{d_h}$$

$$= 141.63 \times \frac{0.6}{0.2} = 425 \text{ W/m}^2\text{-deg}$$

Substituting the relevant values in expression (*i*), we get

$$\frac{1}{U} = \frac{0.08}{0.07 \times 227} + \frac{0.08}{2 \times 48} \log_e\left(\frac{0.08}{0.07}\right) + 0.0002 + \frac{1}{425}$$

$$= 0.00503 + 0.00011 + 0.0002 + 0.00235$$

$$= 0.00769$$

$$\therefore \quad U = 130.04 \text{ W/m}^2\text{-deg}$$

The length of the heat exchanger can then be calculated as

$$Q = U A \theta_m = U (\pi d_o l) \times \theta_m$$

$$67824 = 130.04 \times (\pi \times 0.08 \times l) \times 31.91 ;$$

$$l = 65.07 \text{ m}$$

Comments : The heat exchanger has a long length. It would be appropriate to replace the double pipe heat exchanger by a shell and tube type heat exchanger.

14.5. CORRECTION FACTORS FOR MULTI-PASS ARRANGEMENTS

The relation $\theta_m = (\theta_1 - \theta_2)/\log_e (\theta_1/\theta_2)$ for log-mean temperature difference is essentially applicable for the single-pass heat exchangers. The effect of multi tubes, several shell passes or cross flow in an actual flow arrangement is considered by identifying a correction factor F such that

$$Q = F U A \theta_m \quad \text{...(14.27)}$$

The logarithmic mean temperature difference θ_m corresponds to that for a counter flow double pipe exchanger with the same fluid inlet and outlet temperatures as in the actual complex design. Correction factors for several common arrangements have been given in Figs. 14.18 to 14.21. The data is presented as a function of two non-dimensional temperature ratios P and Z defined in term of the temperatures of the hot and cold fluids. The parameter P is the ratio of the rise in temperature of the cold fluid to the difference in the inlet temperatures of the two fluids

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} \quad \text{...(14.28)}$$

The parameter Z defines the ratio of the temperature drop of the hot fluid to temperature rise in the cold fluid.

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} \quad \text{...(14.29)}$$

For these plots, it is immaterial whether the hot fluid flows through the shell or the tubes. Since no arrangement can be more effective than the conventional counter flow, the correction factor F is always less than unity. Its value is an indication of the performance level of a given arrangement for the given terminal fluid temperatures.

EXAMPLE 14.35.

Calculate the surface area required for a heat exchanger which is required to cool 3600 kg/hr of benzene (c_p = 1.74 kJ/kgK) from 75°C to 45°C. The cooling water (c_p = 4.18 kJ/kgK) at 15°C has a

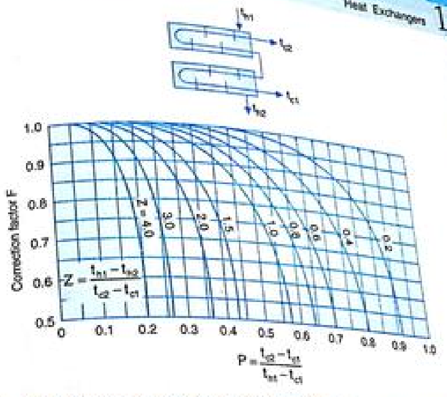


Fig. 14.18. Correction-factor plot for exchanger with one shell pass and two, four or any multiple of tube passes

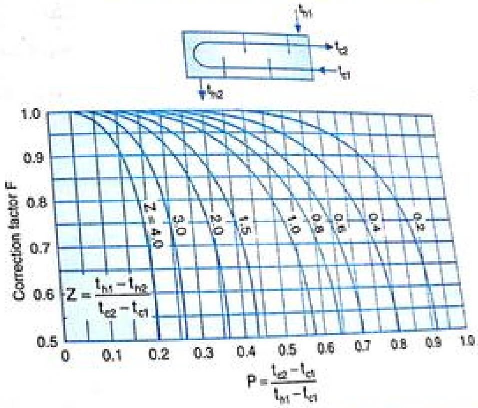


Fig. 14.19. Correction factor plot for exchanger with two shell passes and four, eight or any multiple of tube passes

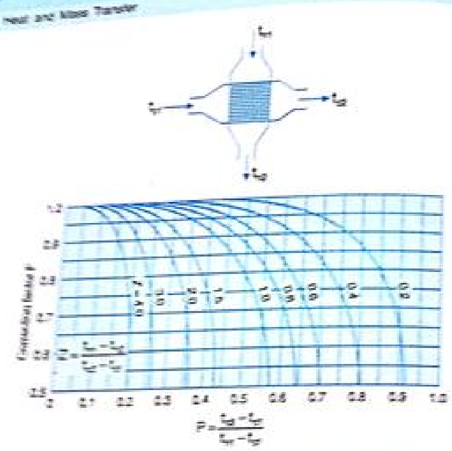


Fig. 14.20. Correction factor plot for single pass cross-flow heat exchanger with both fluids unmixed

flow rate of 2500 kg/hr. Consider the following arrangements :

(i) Single pass counter flow

(ii) 1-4 exchanger: one shell pass and four tube passes

(iii) Cross flow single pass with water mixed and benzene unmixed.

The overall heat transfer coefficient for each configuration is approximated to be 0.3 kW/m²K.

Solution: The unknown temperature t_{c2} of the cold fluid (water) can be obtained from energy balance on both the fluid.

$$m_c c_c (t_{c2} - t_{c1}) = m_h c_h (t_{h1} - t_{h2})$$

$$\therefore \quad t_{c2} = t_{c1} + \frac{m_h c_h}{m_c c_c}(t_{h1} - t_{h2})$$

$$= 15 + \frac{3600 \times 1.74}{2500 \times 4.18}(75 - 45)$$

$$= 32.98°C$$

An energy balance on the hot fluid (benzene) yields the total heat transfer.

$$Q = m_h c_h (t_{h1} - t_{h2})$$

$$= 3600 \times 1.74\,(75 - 45)$$

$$= 187920 \text{ kJ/hr} = 52.2 \text{ kJ/s (kW)}$$

(i) For the single pass counter flow arrangement, the log-mean temperature difference is

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}}$$

where, $\theta_1 = t_{h1} - t_{c2} = 75 - 32.98 = 42.02°C$

$\theta_2 = t_{h2} - t_{c1} = 45 - 15 = 30°C$

$$\theta_m = \frac{42.02 - 30}{\log_e \dfrac{42.02}{30}} = 35.68°C$$

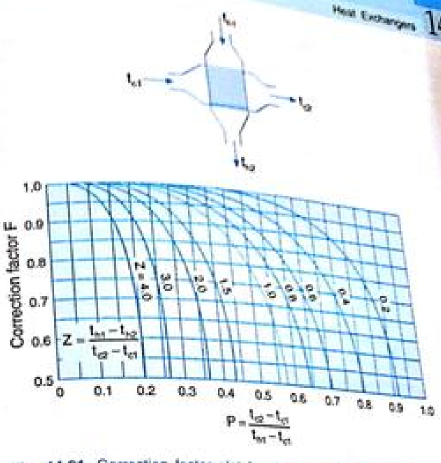


Fig. 14.21. Correction factor plot for single-pass flow heat exchanger, one fluid mixed and the other unmixed

$\therefore$ Required heat exchange area, A

$$= \frac{Q}{U\theta_m} = \frac{52.2}{0.3 \times 35.68} = 4.87 \text{ m}^2$$

(ii) The parameters required for the estimation of correction factor F are :

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{32.98 - 15}{75 - 15} = 0.2996$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{75 - 45}{32.98 - 15} = 1.668$$

Using these parameters with Fig. 14.17, the correction factor F is read as F = 0.92

$\therefore$ Required heat exchange area,

$$A = \frac{Q}{FU\theta_m}$$

$$= \frac{52.2}{0.92 \times 0.3 \times 35.68} = 5.29 \text{ m}^2$$

(iii) Using the parameters P = 0.2996 and Z = 1.7668 with Fig. 14.20, the correction factor F is read as F = 0.94

$\therefore$ Required heat exchange area,

$$A = \frac{Q}{FU\theta_m}$$

$$= \frac{52.2}{0.94 \times 0.3 \times 35.68} = 5.18$$

EXAMPLE 14.36.

A heat exchanger with 2 shell passes and ... passes is used to cool oil (c_p = 3.55 kJ/kg ... 125°C to 50°C, flowing at the rate of 2.5 ... cooling water (c_p = 4.18 kJ/kg K) enters ... at 20°C with a flow rate of 3 kg/s and ... heat transfer coefficient for the exchanger ... estimated at 115 W/m²K. Calcualte (a) ... through the exchanger (b) correction f...

multi-arrangement and (c) heat transfer area to accomplish the specified energy transfer.

Solution : From an energy balance on hot fluid, the heat transfer is given by :

$$Q = m_h\, c_h\, (t_{h1} - t_{h2})$$
$$= 2.5 \times 3.55 \times (125 - 50)$$
$$= 665.62 \text{ kJ/s}$$

(b) The unknown exit temperature of the cold fluid (water) may be found from an energy balance on the two fluids, *i.e.,*

$$m_c\, c_c\, (t_{c2} - t_{c1}) = m_h\, c_h\, (t_{h1} - t_{h2})$$

$$t_{c2} = t_{c1} + \frac{m_h c_h}{m_c c_c}(t_{h1} - t_{h2})$$

$$= 20 + \frac{2.5 \times 3.55}{3 \times 4.18}\,(125 - 50)$$

$$= 73.08°C$$

The parameters required to get the correction factor are :

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{73.08 - 20}{125 - 20} = 0.505$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{125 - 50}{73.08 - 20} = 1.41$$

From Fig. 14.18, $F = 0.88$

(c) For the conventional counter flow arrangement

$$\theta_1 = t_{h1} - t_{c2} = 125 - 73.08 = 51.92°C$$
$$\theta_2 = t_{h2} - t_{c1} = 50 - 20 = 30°C$$

$$\therefore \quad \theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{51.92 - 30}{\log_e \frac{51.92}{30}} = 40°C$$

The heat exchange, $Q = F\,U\,A\,\theta_m$

$\therefore$ Heat surface area,

$$A = \frac{Q}{F U \theta_m}$$

$$= \frac{665.62 \times 10^3}{0.88 \times 115 \times 40} = 164.43 \text{ m}^2$$

EXAMPLE 14.37.

A shell-tube heat exchanger is to be designed for heating 9000 kg/hr of water from 15°C to 85°C by hot engine oil (c_p = 2.35 kJ/kg K) flowing through the shell of the heat exchanger. The oil makes a single pass, entering at 150°C and leaving at 95°C with an average heat transfer coefficient of 400 W/m²K. The water flows through 10 thin-walled tubes of 25 mm diameter with each tube making 8-passes through the shell. Calculate the length of tube required for the heat exchanger to accomplish the specified water heating. The heat transfer coefficient on the water side is 3000 W/m²K.

Solution : If thermal resistance of the tube is negligible, then the overall heat transfer coefficient is given by :

$$\frac{1}{U} = \frac{1}{h_i} + \frac{1}{h_o}$$

$$\text{or} \quad U = \frac{h_i \times h_o}{h_i + h_o}$$

$$= \frac{3000 \times 400}{3000 + 400} = 352.94 \text{ W/m}^2\text{K}$$

The parameters required to get the correction factor are :

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{85 - 15}{150 - 15} = 0.158$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{150 - 95}{85 - 15} = 0.718$$

From Fig. 14.17, $F = 0.91$

For the conventional counter flow arrangement;

$$\theta_1 = t_{h1} - t_{c2} = 150 - 85 = 65°C$$
$$\theta_2 = t_{h2} - t_{c1} = 95 - 15 = 80°C$$

$$\therefore \quad \theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{65 - 85}{\log_e \frac{\theta_1}{\theta_2}} = 74.62°C$$

From an energy balance on the cold fluid (water), the required heat transfer is given by,

$$Q = m_c\, c_c\, (t_{c1} - t_{c2})$$

$$= 9000 \times 4.18 \times (85 - 15)$$
$$= 2633400 \text{ kJ/hr}$$
$$= 7.315 \times 103 \text{ J/s}$$

Also, $Q = F\,U\,A\,\theta_m$

$\therefore$ Heat surface area,

$$A = \frac{Q}{F U \theta_m} = \frac{7.315 \times 10^5}{0.91 \times 352.94 \times 74.62}$$
$$= 30.52 \text{ m}^2$$

The heating surface area also equals $n \pi d l$ where n is the number of tubes, d and l denote the tube diameter and length respectively

$\therefore$ Required length of the tube

$$= \frac{30.52}{10 \times \pi \times 0.025} = 38.88 \text{ m}$$

With eight passes, the shell length is

$$= 38.88/8 = 4.86 \text{ m}$$

EXAMPLE 14.38.

A steam power plant of large capacity incorporates a shell and tube type heat exchanger having 30000 thin wall tubes of 25 mm diameter. The steam condenses on the outside surface of these tubes with convection coefficient 10 kW/m²-deg. Water serves as the coolant entering the tubes at 20°C at mass flow rate of 30 × 10³ kg/s. If the condenser (heat exchanger) arrangement involves one shell pass and two passes and the heat transfer rate is 2000 MW, determine

(a) temperature of cooling water at exit from the condenser ;

(b) overall heat transfer coefficient for heat exchanger ;

(c) heat transfer area and length of tube per pass.

Assume the following data :

$$c_p = 4186 \text{ J/kgK}$$
$$\mu = 860 \times 10^{-6} \text{ Ns/m}^2$$
$$k = 0.65 \text{ W/m-deg}$$
$$Pr = 5.82$$

Solution : The heat transfer rate through the heat exchanger (condenser) is,

$$Q = m_c\, c_p\, (t_{c2} - t_{c1})\;;$$
$$2000 \times 10^6 = 30 \times 10^3 \times 4186\,(t_{c2} - 20)$$

$\therefore$ Temperature of cooling water coming out of condenser,

$$t_{c2} = \frac{2000 \times 10^6}{30 \times 10^3 \times 4186} + 20$$
$$= 35.93° \text{ C}$$

(b) Mass flow rate of water per tube

$$= \frac{30 \times 10^3}{30000} = 1 \text{ kg/s}$$

Reynolds number Re

$$= \frac{4m}{\pi d \mu} = \frac{4 \times 1}{\pi \times 0.025 \times 860 \times 10^{-6}}$$
$$= 59250$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies for calculating the heat transfer coefficient on the water side.

$$Nu = 0.023\,(Re)^{0.8}\,(Pr)^{0.4}$$
$$= 0.023\,(59250)^{0.8} \times (5.82)^{0.4}$$
$$= 306.09$$

$$h_i = Nu \times \frac{k}{d} = 306.9 \times \frac{0.615}{0.025}$$
$$= 530 \text{ W/m}^2\text{-deg}$$

The overall heat transfer coefficient is given by

$$\frac{1}{U} = \frac{1}{h_o} + \frac{d_o}{d_i} \times \frac{1}{h_i}$$

$$= \frac{1}{h_o} + \frac{1}{h_i} \text{ as } d_o = d_i$$

$$U = \frac{h_o h_i}{h_o + h_i} = \frac{10000 \times 7530}{10000 + 7530}$$

$$= 4295.5 \text{ W/m}^2\text{-deg}$$

(c) The logarithmic mean temperature difference for counter flow arrangement

$$\theta_1 = t_{h1} - t_{c2} = 50 - 35.93 = 14.07° \text{ C}$$
$$\theta_2 = t_{h2} - t_{c1} = 50 - 20 = 30°C$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{14.07 - 30}{\log_e \frac{14.07}{30}} = 21.04° \text{ C}$$

The parameters required to obtain the correction factor F are

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{35.93 - 20}{50 - 20} = 0.53$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{50 - 50}{35.93 - 20} = 0$$

Using these parameters with Fig. 14.17, the corrector factor is read as $F = 1.0$

Heat exchange, $Q = F\, U\, A\, \theta_m$

$\therefore$ Heating surface area A

$$= \frac{Q}{F U \theta_m} = \frac{2000 \times 10^6}{1 \times 4295.5 \times 21.04}$$
$$= 22129 \text{ m}^2$$

The heating surface area also equals ($N \times \pi\, d\, l$) where d and l represent the tube diameter and length respectively, and N is the number of tubes.

$\therefore$ Total length of tube

$$= \frac{A}{N \times \pi d} = \frac{22129}{30000 \times \pi \times 0.025}$$
$$= 9.397 \text{ m}$$

Since there are two tube passes,

$$\text{Tube length per pass} = \frac{0.397}{2} = 4.698 \text{ m}$$

EXAMPLE 14.39.

A steam condenser employed in a steam power plant is to handle 36000 kg/hr of dry and saturated steam at 50°C. The outer and inner diameters of the tubes are 25 mm and 22.5 mm respectively. The cooling water enters the tubes at 15° C, flows with an average velocity of 2 m/s and leaves at 25°C. If the heat transfer coefficient on the steam side is 5000 W/m²-deg, make calculations for the followings :

(a) mass flow rate of water and number of tubes required for water flow ;

(b) overall heat transfer coefficient and heat transfer surface area ;

(c) number of tube passes if the length of each tube per pass is not to exceed 2.5 m.

Ignore thermal resistance of wall material and assume that the steam loses only its latent heat which is equal to 2375 kJ/kg. The thermo-physical properties of water at the mean bulk temperature 20° C are :

$$\rho = 998.8 \text{ kg/m}^3$$
$$c_p = 4186 \text{ J/kgK}$$
$$\nu = 1 \times 10^{-6} \text{ m}^2\text{/s}$$
$$k = 0.6 \text{ W/m-deg}$$

Solution : From energy balance,

heat gained by water

= heat loss by steam

$$m_c\, c_p\, (t_{c2} - t_{c1}) = m_f \times h_{fg}$$
$$m_c \times 4.816\ (25 - 15)$$
$$= \frac{36000}{3600} \times (2375 \times 10^3)$$
$$= 2375 \times 10^4$$

$\therefore$ Mass flow rate of cooling water,

$$m_c = \frac{2375 \times 10^4}{4186 \times 10} = 567.36 \text{ kg/s}$$

Mass flow rate through each tube, m

$$= \rho A V = 998.8 \times \frac{\pi}{4}(0.0225)^2 \times 2$$
$$= 0.7938 \text{ m}^2$$

$$\therefore \text{Number of tubes} = \frac{567.36}{0.7938} = 715$$

(b) Reynolds number, Re

$$= \frac{4m}{\pi d_i \mu} = \frac{4m}{\pi d_i \rho \nu}$$
$$= \frac{4 \times 0.7938}{\pi \times 0.0225 \times 999.8 \times 10^{-6}}$$
$$= 44997$$

Prandtl number, Pr

$$= \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k}$$
$$= \frac{998.9 \times 10^{-6} \times 4186}{0.6} = 6.97$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies for calculating the convective heat transfer coefficient on the water side.

$$Nu = \frac{h_i d_i}{k}$$
$$= 0.023\ (Re)^{0.8}\ (Pr)^{0.4}$$

$$= 0.023\ (44997)^{0.8} \times (6.97)^{0.4}$$
$$= 263.96$$

$$h_i = Nu \frac{k}{d_i} = 263.96 \times \frac{0.6}{0.0225}$$
$$= 7039 \text{ W/m}^2\text{-deg}$$

The overall heat transfer coefficient based on outside surface is

$$\frac{1}{U_s} = \frac{d_0}{d_i} \times \frac{1}{h_i} + \frac{1}{h_0}$$
$$= \frac{0.025}{0.0225} \times \frac{1}{7039} + \frac{1}{5000}$$
$$= 0.0003578$$

$$U_s = \frac{1}{0.0003578} = 2795 \text{ W/m}^2\text{-deg}$$

The logarithmic mean temperature difference for counter flow arrangement,

$$\theta_1 = t_{h2} - t_{c1} = 50 - 15 = 35° \text{ C}$$
$$\theta_2 = t_{h1} - t_{c2} = 50 - 25 = 25° \text{ C}$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{35 - 25}{\log_e \frac{35}{25}}$$
$$= 29.72°\text{C}$$

The parameters required to obtain the correction factor F are :

$$F = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{25 - 15}{50 - 15} = 0.2857$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{50 - 50}{25 - 15} = 0$$

Using these parameters with Fig. 14.17, the correction factor is read as $F = 1.0$.

Heat exchange, $Q = F\, U_0\, A_0\, \theta_m$

$\therefore$ Heating surface area, A_0

$$= \frac{Q}{F U_0 \theta_m}$$
$$= \frac{2375 \times 10^4}{1 \times 2795 \times 29.72} = 285.95 \text{ m}^2$$

(c) The heating surface area A_0 also equals ($N \times \pi\, d_0\, l$) where d_0 and l represent the outside tube diameter and length respectively, and N is the number of tubes.

$\therefore$ length of tube

$$= \frac{A_0}{N \times \pi d_0}$$
$$= \frac{285.91}{715 \times \pi \times 0.025} = 5.095 \text{ m}$$

Since the length of tube is limited to 2.5 m per pass,

$$\text{Number of tube passes} = \frac{5.095}{2.5} = 2$$

14.6. EFFECTIVENESS AND NUMBER OF TRANSFER UNITS (NTU)

The concept of log-mean temperature difference for estimating/analysing the performance of a heat exchanger unit is quite useful only when the inlet and outlet temperatures of the fluids are either known or can be easily determined from the relevant data. The usual design is however, based on known fluid inlet temperature and estimated heat transfer coefficients. The unknown parameters may be the outlet conditions and heat transfer or the surface area required for a specified heat transfer. From energy balance, the heat given up by the hot fluid is picked up

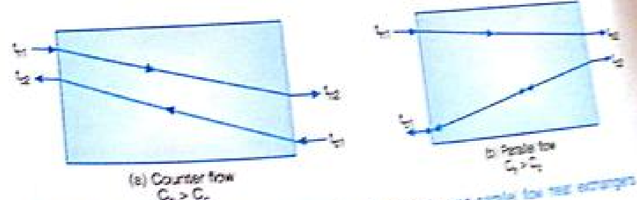


Fig. 14.22. Typical temperature profiles for counter-flow and parallel flow heat exchangers

by the coolant on being transferred through the heat exchanger. Thus

$$Q = m_c c_c (t_{c2} - t_{c1})$$
$$= m_h c_h (t_{h1} - t_{h2}) = U A \theta_m$$

where $$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{(t_{h1} - t_{c2}) - (t_{h2} - t_{c1})}{\log_e \left[\frac{t_{h1} - t_{c2}}{t_{h2} - t_{c1}}\right]}$$

The outlet conditions for each fluid can be worked out by eliminating Q between the above equations. The resulting equation however becomes unwieldly requiring a trial-and-error iteration approach owing to the logarithmic function in θ_m.

An estimate analysis of the heat exchanger can be made more conveniently by the NTU approach which is based on the concept of capacity ratio, effectiveness and number of transfer units. The approach facilitates the comparison between the various types of heat exchangers which may be used for a particular application.

Capacity ratio (C) : The product $m\,c$ (mass × specific heat) of a fluid flowing in a heat exchanger is repeatedly encountered and is termed as the *capacity rate*. It indicates the capacity of the fluid to store energy at a given rate.

Capacity rate of the hot fluid, $C_h = m_h c_h$

Capacity rate of the cold fluid, $C_c = m_c c_c$

The *capacity ratio* C is defined as the ratio of the minimum to maximum capacity rate. In parallel or counter flow heat exchangers, the hot or cold fluid may have the minimum value.

$$\text{if } m_h c_h > m_c c_c : C = \frac{m_c c_c}{m_h c_h} \qquad \text{...(14.30)}$$

$$\text{if } m_h c_h < m_c c_c : C = \frac{m_h c_h}{m_c c_c} \qquad \text{...(14.31)}$$

The temperature distribution along the exchanger length would be different depending on capacity ratio. The relative temperature change of the two fluids is inversely related to their capacity rates, the one with a smaller value of capacity rate experiencing the greater change in temperature. When $m_h c_h > m_c c_c$ in the counter-flow arrangement, the temperature tend to converge at the inlet. However, the temperature would be diverging when $m_c c_c > m_h c_h$. Further, in a parallel flow arrangement, t_{c2} approaches t_{h2} for an infinitely long heat exchanger ; consequently the counter-flow configuration is more effective in its operation.

Heat exchanger effectiveness (ϵ) : The effectiveness of a heat exchanger is defined as the ratio of the energy actually transferred to the maximum theoretical energy transfer.

$$\epsilon = \frac{Q_{act}}{Q_{max}}$$

$$= \frac{\text{actual heat transfer}}{\text{maximum possible heat transfer}} \qquad \text{...(14.32)}$$

The actual energy transfer is given by the product of the capacity rate and the temperature difference for either fluid, *i.e.*,

$$Q_{act} = m_h c_h (t_{h1} - t_{h2})$$
$$= m_c c_c (t_{c2} - t_{c1})$$

A maximum possible heat transfer (Q_{max}) rate is achieved if a fluid undergoes temperature change equal to he maximum temperature difference available. Both for the parallel and counter-flow exchanger, the maximum temperature difference equals the inlet temperature of hot fluid minus the inlet temperature of the cold fluid, *i.e.*,

maximum available temperature difference

$$= (t_{h1} - t_{c1})$$

Further, the maximum possible heat transfer occurs when the fluid of a small heat capacity rate undergoes the maximum temperature difference available. The maximum possible energy transfer, therefore, becomes :

$$Q_{max} = C_{min} (t_{h1} - t_{c1}) \qquad \text{...(14.33)}$$

The effectiveness of heat exchanger is then

$$\epsilon = \frac{m_h c_h (t_{h1} - t_{h2})}{C_{min} (t_{h1} - t_{c1})}$$

$$= \frac{C_h (t_{h1} - t_{h2})}{C_{min} (t_{h1} - t_{c1})}$$

$$= \frac{m_c c_c (t_{c2} - t_{c1})}{C_{min} (t_{h1} - t_{c1})}$$

$$= \frac{C_c (t_{c2} - t_{c1})}{C_{min} (t_{h1} - t_{c1})} \qquad \text{...(14.34)}$$

Since either the hot or cold fluid may have the minimum value of capacity heat rate, there are two possible values of effectiveness

$$C_h < C_c : \epsilon_h = \frac{(t_{h1} - t_{h2})}{(t_{h1} - t_{c1})}$$

$$C_c < C_h : \epsilon_c = \frac{(t_{c2} - t_{c1})}{(t_{h1} - t_{c1})} \qquad \text{...(14.35)}$$

The subscript on ϵ designates the fluid which has the minimum heat capacity rate. Apparently, the effectiveness is simply a ratio of the temperature change of the fluid with the smaller heat capacity to the maximum temperature difference available in the heat exchanger.

Number of transfer units (NTU) : The number of transfer units (NTU) is a measure of the size of heat exchanger; it provides some indication of the size of the heat exchanger. It is defined as :

$$NTU = \frac{UA}{m_c c_c} \qquad \text{when } m_h c_h > m_c c_c$$

$$NTU = \frac{UA}{m_h c_h} \qquad \text{when } m_h c_h < m_c c_c \qquad \text{...(14.36)}$$

The denominator is always the smaller thermal capacity rate and therefore

$$NTU = \frac{UA}{(mc)_{min}} = \frac{UA}{C_{min}} \qquad \text{...(14.37)}$$

14.6.1. Effectiveness for the Parallel Flow Heat Exchanger

The i.eat exchange through an incremental area dA (Fig. 14.23) of the exchanger is

$$dQ = U\,dA\,(t_h - t_c)$$
$$= -m_h c_h\,dt_h = m_c c_c\,dt_c$$
$$= -C_h\,dt_h = C_c\,dt_c$$

From expression (ii)

$$dt_h = \frac{-dQ}{C_h} \quad \text{and} \quad dt_c = \frac{dQ}{C_c}$$

$$\therefore\ d(t_h - t_c) = -dQ \left[\frac{1}{C_h} + \frac{1}{C_c}\right]$$

Substitution of the value of dQ from expression (i) and subsequent rearrangement gives

$$\frac{d(t_h - t_c)}{(t_h - t_c)} = -U\,dA \left[\frac{1}{C_h} + \frac{1}{C_c}\right]$$

Through integration, we obtain

$$\int \frac{d(t_h - t_c)}{(t_h - t_c)} = -U\left[\frac{1}{C_h} + \frac{1}{C_c}\right] \int dA$$

The integration limits of the temperature difference are from $(t_{h1} - t_{c1})$ to $(t_{h2} - t_{c2})$ and those of area from 0 to A

$$\therefore\ \log_e \frac{(t_{h2} - t_{c2})}{(t_{h1} - t_{c1})} = -UA \left[\frac{1}{C_h} + \frac{1}{C_c}\right] \qquad \text{...(14.38)}$$

From the definition of effectiveness,

$$\epsilon = \frac{m_h c_h (t_{h1} - t_{h2})}{C_{min} (t_{h1} - t_{c1})}$$

$$= \frac{C_h (t_{h1} - t_{h2})}{C_{min} (t_{h1} - t_{c1})}$$

$$= \frac{m_c c_c (t_{c2} - t_{c1})}{C_{min} (t_{h1} - t_{c1})}$$

$$= \frac{C_c (t_{c2} - t_{c1})}{C_{min} (t_{h1} - t_{c1})}$$

the values of outlet temperature t_{h2} and t_{c2} are worked out as :

$$t_{h2} = t_{h1} - \frac{C_{min}}{C_h} (t_{h1} - t_{c1})\,\epsilon$$

$$\text{and} \quad t_{c2} = t_{c1} + \frac{C_{min}}{C_c} (t_{h1} - t_{c1})\,\epsilon$$

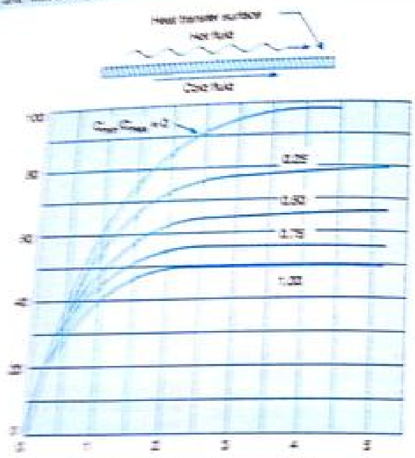


Fig. 14.23. Effectiveness for parallel flow heat exchanger

$$(t_{h2}-t_{c2}) = (t_{h1}-t_{c1}) - C_{min}(t_{h1}-t_{c1})\,\varepsilon\left[\frac{1}{C_h}+\frac{1}{C_c}\right]$$

$$= (t_{h1}-t_{c1})\left[1-\varepsilon\, C_{min}\left(\frac{1}{C_h}+\frac{1}{C_c}\right)\right]$$

Substituting this value in equation 14.38, we obtain

$$\ln\left[1-\varepsilon\, C_{min}\left(\frac{1}{C_h}+\frac{1}{C_c}\right)\right] = -UA\left(\frac{1}{C_h}+\frac{1}{C_c}\right)$$

or $1-\varepsilon\, C_{min}\left(\frac{1}{C_h}+\frac{1}{C_c}\right) = \exp\left[-UA\left(\frac{1}{C_h}+\frac{1}{C_c}\right)\right]$

or $$\varepsilon = \frac{1-\exp\left[-UA\left(\frac{1}{C_h}+\frac{1}{C_c}\right)\right]}{C_{min}\left(\frac{1}{C_h}+\frac{1}{C_c}\right)}$$

$$= \frac{1-\exp\left[-UA/C_h\{1+C_h/C_c\}\right]}{C_{min}/C_h\{1+C_h/C_c\}} \quad ...(14.39)$$

If C_h is assumed minimum, then $C_h < C_c$ and therefore $C_h = C_{min}$ and $C_c = C_{max}$. Then

$$\varepsilon = \frac{1-\exp\left[-UA/C_{min}\{1+C_{min}/C_{max}\}\right]}{C_{min}/C_{min}\{1+C_{min}/C_{max}\}}$$

The dimensionless ratio UA/C_{min} is called the number of transfer units (NTU) and the factor C_{min}/C_{max} represents the capacity ratio (C).

Therefore,

$$\varepsilon_h = \frac{1-\exp[-NTU(1+C)]}{(1+C)}$$

where ε_h is the effectiveness of the parallel flow exchanger with hot fluid having the minimum capacity rate. The same relationship would result when the analysis is made with the cold fluid having minimum capacity rate. The suffix h can, therefore, be dropped to give effectiveness of a parallel flow exchanger as

$$\varepsilon = \frac{1-\exp[-NTU(1+C)]}{(1+C)} \quad ...(14.40)$$

14.6.2. Effectiveness for the Counterflow Heat Exchanger

The heat exchange through an incremental area dA (Fig. 14.24) of the exchanger is

$$dQ = U\,dA\,(t_h - t_c) \quad ...(a)$$
$$= C_h(-dt_h) = C_c(-dt_c) \quad ...(b)$$

From expression (b),

$$dt_h = \frac{-dQ}{C_h} \quad \text{and} \quad dt_c = \frac{-dQ}{C_c}$$

$$\therefore d(t_h - t_c) = dQ\left(\frac{1}{C_c}-\frac{1}{C_h}\right)$$

Substitution of the value of dQ from expression (a) and subsequent rearrangement gives

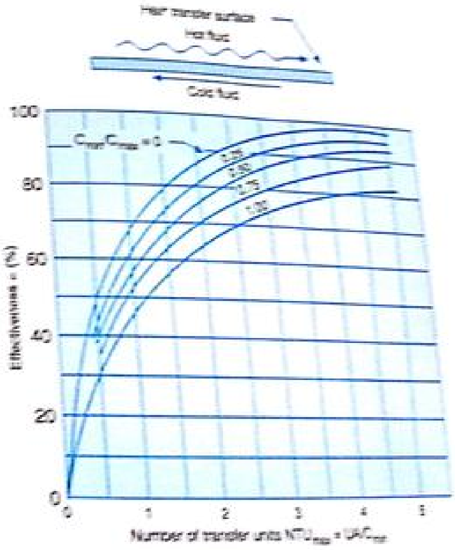


Fig. 14.24. Effectiveness of counter-flow heat exchanger

$$\frac{d(t_h - t_c)}{(t_h - t_c)} = U\,dA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

Through integration we obtain :

$$\int \frac{d(t_h - t_c)}{(t_h - t_c)} = U\left\{\frac{1}{C_c} - \frac{1}{C_h}\right\} \int dA$$

The integration limits for the temperature difference are from $(t_{h1} - t_{c2})$ to $(t_{h2} - t_{c1})$ and those of area from 0 to A

$$\therefore \log_e \frac{t_{h2} - t_{c1}}{t_{h1} - t_{c2}} = UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

$$\text{or } \log_e \frac{t_{h1} - t_{c2}}{t_{h2} - t_{c1}} = -UA\left\{\frac{1}{C_c} - \frac{1}{C_h}\right\}$$

$$\text{or } \frac{t_{h1} - t_{c2}}{t_{h2} - t_{c1}} = \exp\left[-UA\left\{\frac{1}{C_c} - \frac{1}{C_h}\right\}\right] \quad \text{...(14.41)}$$

From the definition of effectiveness

$$\epsilon = \frac{C_h(t_{h1} - t_{h2})}{C_{min}(t_{h1} - t_{c1})} = \frac{C_c(t_{c2} - t_{c1})}{C_{min}(t_{h1} - t_{c1})}$$

the values of outlet temperature t_{h2} and t_{c2} are worked out as,

$$t_{h2} = t_{h1} - \frac{C_{min}}{C_h}(t_{h1} - t_{c1})\,\epsilon$$

$$\text{and} \quad t_{c2} = t_{c1} - \frac{C_{min}}{C_c}(t_{h1} - t_{c1})\,\epsilon$$

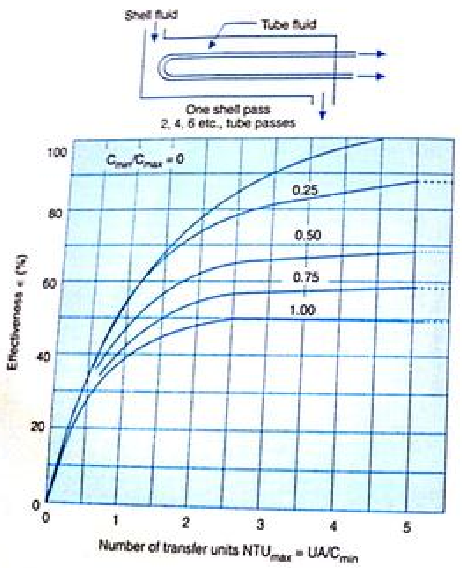


Fig. 14.25. Effectiveness of 1-2 parallel counter flow heat exchanger

Substituting these values in equation 14.41, we get,

$$\frac{t_{h1} - t_{c1} - (C_{min}/C_c)(t_{h1} - t_{c1})\epsilon}{t_{h1} - (C_{min}/C_h)(t_{h1} - t_{c1})\epsilon - t_{c1}} = \exp\left[-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right]$$

$$\text{or } \frac{(t_{h1} - t_{c1})[1 - (C_{min}/C_c)\epsilon]}{(t_{h1} - t_{c1})[1 - (C_{min}/C_h)\epsilon]} = \exp\left[-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right]$$

$$\text{or } \frac{[1 - (C_{min}/C_c)\epsilon]}{[1 - (C_{min}/C_h)\epsilon]} = \exp\left[-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right]$$

$$\text{or } \epsilon\left[\frac{C_{min}}{C_c} - \frac{C_{min}}{C_h}\exp\left\{-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right\}\right] = 1 - \exp\left[-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right]$$

$$\therefore \epsilon = \frac{1 - \exp[-UA\{1/C_c - 1/C_h\}]}{C_{min}[1/C_c - 1/C_h \exp\{-UA(1/C_c - 1/C_h)\}]}$$

$$= \frac{1 - \exp[-UA/C_c\{1 - C_c/C_h\}]}{C_{min}/C_c[1 - (C_c/C_h)\exp\{-UA/C_c(1 - C_c/C_h)\}]} \quad \text{...(14.42)}$$

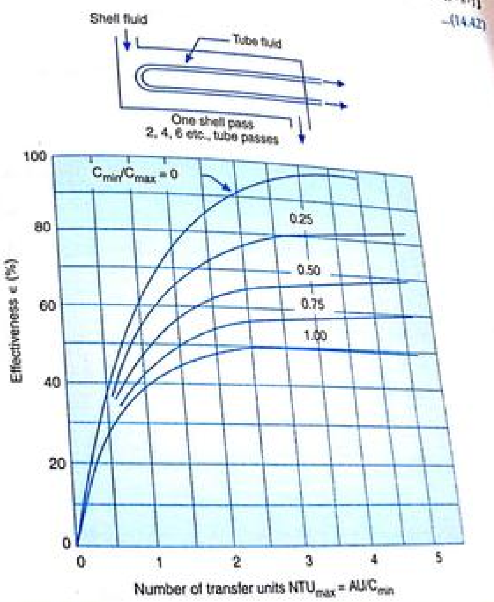


Fig. 14.26. Effectiveness for 2-4 multi pass counter flow-heat exchanger

If C_c is assumed minimum, then $C_c < C_h$ and therefore $C_c = C_{min}$ and $C_h = C_{max}$. Thus

$$\epsilon_c = \frac{1-\exp\left[-UA/C_{min}\{1-C_{min}/C_{max}\}\right]}{C_{min}/C_{min}\left[1-(C_{min}/C_{max})\exp\{-UA/C_{min}(1-C_{min}/C_{max})\}\right]}$$

The dimensionless ratio UA/C_{min} is called the number of transfer units (NTU) and the factor C_{min}/C_{max} defines the capacity ratio, (C).

Therefore,

$$\epsilon_c = \frac{1-\exp\left[-NTU(1-C)\right]}{1-C\exp\left[-NTU(1-C)\right]}$$

where ϵ_c is the effectiveness of the counter flow exchanger with cold fluid having the minimum capacity rate. The same relationship would result when the analysis is made with hot fluid having minimum capacity rate. The suffix c can, therefore, be dropped to give effectiveness of a counter flow heat exchanger as

$$\epsilon = \frac{1-\exp\left[-NTU(1-C)\right]}{1-C\exp\left[-NTU(1-C)\right]} \quad ...(14.43)$$

It is to be noted that the expressions for the effectiveness contain only the overall heat transfer coefficient, area, fluid properties and flow rates.

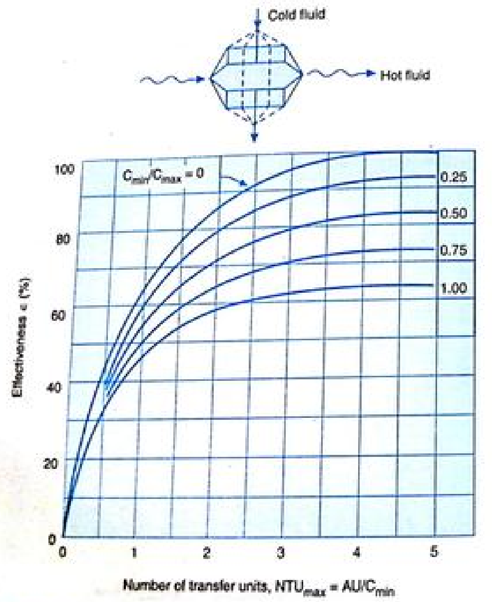


Fig. 14.27. Effectiveness for cross flow heat exchangers with both fluids mixed

14.6.3. Limiting values of capacity ratio, C

Two limiting cases of practical interest are :

(i) During the process of boiling and condensation, only a phase change takes place and one fluid remains at constant temperature throughout the exchanger.

By definition, the specific heat represents the change of enthalpy with respect to temperature, *i.e.*, $c_p = dh/dt$. With temperature difference dt being zero, the effective specific heat and consequently the heat capacity tends to infinity. In that case $C_{max} = \infty$ and $C_{min}/C_{max} = 0$. The expression for effectiveness then reduces to

$$\epsilon = 1 - \exp(-NTU) \quad ...(14.44)$$

both for parallel and counter flow configurations.

(ii) In a gas turbine recuperator, the exhaust gases after expansion in the turbine are used to heat the compressed air. Both the fluids have approximately equal thermal capacities and so the capacity ratio,

$$C = C_{min}/C_{max}$$

becomes very close to unity.

Then for the parallel flow configuration,

$$\epsilon = \frac{1-\exp(-2NTU)}{2} \quad ...(14.45)$$

Obviously, no matter how large the exchanger be or how high be the of overflow

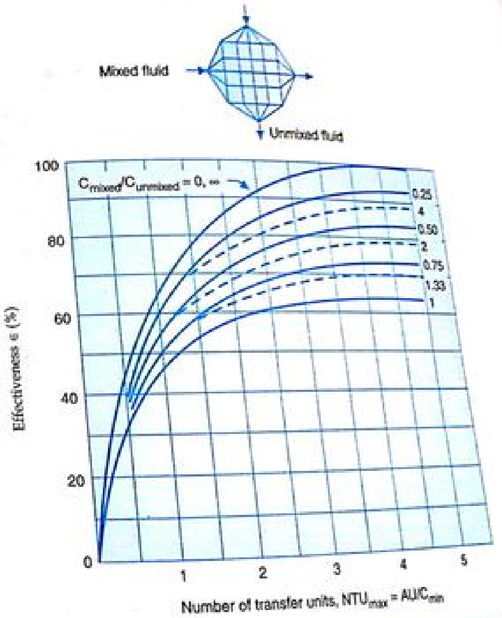


Fig. 14.28. Effectiveness for cross flow heat exchangers with one fluid mixed

at transfer coefficient, the maximum effectiveness for parallel flow heat exchanger is only 5%.

For counter flow arrangement, effectiveness becomes indeterminate and that necessitates fresh analysis.

With $C = C_{min}/C_{max} \to 1$, it is possible to write that

$$(1 - C) = 0\,; \quad NTU\,(1 - C) << 1$$

and $\exp\,[- NTU\,(1 - C)] = - NTU\,(1 - C)$

Substitution of these results in equation 14.43 gives ;

$$\epsilon = \frac{1-[1-NTU(1-C)]}{1-C[1-NTU(1-C)]}$$

$$= \frac{NTU(1-C)}{1-C[1+NTU]}$$

$$= \frac{NTU}{1+NTU} \qquad ...(14.46)$$

The limiting value of effectiveness for counter flow exchanger is 100% and apparently counter flow units are more advantageous for gas turbine heat exchangers.

Graphs of effectiveness versus the *NTU*-parameters have been published for various heat exchanger configuration; and graphs of some of the more common heat exchanger configurations are shown in Figs. 14.23 to 14.28. These curves indicate the relationship between

- effectiveness ϵ
- C_{min}/C_{max}
- $UA/C_{min} = NTU_{max}$

When any two of three parameters are known, the third can be read from these graphs. That serves to completely define the heat exchanger and its heat transfer performance.

EXAMPLE 14.40.

Hot water having specific heat 4200 J/kgK flows through a heat exchanger at the rate of 4 kg/min with an inlet temperature of 100°C. A cold fluid having a specific heat 2400 J/kg K flows in at a rate of 8 kg/min and with inlet temperature 20°C. Make calculations for the maximum possible effectiveness if the fluid flow conforms to (a) parallel flow arrangement (b) counter flow arrangement.

Solution : Thermal capacity rates of the hot and cold fluids are :

$$C_h = m_h\, c_h = \frac{4}{60} \times 4200 = 280 \text{ W/K}$$

$$C_c = m_c\, c_c = \frac{8}{60} \times 2400 = 320 \text{ W/K}$$

Obviously,

$$C_{min} = 280 \text{ W/K}$$

and C_{max} 320 W/K

$$\text{Capacity ratio, } C = \frac{C_{min}}{C_{max}} = \frac{280}{320} = 0.875$$

The effectiveness for a parallel flow heat exchanger is

$$\epsilon = \frac{1-\exp[-NTU(1+C]}{1+C}$$

Effectiveness will be maximum when $NTU = \infty$ for the given value of capacity ratio C.

$\therefore$ Maximum possible effectiveness

$$= \frac{1-\exp[-\infty]}{1+C}$$

$$= \frac{1}{1+C} = \frac{1}{1+0.875} = \mathbf{0.533}$$

(*b*) The effectiveness for a counter flow heat exchanger is

$$\epsilon = \frac{1-\exp[-NTU(1-C)]}{1-C\exp[-NTU(1-C)]}$$

Effectivenss will be maximum when $NTU = \infty$ for the given value of capacity ratio C

$\therefore$ Maximum possible effectiveness

$$= \frac{1-\exp(-\infty)}{1-C\exp(-\infty)}$$

$$= \frac{1-0}{1-0} = \mathbf{1}$$

EXAMPLE 14.41.

A counter flow heat exchanger is used to cool 2000 kg/hr of oil (c_p = 2.5 kJ/kg K) from 105°C to 30°C by the use of water entering at 15°C. If the overall heat transfer coefficient is expected to be 1.5 kW/m²K, make calculations for the water flow rate, the surface area required and the effectiveness of heat exchanger. Presume that the exit temperature of the water is not to exceed 80°C. Use NTU-effectiveness approach.

Solution : The mass flow rate of water can be determined from an energy balance on the two fluids, *i.e.*,

heat lost by oil (hot fluid)

= heat gained by water (coolant)

$$m_h\, c_h\,(t_{h1} - t_{h2}) = m_c\, c_c\,(t_{c2} - t_{c1})$$

$$2000 \times 2.5\,(105 - 30) = m_c \times 4.18\,(80 - 15)$$

$\therefore$ mass flow rate of coolant (water) m_c

$$= \frac{2000\times2.5\times(105-30)}{4.18\times(80-15)}$$

$$= 1380.2 \text{ kg/hr}$$

(*b*) Thermal capacity of the water stream (coolant)

$$C_c = m_c\, c_c$$
$$= 1380.2 \times 4.18 = 5769.24 \text{ kJ/hrK}$$

Thermal capacity of the oil stream (hot fluid)

$$C_h = m_h\, c_h$$
$$= 2000 \times 2.5 = 5000 \text{ kJ/hrK}$$

Obviously,

$$C_{min} = 5000 \text{ kJ/hrK}$$

and $C_{max} = 5769.24$ kJ/hrK

$$\text{Capacity ratio } C = \frac{C_{min}}{C_{max}} = \frac{5000}{5769.24} = 0.867$$

When hot fluid has the minimum heat capacity, then

Effectiveness ϵ

$$= \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}} = \frac{105-300}{105-15} = 0.833$$

The number of transfer units (NTU) can be computed from the following expression for effectiveness of a counter flow exchanger

$$\epsilon = \frac{1-\exp[-NTU(1-C)]}{1-C\exp[-NTU(1-C)]}$$

Rearranging

$$\frac{\epsilon-1}{\epsilon\, C-1} = \exp\,[-NTU\,(1-C)]$$

$$\frac{0.833-1}{0.833\times0.867-1} = \exp\,[-NTU\,(1-0.867)]$$

$$\text{or } \log_e \frac{0.167}{0.278} = 0.133 \text{ NTU or NTU} = 3.84$$

Alternatively using the parameters $C = 0.867$ and $\epsilon = 0.833$ with Fig. 14.23, we get; NTU = 3.84

$$\text{But NTU} = \frac{UA}{C_{min}}$$

Therefore the heat transfer area is

$$A = \frac{NTU \times C_{min}}{U}$$

$$= \frac{3.84\times5000}{1.5\times3600} = 3.55 \text{ m}^2$$

EXAMPLE 14.42.

Sketch the temperature-length curves for a counter flow and parallel flow heat exchangers for the cases when (i) $C_c > C_h$ (ii) $C_c = C_h$ and (iii) $C_c < C_h$

A counter-flow exchanger of surface area 8 m² is to be used to heat a process liquid by using a high temperature water available from another part of the plant. If the overall coefficient of heat transfer is 450 W/m²K, find the exit temperatures of the process liquid and water stream from the data given below :

	hot fluid (water)	cold fluid (process liquid)
inlet temperature K	365	300
mass flow rate kg/s	1.0	3.0
specific heat kJ/kgK	4.2	2.1

Solution : Thermal capacity rate of the hot (water stream) and cooling (process liquid) fluid are :

$$C_h = m_c\, c_h = 1.0 \times 4.2 = 4.2 \text{ kJ/sK}$$
$$C_c = m_c\, c_c = 3.0 \times 2.1 = 6.3 \text{ kJ/sK}$$

Obviously,

$$C_{min} = 4.2 \text{ kJ/sK}$$

and $C_{max} = 6.2$ kJ/sK

Capacity ratio, C

$= C_{min}/C_{max} = 4.2/6.3 = 0.667$

Number of transfer units, NTU

$= \frac{UA}{C_{min}} = \frac{450 \times 8}{4.2 \times 10^3} = 0.857$

The effectiveness for a counter flow heat exchanger is

$$\varepsilon = \frac{1-\exp[-NTU(1-C)]}{1-C\exp[-NTU(1-C)]}$$

$$= \frac{1-\exp[-0.857(1-0.667)]}{1-0.667\exp[-0.857(1-0.667)]}$$

$$= \frac{1-\exp(-0.285)}{1-0.667\exp(-0.285)} = 0.495$$

Alternatively using the parameters C = 0.667 and NTU = 0.857 with Fig. 14.25, we get ε = 0.495

The hot fluid has the minimum thermal capacity and therefore in terms of temperature difference

$$\varepsilon = \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}}$$

$$\text{or } 0.495 = \frac{365-t_{h2}}{365-200}$$

∴ Outlet temperature of the hot fluid (water),

$t_{h2} = 365 - 0.495(365 - 200)$

$= 283.32$ K

From energy balance on the two fluids,

$m_h C_h (t_{h1} - t_{h2}) = m_c C_c (t_{c2} - t_{c1})$

or $10 \times 4.2 \times (365 - 283.32)$

$= 5.6 \times 2.1 (t_{c2} - 200)$

∴ Outlet temperature of the cold fluid (process liquid),

$$t_{c2} = 200 + \frac{(10 \times 4.2) \times (365-283.32)}{5.6 \times 2.1}$$

$= 491.46$ K

EXAMPLE 14.43.

Define effectiveness and NTU of a heat exchanger.

In an open heart surgery under hypothermic conditions, the patients blood is cooled before the surgery and rewarmed afterwards. The task is accomplished by a concentric tube counter-flow heat exchanger of length 500 mm with a thin walled inner tube of 60 mm diameter. The blood enters the heat exchanger at 20°C and 0.05 kg/s is warmed by water at 60°C and 0.12 kg/s. Determine the temperature of blood at exit from the heat exchanger and the heat flow rate. Assume the following data :

c_p of blood = 3500 J/kgK

and c_p of water = 4186 J/kgK

Overall heat transfer coefficient U_o

= 475 W/m²K

Solution : Thermal capacity rates for the hot fluid (water) and cold fluid (blood) are :

$C_h = m_h c_h$

$= 0.12 \times 4186 = 502.32$ W/K

$C_c = m_c c_c$

$= 0.05 \times 3500 = 175$ W/K

Obviously,

$C_{max} = 502.32$ W/K

and $C_{min} = 175$ W/K

Transfer units NTU

$$= \frac{AU}{C_{min}} = \frac{(\pi \times 0.06 \times 0.5) \times 475}{175}$$

$= 0.256$

The effectiveness for a counter flow arrangement is

$$\varepsilon = \frac{1-\exp[-NTU(1-C)]}{1-C\exp[-NTU(1-C)]}$$

$$= \frac{1-\exp[-0.256(1-0.348)]}{1-0.348\exp[-0.256(1-0.348)]}$$

$$= \frac{1-\exp(-0.1669)}{1-0.348\exp(-0.1669)}$$

$= 0.1598$

Further, $\varepsilon = \frac{t_{c2}-t_{c1}}{t_{h1}-t_{c1}} = \frac{t_{c2}-20}{60-20}$

∴ Temperature of blood at exit from heat exchanger,

$t_{c2} = 0.1598(60-20) + 20 = 26.39°C$

Heat flow rate Q

$= m_c c_c (t_{c2} - t_{c1})$

$= 0.05 \times 3500 (26.39 - 20)$

$= 1118.25$ W

EXAMPLE 14.44.

A home air-conditioning system uses a counter-flow heat exchanger to cool 0.8 kg/s of air from 45°C to 35°C. The cooling is accomplished by a stream of cooling water that enters the system with 0.5 kg/s flow rate and 8°C temperature. If the overall heat transfer coefficient is 35 W/m²K, what heat exchanger area is required ? If the same air flow rate is maintained while the water flow rate is reduced to half, how much will be the percentage reduction in heat transfer ? Use effectiveness–NTU approach.

Solution : Thermal capacity rates for the hot fluid (air) and cold fluid (water) are :

$C_h = m_h c_h = 0.8 \times 1005 = 804$ W/K

$C_c = m_c c_c = 0.75 \times 4186$

$= 3139.5$ W/K

Obviously

$C_{max} = 3139.5$ W/K

and $C_{min} = 804$ W/K

Capacity ratio, $C = \frac{C_{min}}{C_{max}} = \frac{804}{3139.5} = 0.256$

Effectiveness, ε

$$= \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}} = \frac{45-35}{45-8} = 0.81$$

The effectiveness for a counter flow arrangement is,

$$\varepsilon = \frac{1-\exp[-NTU(1-C)]}{1-C\exp[-NTU(1-C)]}$$

Rearranging :

$$\frac{\varepsilon-1}{\varepsilon C-1} = \exp[-NTU(1-C)]$$

$$\frac{0.81-1}{0.81 \times 0.256-1} = \exp[-NTU(1-0.256)]$$

or $\log_e \frac{0.19}{0.793} = -0.744$ NTU

or NTU = 1.92

Therefore, the required transfer area is,

$$A = \frac{NTU \times C_{min}}{U} = \frac{1.92 \times 804}{35}$$

$= 44.10$ m²

(ii) Since the water flow rate is reduced to half,

$m_c = \frac{1}{2} \times 0.75 = 0.375$ kg/s

$C_c = m_c c_c = 0.375 \times 4186$

$= 1569.75$ W/K

$C_h = m_h c_h$

$= 804$ W/K, as in previous case

$$C = \frac{C_{min}}{C_{max}} = \frac{804}{1569.75} = 0.512$$

Transfer units NTU $= \frac{UA}{C_{min}}$

Since all the parameters in the above expression for NTU remain same, NTU will be same as in the original situation. That is

NTU = 1.92

Invoking the effectiveness relation for counter flow heat exchanger

$$\varepsilon = \frac{1-\exp[-NTU(1-C)]}{1-C\exp[-NTU(1-C)]}$$

$$= \frac{1-\exp[-1.92(1-0.512)]}{1-0.512\exp[-1.92(1-0.512)]}$$

$$= \frac{1-\exp(-0.936)}{1-0.512\exp(-0.936)}$$

$$= \frac{1-0.392}{1-0.512 \times 0.392} = 0.76$$

Also, $\varepsilon = \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}}$

$0.76 = \frac{\Delta t_h}{45-8}$

$\Delta t_h = 28.12$

The rate of heat flow is proportional to Δt_h. Hence percentage change in heat transfer is

$$= \frac{(\Delta t_h)_{old}-(\Delta t_h)_{new}}{(\Delta t_h)_{old}}$$

$$= \frac{30 - 28.12}{30}$$
$$= 0.0627 \text{ or } 6.27\%$$

EXAMPLE 14.45.

In a surface condenser, the water flowing through a series of tubes at the rate of 200 kg/hr is heated from 15°C to 75°C. The steam condenses on the outside surface of tubes at atmospheric pressure and the overall coefficient of heat transfer is estimated at 860 kJ/m²-hr-deg. Use NTU method to work out the length of tube and the steam condensation rate. Presume that the tube is 25 mm in diameter.

At the condensing pressure, steam has saturation temperature t_s = 100°C and the latent heat of vaporisation h_{fg} = 2160 kJ/kg. Further, the steam is initially just saturated and the condensate leaves the exchanger without subcooling, i.e., only the latent heat of condensing steam is transferred to water. Take specific heat of water as 4 kJ/kgK.

Solution : During condensation, only a phase change occurs and steam remains at constant temperature throughout the exchanger. With its temperature difference being zero, the effective specific heat and consequently the heat capacity tends to infinity. Obviously then the cooling water has minimum value of thermal capacity.

$$C_{min} = m_c\, c_c = 200 \times 4 = 800 \text{ kJ/hrK}$$

Effectiveness,

$$\epsilon = \frac{C_h(t_{h1} - t_{h2})}{C_{min}(t_{h1} - t_{c1})}$$
$$= \frac{C_c(t_{c2} - t_{c1})}{C_{min}(t_{h1} - t_{c1})}$$
$$= \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{75 - 15}{100 - 15} = 0.706$$

Also, $\epsilon = 1 - \exp(-NTU)$ or $0.706 = 1 - \exp(-NTU)$ or $NTU = 1.224$

Again, $NTU = \dfrac{UA}{C_{min}} = \dfrac{U(\pi d l)}{C_{min}}$

∴ length of condenser tube, l

$$= \frac{C_{min} \times NTU}{U \pi d}$$
$$= \frac{800 \times 1.224}{860 \times \pi \times 0.025} = 14.5 \text{ m}$$

(ii) From energy balance on steam and the cooling water,

$$m\, h_{fg} = m_c\, c_c\, (t_{c2} - t_{c1})$$
$$m \times 2160 = 200 \times 4 \times (75 - 15)$$

∴ Steam condensation rate, m

$$= \frac{200 \times 4 \times (75 - 15)}{2160}$$
$$= \mathbf{22.22 \text{ kg/hr}}$$

EXAMPLE 14.46.

A shell-and-tube heat exchanger is designed as an ammonia condenser with ammonia vapour entering the shell at 60°C as a saturated vapour. Water enters the single-pass tube arrangement at 25°C and gets heated to 50°C and the total heat transfer rate is 250 kW. Calculate the area of the heat exchanger to achieve 60 per cent effectiveness with overall heat transfer coefficient 1000 W/m²-deg. How would the heat transfer rate be affected if the water flow rate is reduced in half? It may be presumed that heat exchanger area and heat transfer coefficient remain the same.

Solution : From energy balance,

$$Q = 250 \times 10^3$$
$$= m_c\, c_c\, (t_{c2} - t_{c1})$$
$$= m_c \times 4180 \times (50 - 25)$$

∴ Mass flow rate of water, m_c

$$= \frac{250 \times 10^3}{4180 \times 25} = 2.392 \text{ kg/s}$$

During vapour condensation, only a phase change occurs and vapour remains at constant temperature throughout the exchanger. With its temperature fall being zero, the effective specific heat and consequently the heat capacity tends to infinity. Obviously then water is the minimum fluid, *i.e.*, the fluid having minimum value of thermal capacity.

$$C_{min} = m_c\, c_c$$
$$= 2.392 \times 4180$$
$$= 9998.56 \text{ W per degree kelvin}$$

Again with boiling or condensation process $C_{min}/C_{max} \to 0$ and heat exchanger effectiveness is prescribed by the equation

$$\epsilon = 1 - \exp(-NTU)$$

or $0.6 = 1 - \exp(-NTU)$

Solution gives:

$$NTU = 0.916$$

Further, $NTU = \dfrac{UA}{C_{min}}$

∴ Area of heat exchanger,

$$A = NTU \times \frac{C_{min}}{U}$$
$$= 0.916 \times \frac{9998.56}{1000} = 9.16 \text{ m}^2$$

(ii) When the mass flow rate of water is reduced in half, the changed value of NTU is

$$NTU = \frac{UA}{C_{min}} = \frac{1000 \times 9.16}{(9998.56/2)} = 1.832$$

Effectiveness, ϵ

$$= 1 - \exp(-NTU)$$
$$= 1 - \exp(-1.832)$$
$$= 1 - \frac{1}{e^{1.832}} = 0.84$$

In terms of temperature differences and thermal capacity rates, effectiveness is given by

$$\epsilon = \frac{C_h(t_{h1} - t_{h2})}{C_{min}(t_{h1} - t_{c1})}$$
$$= \frac{C_c(t_{c2} - t_{c1})}{C_{min}(t_{h1} - t_{c1})}$$
$$= \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} \qquad (\because C_c = C_{min})$$

∴ New water temperature difference,

$$(t_{c2} - t_{c1}) = \epsilon \times (t_{h1} - t_{c1})$$
$$= 0.84 \times (60 - 25) = 29.4°\text{C}$$

New heat transfer rate, Q

$$= m_c\, c_c\, (t_{h2} - t_{c1})$$
$$= C_{min}\, (t_{c2} - t_{c1})$$
$$= \frac{9998.56}{2}(29.4) = 146979 \text{ W}$$

Thus by reducing the water flow rate to half, the heat transfer rate has been reduced from 250 kW to 147 kW or by

$$\frac{250 - 147}{257} \times 100 = 41.2 \text{ percent}$$

EXAMPLE 14.47.

A tube type heat exchanger is used to cool hot water from 80°C to 60°C. The task is accomplished by transferring heat to cold water that enters the heat exchanger at 20°C and leaves at 40°C. Should this exchanger operate under counter flow or parallel flow conditions? Also determine the exit temperatures if the flow rates of fluids are doubled.

Solution : The outlet temperature of cold fluid is less than the outlet temperature of hot fluid. Such a temperature profile is possible in parallel flow arrangement, and hence the exchanger should operate in a parallel flow mode.

Making energy balance for the two fluids,

$$m_c\, c_c\, \Delta t_c = m_h\, c_h\, \Delta t_h$$

Since both fluids have equal temperature difference,

$$m_c\, c_c = m_h\, c_h \text{ or } C_c = C_h$$

That is, the thermal capacity rates of the hot and cold fluids are equal. Obviously then

Heat capacity ratio, C

$$= \frac{C_{min}}{C_{max}} = 1$$

Effectiveness, ϵ

$$= \frac{t_{h1} - t_{h2}}{t_{h1} - t_{c1}} = \frac{80 - 60}{80 - 20} = \frac{20}{60} = \frac{1}{3}$$

The effectiveness for a parallel flow heat exchanger is given by

$$\epsilon = \frac{1}{3} = \frac{1 - \exp[-NTU(1 + C)]}{1 + C}$$
$$= \frac{1 - \exp[-NTU(1 + C)]}{1 + 1}$$
$$\exp(-2\, NTU) = 1 - \frac{2}{3} = \frac{1}{3}$$
$$\text{or } -2\, NTU = \log_e \frac{1}{3} = 1.0986;$$
$$NTU = 0.549$$

When flow rates of the fluids are doubled, the thermal capacity rates of the hot and cold

fluids will still be equal and accordingly the heat capacity ratio C will be unity. Also

$$(mc)_{new} = 2\,(mc)_{old}$$

Hence,

$$(NTU)_{new} = \frac{UA}{C_{min}} = \frac{1}{2}\,(NTU)_{old}$$

$$= \frac{1}{2} \times 0.549 = 0.2745$$

$\therefore$ New effectiveness,

$$\epsilon = \frac{1-\exp[-0.2745(1+1)]}{1+1}$$

$$= 0.2112$$

In terms of temperatures,

$$\epsilon = \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}} = \frac{t_{c2}-t_{c1}}{t_{h1}-t_{c1}}$$

$$\therefore \quad t_{h2} = t_{h1} - \epsilon\,(t_{h1} - t_{c1})$$

$$= 80 - 0.2112\,(80 - 20)$$

$$= 67.328°C$$

$$t_{c2} = t_{c1} + \epsilon\,(t_{h1} - t_{c1})$$

$$= 20 + 0.2112\,(80 - 20)$$

$$= 32.672°C$$

EXAMPLE 14.48.

The engine oil at 150°C is cooled to 80°C in a parallel flow heat exchanger by water entering at 25°C and leaving at 60°C. Estimate the exchanger effectiveness and the number of transfer units. If the fluid flow rates and the inlet conditions remain unchanged, work out the lowest temperature to which the oil may be cooled by increasing length of the exchanger.

Solution : From energy balance on the hot (oil) and cold (water) fluids,

$$m_c\,c_c\,(t_{c2} - t_{c1}) = m_h\,c_h\,(t_{h1} - t_{h2})$$

$$\frac{m_h c_h}{m_c c_c} = \frac{t_{c2}-t_{c1}}{t_{h1}-t_{h2}} = \frac{60-25}{150-80} = 0.5$$

Apparently, the hot fluid has the minimum thermal capacity and $C_{min}/C_{max} = 0.5$

Effectiveness, ϵ

$$= \frac{C_h(t_{h1}-t_{h2})}{C_{min}\,(t_{h1}-t_{c1})}$$

$$= \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}} = \frac{150-80}{150-25} = 0.56$$

In terms of capacity ratio and number of transfer units,

$$\epsilon = \frac{1-\exp[-NTU(1+C)]}{1+C}$$

$$0.56 = \frac{1-\exp[-NTU(1+0.5)]}{1+0.5}$$

$$= \frac{1-\exp[-1.5NTU]}{1.5}$$

$$\exp[-1.5\,NTU] = 1 - 1.5 \times 0.56 = 0.16$$

$\therefore$ Number of transfer units, $NTU = 1.221$

(b) The exit temperature of oil would be minimum corresponding to the situation when the exchanger is increased infinitely or $NTU \to \infty$.

$$\therefore \quad \epsilon = \frac{1-\exp[-NTU(1+C)]}{1+C}$$

$$= \frac{1}{1+C} = \frac{1}{1+0.5} = 0.667$$

Also, $\epsilon = \dfrac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}}$

or $0.667 = \dfrac{150-t_{h2}}{150-25}$

Therefore, the minimum possible exit temperature of the oil is

$$t_{h2} = 150 - 0.667\,(150 - 25) = 66.62°C$$

EXAMPLE 14.49.

Water (sp heat = 4 kJ/kg K) enters a cross flow exchanger (both fluids unmixed) at 15°C and flows at the rate of 7.5 kg/s. It cools air (c_p = 1 kJ/kg K) flowing at the rate of 10 kg/s from an inlet temperature of 120°C. For an overall heat transfer coefficient 780 kJ/m²-hr-K and an exchanger surface area of 240 m², determine the total heat transfer and the outlet temperature of air. How these results would be affected if a 1 - 10 (one-shell pass and ten-tube pass) heat exchanger is employed ?

Solution : Thermal capacity rates of the hot fluid (air) and cold fluid (water) are :

$$C_h = m_h\,c_h = 10 \times 1 = 10 \text{ kJ/sK}$$

$$C_c = m_c\,c_c = 7.5 \times 4 = 30 \text{ kJ/sK}$$

Obviously,

$$C_{min} = 10 \text{ kJ/sK and } C_{max} = 30 \text{ kJ/sK}$$

Capacity ratio, C

$$= \frac{C_{min}}{C_{max}} = \frac{10}{30} = 0.333$$

Number of transfer units,

$$NTU = \frac{UA}{C_{min}} = \frac{780 \times 240}{10 \times 3600} = 5.2$$

Using the parameters $C = 0.333$ and $NTU = 5.2$ with Fig. 14.26, we read $\epsilon = 0.94$

Also, $\epsilon = \dfrac{\text{actual heat transfer}}{\text{maximum possible heat transfer}}$

$$= \frac{Q}{C_{min}\,(t_{h1}-t_{c1})}$$

$$\therefore \quad Q = \epsilon\,C_{min}\,(t_{h1} - t_{c1})$$

$$= 0.94 \times 10 \times (120 - 15)$$

$$= 987 \text{ kJ/s}$$

An energy balance on the hot fluid gives,

$$Q = m_h\,c_h\,(t_{h1} - t_{h2})$$

$$987 = 10 \times 1 \times (120 - t_{h2})$$

$\therefore$ Exit temperature of the hot fluid (air),

$$t_{h2} = 120 - \frac{987}{10 \times 1} = 21.3°C$$

(2) The non-dimensional parameters C and NTU remain unchanged at 0.333 and 5.40 respectively. Using these parameters with Fig. 14.24, $\epsilon = 0.83$

Heat transfer,

$$Q = \epsilon\,C_{min}\,(t_{h1} - t_{c1})$$

$$= 0.83 \times 10 \times (120 - 15)$$

$$= 871.5 \text{ kJ/s}$$

Also, $Q = m_h\,c_h\,(t_{h1} - t_{h2})$

$$871.5 = 10 \times 1 \times (120 - t_{h2})$$

$\therefore$ Exit temperature of the hot fluid (air)

$$t_{h2} = 120 - \frac{871.5}{10 \times 10} = 32.85°C$$

EXAMPLE 14.50.

A parallel flow heat exchanger uses 1500 kg/hr of cold water entering at 25°C to cool 600 kg/hr of hot water entering at 70°C. The exit temperature on the hot side is required to be 50°C. Neglecting the effects of fouling, make calculations for the area of heat exchanger. It may be presumed that the individual heat transfer coefficients on both sides are 1600 W/m²K. Use the mean temperature difference approach and the effectiveness-NTU approach.

Proceed to calculate the exit temperature of the cold and hot streams if the flow of hot water is doubled, i.e., it becomes 1200 kg/hr. It has been stated that the individual heat transfer coefficients are proportional to 0.8th power of flow rate. For water C = 4180 J/kgK.

Solution : The unknown exit temperature of the cooling water can be found from an energy balance on the two fluids. That is

$$m_c\,c_c\,(t_{c2} - t_{c1}) = m_h\,c_h\,(t_{h1} - t_{h2})$$

$$t_{c2} = t_{c1} + \frac{m_h}{m_c}\,(t_{h1} - t_{h2})$$

(because $c_c = c_h$)

$$= 25 + \frac{600}{1500}\,(70 - 50) = 33°C$$

From energy balance on the hot fluid, the heat transfer rate is

$$Q = m_h\,c_h\,(t_{h1} - t_{h2})$$

$$= \frac{600}{3600} \times 4180 \times (70 - 50)$$

$$= 13933 \text{ J/s}$$

Mean temperature difference approach :

Log mean temperature difference is, θ_m

$$= \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}}$$

where; $\theta_1 = t_{h1} - t_{c1} = 70 - 25 = 45°C$

$\theta_2 = t_{h2} - t_{c2} = 50 - 33 = 17°C$

$$\therefore \quad \theta_m = \frac{45-17}{\log_e \dfrac{45}{17}} = \frac{28}{0.973}$$

$$= 28.77°C$$

$$\text{Now, } \frac{1}{U} = \frac{1}{h_h} + \frac{1}{h_c}$$

$$= \frac{1}{1600} + \frac{1}{1600}$$

$$= \frac{1}{800} \quad \text{or} \quad U = 800 \text{ W/m}^2\text{K}$$

Heat exchange, $Q = U A \theta_m$

$\therefore$ Heating surface area, A

$$= \frac{Q}{U\theta_m} = \frac{13933}{800 \times 28.77} = 0.605\ m^2$$

Effectiveness - NTU approach : Thermal capacity rates of the hot and cold fluids are :

$$C_h = m_h c_h = \frac{600}{3600} \times 4180 = 696.67\ W/K$$

$$C_c = m_c c_c = \frac{1500}{3600} \times 4180 = 1741.67\ W/K$$

Obviously,

$$C_{min} = 696.67\ W/K$$

and $C_{max} = 1741.67\ W/K$

Capacity ratio, C

$$= \frac{C_{min}}{C_{max}} = \frac{696.67}{1741.67} = 0.400$$

Effectiveness, ϵ

$$= \frac{t_{h1} - t_{h2}}{t_{h1} - t_{c1}} = \frac{70-50}{70-25} = 0.444$$

The effectiveness for a parallel flow heat exchanger is given by

$$\epsilon = \frac{1 - \exp[-NTU(1+C)]}{1+C}$$

$$\therefore\ 0.444 = \frac{1-\exp[-NTU(1+0.400)]}{1+0.400}$$

Upon solution we obtain : $NTU = 0.695$

Also, $NTU = \frac{UA}{C_{min}}$

or $\quad 0.695 = \frac{800 \times A}{696.67}$

$\therefore$ Heating surface area, A

$$= \frac{0.695 \times 696.67}{800} = 0.605\ m^2$$

The second part of the problem can be worked out more easily by using the effectiveness - NTU approach.

Since the flow rate is doubled on the hot side,

$$h_h = 1600 \times (2)^{0.8} = 2786\ W/m^2K$$

$$\frac{1}{U} = \frac{1}{h_h} + \frac{1}{h_c}$$

or $\quad U = \frac{h_c h_h}{h_c + h_h} = \frac{1600 \times 2786}{1600 + 2786} = 1016\ W/m^2K$

$$m_h c_h = \frac{1200}{3600} \times 4180 = 1393.33\ W/K$$

$m_c c_c = 1741.67\ W/K$ (No change)

Capacity ratio $C = 1393.33/1741.67 = 0.801$

Transfer units NTU

$$= \frac{UA}{C_{min}} = \frac{1016 \times 0.606}{1393.33} = 0.440$$

Invoking the effectiveness relation for parallel flow heat exchanger,

$$\epsilon = \frac{1-\exp[-NTU(1+C)]}{1+C}$$

$$= \frac{1-\exp[-0.440(1+0.801)]}{1+0.801}$$

$$= 0.305$$

$$\therefore\ \frac{t_{h1} - t_{h2}}{t_{h1} - t_{c1}} = 0.305 \quad \text{or} \quad \frac{70 - t_{h2}}{70-25} = 0.305$$

Upon solution we get ;

$$t_{h2} = 70 - 0.305 \times 45 = 56.27°C$$

From energy balance,

$$m_c c_c (t_{c2} - t_{c1}) = m_h c_h (t_{h1} - t_{h2})$$

or $\quad t_{c2} = t_{c1} + \frac{m_h}{m_c} (t_{h1} - t_{h2})$

(because $c_c = c_h$)

$$= 25 + \frac{1200}{1500}(70 - 56.27)$$

$$= \mathbf{35.98°C}$$

EXAMPLE 14.51.

In a 1-2 shell and tube type steam condenser employed in a large steam power plant, steam (shell side) and cooling water (tube side) are in counter flow. The steam condenses on the outer surface of tubes with condensation temperature 50°C and heat transfer coefficient 11500 W/m²K. The water enters the 30000 thin walled tubes of 25 mm diameter (each executing two passes) with inlet temperature 20°C and mass flow rate 30 × 10³ kg/s. If the heat exchange rate is 2000 MW, make calculations for the outlet temperature of cooling water and the length of tube per pass.

Use LMTD correction factor method and NTU method. Neglect the thermal resistance of tube material and fouling effects and take the following thermo-physical properties of water :

$$c = 4180\ J/kgK$$

$$\mu = 850 \times 10^{-6}\ Ns/m^2$$

and $\quad k = 0.6\ W/m\text{-}deg$

Solution : Refer Fig. 14.29 for the flow arrangement and temperature distribution of hot fluid (steam) and cold fluid (water) in a shell and tube type condenser.

From energy balance,

$$Q = m_c c_c (t_{c2} - t_{c1})$$

$$2000 \times 10^6 = 30 \times 10^3 \times 4180 \times (t_{c2} - 20)$$

$\therefore$ Water outlet temperature, tc_2

$$= \frac{2000 \times 10^6}{30 \times 10^3 \times 4180} + 20 = 35.95°C$$

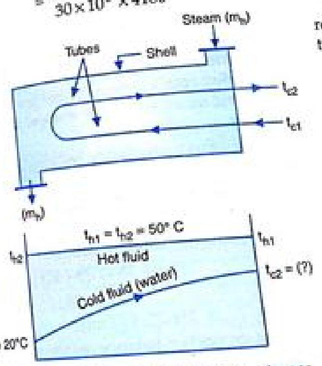


Fig. 14.29. Shell and tube type condenser

(a) LMTD correction factor method

Mass flow rate of water through each tube,

$$m = \frac{30 \times 10^3}{30000} = 1\ kg/s$$

Reynolds number, Re

$$= \frac{\rho V d}{\mu} = \frac{4m}{\pi d \mu}$$

$$= \frac{4 \times 1}{\pi \times 0.025 \times 850 \times 10^{-6}}$$

$$= 59948$$

Prandtl number, Pr

$$= \frac{\mu c_p}{k} = \frac{850 \times 10^{-6} \times 4180}{0.6}$$

$$= 5.92$$

Since $Re > 2300$, the flow is turbulent and accordingly the following correlation applies for calculating the heat transfer coefficient on the water side.

$$N_u = \frac{h_i d_i}{k}$$

$$= 0.023\ (R_e)^{0.8} \times (P_r)^{0.4}$$

$$= 0.023\ (59948)^{0.8} \times (5.92)^{0.4}$$

$$= 311.13$$

$$\therefore\ h_i = N_u \times \frac{k}{d_i} = 311.13 \times \frac{0.6}{0.025}$$

$$= 7467\ W/m^2K$$

Neglecting fouling effects and thermal resistance of tube material, the overall heat transfer coefficient is

$$\frac{1}{U} = \frac{1}{h_i} + \frac{1}{h_o};$$

$$U = \frac{h_o \times h_i}{h_o + h_i} = \frac{11500 \times 7467}{11500 + 7467}$$

$$= 4527\ W/m^2K$$

The logarithmic mean temperature difference for counter flow arrangement,

$$\theta_1 = t_{h2} - t_{c1} = 50 - 20 = 30°C$$

$$\theta_2 = t_{h1} - t_{c2} = 50 - 35.95 = 14.05°C$$

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{30 - 14.05}{\log_e \frac{30}{14.05}}$$

$$= 21.02°C$$

The parameters required to obtain the correction factor F are :

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{35.95 - 20}{50 - 20} = 0.5317$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{50 - 50}{35.95 - 20} = 0$$

Using these parameters with Fig. 14.27, the correction factor is read as $F = 1.0$.

Heat exchange, $Q = F\,U\,A\,\theta_m$

$\therefore$ Heat surface area, A

$$= \frac{Q}{F\,U\,\theta_m} = \frac{2000 \times 10^6}{1 \times 4527 \times 21.02}$$
$$= 21018 \text{ m}^2$$

The heating surface area also equals $(N \times \pi d l)$ where d and l represent the tube diameter and length respectively and N is the number of tubes.

$\therefore$ Total length of tube l

$$= \frac{A}{N \times \pi d} = \frac{21018}{30000 \times \pi \times 0.025}$$
$$= 8.924 \text{ m}$$

Since there are two tube passes,

$$\text{Tube length per pass} = \frac{8.924}{2} = 4.462 \text{ m}$$

(b) NTU method

Since the heat exchanger is a condenser,

$$C_h = C_{max} = \infty$$
$$C_{min} = C_c = m_c\,c_c = 30 \times 10^3 \times 4180$$
$$= 12558 \times 10^4$$

In terms of temperature differences and thermal capacity rates, effectiveness is

$$\epsilon = \frac{C_h(t_{h1} - t_{h2})}{C_{min}(t_{h1} - t_{c1})}$$
$$= \frac{C_c(t_{c2} - t_{c1})}{C_{min}(t_{h1} - t_{c1})}$$
$$= \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} \qquad (\because C_c = C_{min})$$
$$= \frac{35.95 - 20}{50 - 20} = 0.5317$$

With boiling or condensation process, $C_{min}/C_{max} \to 0$ and heat exchanger effectiveness is prescribed by the equation

$$\epsilon = 1 - \exp(-NTU);$$
$$0.5317 = 1 - \exp(-NTU)$$
$$\text{or } \exp(-NTU) = 1 - 0.5317 = 0.4683$$
$$\text{or } -NTU = \log_e 0.4683 = -0.7586;$$
$$\therefore \; NTU = 0.7586$$

But $NTU = \dfrac{UA}{C_{min}} = \dfrac{4527\,A}{1255 \times 10^4}$

$\therefore$ Heating surface area, A

$$= \frac{0.7586 \times 12558 \times 10^4}{4527}$$
$$= 21044 \text{ m}^2$$

Total length of tube, l

$$= \frac{A}{n \times \pi d} = \frac{21044}{30000 \times \pi \times 0.025}$$
$$= 8.936 \text{ m}$$

$\therefore$ Tube length per pass

$$= \frac{8.936}{2} = 4.468 \text{ m}$$

EXAMPLE 14.52.

A parallel flow heat exchanger is to be designed to cool oil (c_p = 2.0 kJ/kg K) from 120°C to 85°C by the flow of water. The water flows at the rate of 75 kg/min and gets heated from 40°C to 75°C. What heat exchanger area is required for an overall heat transfer coefficient of 0.35 kW/m²K?

A change in the operating conditions occurs and the water flow rate drops to 50 kg/min for the same oil flow rate. Workout the exit temperature of the oil and water under the changed conditions. Comment upon the result.

Solution : From an energy balance on the hot (oil) and cold (water) fluids:

$$m_h\,c_h\,(t_{h1} - t_{h2}) = m_c\,c_c\,(t_{c2} - t_{c1})$$
$$m_h \times 2.0 \times (120 - 85) = 75 \times 4.18 \times (75 - 40)$$

Therefore the oil flow rate is,

$$m_h = \frac{75 \times 4.18 \times (75 - 40)}{2.0 \times (120 - 85)}$$
$$= 156.75 \text{ kg/min}$$

From an energy balance on the cold fluid, the heat transfer is :

$$Q = m_c\,c_c\,(t_{c2} - t_{c1})$$
$$= 75 \times 4.18 \times (75 - 40)$$
$$= 10972.5 \text{ kJ/min} = 182.27 \text{ kJ/s}$$

For the parallel flow arrangement, the end temperature difference are :

$$\theta_1 = t_{h1} - t_{c1} = 120 - 40 = 80°\text{C}$$
$$\theta_2 = t_{h2} - t_{c2} = 85 - 75 = 10°\text{C}$$

$$\therefore \quad \theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}} = \frac{80 - 10}{\log_e \frac{80}{10}}$$
$$= 33.67°\text{C}$$

Now, the heat exchange, $Q = U\,A\,\theta_m$

$\therefore$ Heating surface area, A

$$= \frac{Q}{U\,\theta_m} = \frac{182.87}{0.35 \times 33.67}$$
$$= 15.52 \text{ m}^2$$

(b) Thermal capacity rate of the hot (oil) and cold (water) fluids are :

$$C_h = m_h\,c_h = \frac{156.75}{60} \times 2.0$$
$$= 5.22 \text{ kJ/sK}$$

$$C_c = m_c\,c_c = \frac{50}{60} \times 4.18 = 3.48 \text{ kJ/sK}$$

Obviously,

$C_{min} = 3.48$ kJ/sK

and $C_{max} = 5.22$ kJ/sK

$$\text{Capacity ratio, } C = \frac{C_{min}}{C_{max}} = \frac{3.48}{5.22} = 0.66$$

Number of transfer units, NTU

$$= \frac{UA}{C_{min}} = \frac{0.35 \times 15.52}{3.48} = 1.56$$

The effectiveness for a parallel flow heat exchanger is given by

$$\epsilon = \frac{1 - \exp[-NTU\,(1 + C)]}{1 + C}$$
$$= \frac{1 - \exp[-1.56(1 + 0.66)]}{1 + 0.66}$$
$$= 0.557$$

Alternatively using the parameters C = 0.66 and NTU = 1.56 with Fig. 14.22, we get; $\epsilon = 0.55$

The cold fluid has the minimum thermal capacity and therefore in terms of temperature differences

$$\epsilon = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} \text{ or } 0.557 = \frac{t_{c2} - 40}{120 - 40}$$

$\therefore$ Outlet temperature of the cold fluid (water),

$$t_{c2} = 40 + 0.557 \times (120 - 40)$$
$$= 84.56°\text{C}$$

From energy balance on the two fluids,

$$m_h\,c_h\,(t_{h1} - t_{h2}) = m_c\,c_c\,(t_{c2} - t_{c1})$$
$$156.75 \times 2.0 \times (120 - t_{h2})$$
$$= 50 \times 4.18 \times (84.56 - 40)$$

$\therefore$ Outlet temperature of the hot fluid (oil),

$$t_{h2} = 120 - \frac{50 \times 4.18 \times (84.56 - 40)}{156.75 \times 20}$$
$$= 90.29°\text{C}$$

The heat transfer under changed flow conditions is

$$t_{h2} = m_c\,c_c\,(t_{c2} - t_{c1})$$
$$= \frac{50}{60} \times 4.18 \times (84.56 - 40)$$
$$= 155.21 \text{ kJ/s}$$

% age reduction in mass flow rate of coolant

$$\frac{75 - 50}{75} \times 100 = 33\%$$

% age reduction in the heat exchange rate,

$$= \frac{182.87 - 155.21}{182.87} \times 100 = 15\%$$

Obviously the heat exchanger works more effectively at reduced rate of the coolant.

EXAMPLE 14.53.

A single-shell pass, four-tube counter flow heat exchanger is used as an economiser on a steam generator. Flue gases (c_p = 1.06 kJ/kg K) enter the exchanger at 250°C and leave at 150°C with a flow rate of 0.5 kg/s. The feed water enters at 125°C at the rate of 0.35 kg/s. Determine the number of transfer units and the effectiveness of the heat exchanger.

(b) A change in the operating conditions occurs; the feed heater must be by-passed so the water enters at 60°C with a flow rate of 0.30 kg/s. Workout the exit temperature of water when the exchanger operates under the changed conditions. Presume that the effectiveness of exchanger remains unchanged.

Solution : From energy balance on the hot fluid (flue gases) and the cold fluid (water);

$$m_h c_h (t_{h1} - t_{h2}) = m_c c_c (t_{c2} - t_{c1})$$
$$0.5 \times 1.05 \times (250 - 150) = 0.35 \times 4.18 \times (t_{c2} - 125)$$

$\therefore$ Outlet temperature of water,

$$t_{c2} = 125 + \frac{0.5 \times 1.05 \times (250 - 150)}{0.35 \times 4.18} = 160.85°C$$

The parameters required to get the correction factor are :

$$P = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{c1}} = \frac{160.85 - 125}{250 - 125} = 0.287$$

$$Z = \frac{t_{h1} - t_{h2}}{t_{c2} - t_{c1}} = \frac{250 - 125}{250 - 125} = 0.279$$

Using these parameters with Fig. 14.17, we read $F = 0.78$.

For the conventional counter flow arrangement,

$$\theta_1 = t_{h1} - t_{c2} = 250 - 160.85 = 89.15°C$$
$$\theta_2 = t_{h2} - t_{c1} = 150 - 125 = 25°C$$

$$\therefore \quad \theta_m = \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}} = \frac{89.15 - 25}{\log_e \dfrac{89.15}{25}} = 50.47°C$$

For an energy balance on the hot fluid (air), the required heat exchange is,

$$Q = m_h c_h (t_{h1} - t_{h2}) = 0.5 \times 1.05 \times (250 - 150) = 52.5 \text{ kJ/s}$$

Also, $Q = F U A \theta_m$

$$\therefore \quad UA = \frac{Q}{F\theta_m} = \frac{52.5}{0.78 \times \times 50.47} = 1.33 \text{ kJ/sK}$$

Thermal capacity rates of the hot fluid (air) and cold fluid (water) are :

$$C_h = m_h c_h = 0.50 \times 1.05 = 0.525 \text{ kJ/sK}$$
$$C_c = m_c c_c = 0.35 \times 4.18 = 1.463 \text{ kJ/sK}$$

Obviously,

$$C_{min} = 0.525 \text{ kJ/sK}$$

and $C_{max} = 1.463$ kJ/sK

Number of transfer units, NTU

$$= \frac{UA}{C_{min}} = \frac{1.33}{0.525} = 2.53$$

Capacity ratio, C

$$= \frac{C_{min}}{C_{max}} = \frac{0.525}{1.463} = 0.359$$

Using these parameters with Fig. 14.24, we read : effectiveness $\epsilon = 0.78$

(b) With effectiveness remaining constant, the heat transfer with new set of operating conditions is

$$Q = \epsilon C_{min} (t_{h1} - t_{c1}) = 0.78 \times 0.525 (250 - 60) = 77.80 \text{ kJ/s}$$

An energy balance on water under the changed conditions gives :

$$Q = m_c c_c (t_{c2} - t_{c1})$$
$$77.80 = 0.30 - 4.18 \times (t_{c2} - 60)$$

$\therefore$ Outlet temperature of water,

$$t_{c2} = 60 + \frac{77.80}{0.30 \times 4018} = 122.04°C$$

EXAMPLE 14.54.

A counter flow heat exchanger is to be designed to cool 3600 kg/hr of oil (c_p = 2 kJ/kg K) from 150°C to 80°C with water (c_p = 4.18 kJ/kg K) whose temperature gets raised from 25°C to 90°C. For an overall heat transfer coefficient of 1800 kJ/m²-hr-K, make calculations for the mass flow rate of water and the exchanger surface area.

(b) To what temperature the oil would be cooled if the mass flow rate of water is doubled ? The values of oil flow rate, inlet oil temperature, inlet water temperature and the overall coefficient of heat transfer remain unchanged.

Solution : From the energy balance on the hot (oil) and cold (water) fluids,

$$m_h c_h (t_{h1} - t_{h2}) = m_c c_c (t_{c2} - t_{c1})$$
$$3600 \times 2 \times (150 - 80) = m_c \times 4.18 \times (90 - 25)$$

$\therefore$ Mass flor rate of cooling water,

$$m_c = \frac{3600 \times 2 \times (150 - 80)}{4.18 \times (90 - 25)} = 1855 \text{ kg/hr}$$

From energy balance on the hot fluid, the heat exchange rate is,

$$Q = m_c c_h (t_{h1} - t_{h2}) = 3600 \times 2 \times (150 - 80) = 504000 \text{ kJ/hr}$$

The heat exchange is also given by

$$Q = U A \theta_m = UA \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$$

For the counter flow arrangement

$$\theta_1 = t_{h1} - t_{c2} = 150 - 90 = 60°C$$
$$\theta_2 = t_{h2} - t_{c1} = 80 - 25 = 55°C$$

$$\therefore \quad \theta_m = \frac{\theta_1 - \theta_2}{\log_e \dfrac{\theta_1}{\theta_2}} = \frac{60 - 55}{\log_e \dfrac{60}{55}} = 57.47°C$$

$\therefore$ Heat exchange surface, A

$$= \frac{Q}{U\theta_m} = \frac{504000}{1800 \times 57.47} = 4.872 \text{ m}^2$$

(b) Mass flow rate of oil

$$= 3600 \text{ kg/hr}$$

mass flow rate of water

$$= \text{twice the original value} = 2 \times 1855 = 3710 \text{ kg/hr}$$

From energy balance,

$$3600 \times 2 \times (150 - t_{h2}) = 3710 \times 4.18 (t_{c2} - 25)$$
$$t_{h2} + 2.16\, t_{c2} = 204.5 \quad \ldots(i)$$

Further,

$$Q = U A \theta_m = m_h c_h (t_{h1} - t_{h2})$$

$$\therefore \quad UA \frac{(t_{h1} - t_{c2}) - (t_{h2} - t_{c1})}{\log_e\left[(t_{h1} - t_{c2})/(t_{h2} - t_{c1})\right]} = m_h c_h (t_{h1} - t_{h2})$$

$$\text{or } 1800 \times 4.872 \times \frac{(150 - t_{c2}) - (t_{h2} - 25)}{\log_e\left[(150 - t_{c2})/(t_{h2} - 25)\right]} = 3600 \times 2 \times (150 - t_{h2})$$

$$t_{h2} + 1.2175 \times \frac{(150 - t_{c2}) - (t_{h2} - 25)}{\log_e\left[(150 - t_{c2})/(t_{h2} - 25)\right]} = 150 \quad \ldots(ii)$$

The expressions (i) and (ii) have two unknowns t_{h2} and t_{c2}; the solution is obtained by the iterative procedure outlined below :

(i) Assume t_{h2}

(ii) Use expression (i) to obtain t_{c2} for the assumed value of t_{h2}

(iii) Substitute t_{h2} and t_{c2} to check the validity of expression (ii)

Iteration number	Assumed t_{h2}	Expression (i) gives t_{c2}	Expression (ii) results in
1	70°C	62.08°C	148 ≠ 150
2	72°C	61.16°C	152 ≠ 150
3	71°C	61.62°C	149.5= 150

Therefore the outlet temperature of the hot fluid (oil) is 71°C and that of the cold fluid (water) is 61.62°C.

NTU-Effectiveness Method : Thermal capacity rates of the hot (oil) and cold (water) fluid are

$$C_h = m_h c_h = 3600 \times 2 = 7200 \text{ kJ/hrK}$$
$$C_c = m_c c_c = 3710 \times 4.2 = 15582 \text{ kJ/hrK}$$

Obviously,

$$C_{min} = 7200 \text{ kJ/hr}$$

and $C_{max} = 15582$ kJ/hrK

$$\text{Capacity ratio } C = \frac{C_{min}}{C_{max}} = \frac{7200}{15582} = 0.462$$

Number of transfer units, NTU

$$= \frac{UA}{C_{min}} = \frac{1800 \times 4.872}{7200} = 1.2175$$

Effectiveness $\in$

$$= \frac{1-\exp\left[-NTU(1-C)\right]}{1-C\exp\left[-NTU(1+C)\right]}$$

$$= \frac{1-\exp\left[-1.2175(1-0.462)\right]}{1-0.462\exp\left[-1.2175(1-0.662)\right]}$$

$$= 0.632$$

In terms of temperature differences,

$$\in = \frac{C_h(t_{h1}-t_{h2})}{C_{min}(t_{h1}-t_{c1})}$$

$$= \frac{C_c(t_{c2}-t_{c1})}{C_{min}(t_{h1}-t_{c1})}$$

Since $C_h = C_{min}$, we have :

$$0.632 = \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}} = \frac{150-t_{h2}}{150-25}$$

Therefore, the outlet temperature of the hot fluid is :

$$t_{h2} = 150 - (150 - 25) \times 0.632$$
$$= 70.96°C$$

EXAMPLE 14.55.

In what situations does the 'effectiveness' approach to heat exchanger calculations have advantage over the 'log mean-temperature-difference' approach?

Oil with a mean specific heat of 2.5 kJ/kgK is to be cooled from 110°C to 30°C in a single pass counter flow heat exchanger. The coolant is water which enters at 20°C and leaves at 80°C and the overall heat transfer coefficient for this type of heat exchanger is 1.5 kW/m²K. If the water flow rate is 1500 kg/hr, determine the quantity of oil that can be cooled per hour and the heat exchanger area. (b) What would be the fluid exit temperatures when the water flow rate is decreased to 1000 kg/hr for the same oil flow rate? Comment upon the results.

The effectiveness of a counter flow heat exchanger is given by the following expression where NTU is the number of transfer units and C is the capacity rate ratio

$$\in = \frac{1-\exp\left[-NTU(1-C)\right]}{1-C\exp\left[-NTU(1-C)\right]}$$

Solution : The mass flow rate of oil can be determined from an energy balance on the two fluids, *i.e.*,

heat lost by oil (hot fluid)

= heat gained by water (coolant)

$$m_h\, c_h\, (t_{h1} - t_{h2}) = m_c\, c_c\, (t_{c2} - t_{c1})$$
$$m_h \times 2.5\, (110 - 30)$$
$$= 1500 \times 4.186\, (80 - 20)$$
$$= 376740 \text{ kJ/hr}$$

Therefore the oil flow rate is,

$$m_h = \frac{376740}{2.5\times 80} = 1884 \text{ kg/hr}$$

Thermal capacity rate of the hot (oil) and cold (water) fluid are :

$$C_h = m_h\, c_h = 1884 \times 2.5$$
$$= 4710 \text{ kJ/hrK}$$
$$C_c = m_c\, c_c = 1500 \times 4.186$$
$$= 6279 \text{ kJ/hrK}$$

Obviously,

$$C_{min} = 4710 \text{ kJ/hrK}$$
and $C_{max} = 6279$ kJ/hrK

$$\text{Capacity ratio } C = \frac{C_{min}}{C_{max}} = \frac{4710}{6279} = 0.75$$

When the hot fluid (oil) has the minimum thermal capacity, then

Effectiveness $\in$

$$= \frac{t_{h1}-t_{h2}}{t_{h1}-t_{c1}} = \frac{110-30}{110-20} = 0.889$$

Rearrangement of the given expression gives :

$$\frac{\in-1}{\in C-1} = \exp\left[-NTU\,(1-C)\right] \quad ...(i)$$

$$\frac{0.889-1}{0.889\times 0.75-1} = \exp\left[-NTU\,(1-0.75)\right]$$

$$\text{or } \log_e\frac{0.111}{0.333} = -0.25\,NTU \text{ or } NTU = 4.39$$

$$\text{But } NTU = \frac{UA}{C_{min}}$$

$\therefore$ Heat transfer area, A

$$= \frac{NTU\times C_{min}}{U}$$

$$= \frac{4.39\times(4710/3600)}{1.5} = 3.83 \text{ m}^2$$

Note : The heat exchange area can be found more conveniently by the *LMTD* approach.

For the counter flow arrangement, the end temperature difference are :

$$\theta_1 = t_{h1} - t_{c2} = 110 - 80 = 30°C$$
$$\theta_2 = t_{h2} - t_{c1} = 30 - 20 = 10°C$$

$$\therefore \quad \theta_m = \frac{\theta_1-\theta_2}{\log_e\frac{\theta_1}{\theta_2}} = \frac{30-10}{\log_e\frac{30}{10}}$$
$$= 18.20°C$$

Now, the heat exchange, $Q = U A\, \theta_m$

$\therefore$ Heating surface area, A

$$= \frac{Q}{U\theta_m} = \frac{376740/3600}{1.5\times 18.20}$$
$$= 3.83 \text{ m}^2$$

(b) The changed thermal capacity rates of the hot (oil) and cold (water) fluid are :

$$C_h = 1884 \times 2.5 = 4710 \text{ kJ/hrK}$$
$$C_c = 1000 \times 4.186 = 4186 \text{ kJ/hrK}$$

$$\text{Capacity ratio } C = \frac{C_{min}}{C_{max}} = \frac{4186}{4710} = 0.889$$

$$NTU = \frac{UA}{C_{min}} = \frac{1.5\times 3.83}{4186/3600} = 4.94$$

Inserting the appropriate values in expression (i),

$$\frac{\in-1}{0.889\in-1} = \exp\left[-4.94\,(1-0.889)\right] = 0.578$$
$$\in - 1 = 0.514 \in - 0.578$$

$$\therefore \text{ Effectiveness } \in = \frac{1-0.578}{1-0.514} = 0.868$$

When the coolant (water) has the minimum thermal capacity, then

$$\in = \frac{t_{c2}-t_{c1}}{t_{h1}-t_{c1}} \text{ or } 0.868 = \frac{t_{c2}-20}{110-20}$$

$\therefore$ Outlet temperature of the coolant (water)

$$t_{c2} = 0.868\,(110 - 20) + 20 = 98.12°C$$

From energy balance on the two fluids,

$$m_h\, c_h\, (t_{h1} - t_{h2}) = m_c\, c_c\, (t_{c2} - t_{c1})$$
$$1884 \times 2.5\, (110 - t_{h2})$$
$$= 1000 \times 4.186\, (98.12 - 20)$$

$$= 327010 \text{ kJ/hr}$$

$\therefore$ Outlet temperature of the hot fluid (oil),

$$t_{h2} = 110 - \frac{327010}{1884\times 2.5} = 40.57°C$$

Percentage reduction in mass flow rate of coolant

$$= \frac{1500-1000}{1500} = 33\%$$

Percentage reduction in the heat exchange rate,

$$= \frac{376740-327010}{376740} = 13.2\%$$

Obviously the heat exchanger works more effectively at reduced flow rate of coolant.

EXAMPLE 14.56.

A concurrent (parallel flow) heat exchanger of 1m length cools oil from 150°C to 100°C by a stream of cooling water that enters the cooler at 15°C and leaves at 25°C. Subsequently the process conditions demand that the oil be cooled to 75°C, and the design engineer suggests that this be done by lengthening the cooler. If the oil and water flow rates, their inlet temperatures and other dimensions remain unchanged, then determine the length and outlet temperature of cooling water of the new cooler.

Solution : Refer Fig. 14.30 for the temperature profiles of oil and water before and after increasing the cooler length.

Case I. Before increasing the length

From energy balance,

$$m_h\, c_h\, (t_{h1} - t_{h2}) = m_c\, c_c\, (t_{c2} - t_{c1})$$

The subscripts h and c represent the hot (oil) and cold (water) fluids respectively.

$$\frac{m_h c_h}{m_c c_c} = \frac{C_h}{C_c} = \frac{t_{c2}-t_{c1}}{t_{h1}-t_{h2}}$$
$$= \frac{25-15}{150-100} = 0.2$$

Obviously oil is the minimum fluid, *i.e.*,

$$C_{min} = m_h\, c_h \text{ and capacity ratio } C = \frac{C_{min}}{C_{max}} = 0.2$$

Heat transfer rate is given by

$$Q = m_h\, c_h\, (t_{h1} - t_{h2}) = U A_i\, \theta_m$$

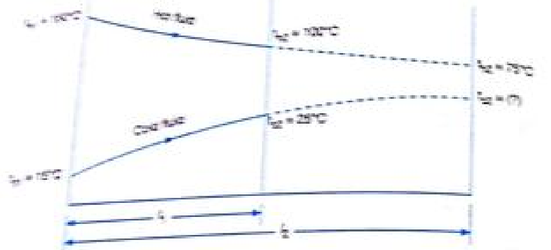


Fig. 14.30.

$$\text{or } t_{h1} - t_{h2} = \frac{UA_1}{C_{min}}\,\theta_m$$

$$= (NTU)_1 \times \theta_m$$

$$= (NTU)_1 \times \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$\text{or } (NTU)_1 = (t_{h1} - t_{h2}) \times \frac{\log_e \frac{\theta_1}{\theta_2}}{\theta_1 - \theta_2}$$

where $\theta_1 = t_{h1} - t_{c1} = 150 - 15 = 135°C$

$\theta_2 = t_{h2} - t_{c2} = 100 - 25 = 75°C$

$$\therefore (NTU)_1 = (150 - 100) \times \frac{\log_e \frac{135}{75}}{135 - 75}$$

$$= 0.49$$

$$\text{or } \frac{UA_1}{C_{min}} = 0.49\ ; \quad \frac{U(\pi d \times l_1)}{C_{min}} = 0.49$$

$$\therefore \frac{U\pi d}{C_{min}} = 0.49 \quad \text{as} \quad l_1 = 1 \text{ m}$$

Case II. After increasing the length,

With increase in length of heat exchanger,

$$(NTU)_2 = \frac{UA_2}{C_{min}} = \frac{U\pi d\, l_2}{C_{min}}$$

$$= \left(\frac{U\pi d}{C_{min}}\right) l_2 = 0.49\, l_2$$

In terms of *NTU* and capacity ratio C, the effectiveness of a parallel flow heat exchanger is

$$\epsilon = \frac{1-\exp[(-NTU)_2(1+C)]}{1+C}$$

$$= \frac{1-\exp[-0.49 l_2(1+0.2)]}{1+0.2}$$

$$= \frac{1-\exp(-0.588 l_2)}{1.2} \qquad ...(i)$$

In terms of capacity rates and temperature, the effectiveness is,

$$\epsilon = \frac{C_h(t_{h1} - t'_{h2})}{C_{min}(t_{h1} - t_{c1})} = \frac{C_c(t'_{c2} - t_{c1})}{C_{min}(t_{h1} - t_{c1})}$$

Since $C_h = C_{min}$; $C_c = C_{max}$ and

$$C = \frac{C_{min}}{C_{max}} = 0.2$$

$$\epsilon = \frac{t_{h1} - t'_{h2}}{t_{h1} - t_{c1}} = \frac{1}{0.2} \times \frac{t'_{c2} - t_{c1}}{t_{h1} - t_{c1}}$$

Substituting the relevant data,

$$\epsilon = \frac{150-75}{150-15} = \frac{1}{0.2} \times \frac{t'_{c2} - 15}{150-15} \qquad ...(ii)$$

$$\therefore t'_{c2} = \left(\frac{150-75}{150-15}\right) \times 0.2 \times (150 - 15) + 15$$

$$= 30°C$$

Equating expression (i) and (ii)

$$\epsilon = \frac{1-\exp(-0.588 l_2)}{1.2} = \frac{150-75}{150-15}$$

or $\exp(-0.588\, l_2)$

$$= 1 - \frac{150-75}{150-15} \times 1.2 = 0.333$$

or $-0.588\, l_2 = \log_e 0.333 = -1.0996$

$\therefore$ Length of the new cooler, l_2

$$= \frac{1.0996}{0.588} = 1.87 \text{ m}$$

EXAMPLE 14.57. *In a double pipe heat exchanger $m_h\, c_{ph} = 0.5\, m_c\, c_{pc}$ and the inlet temperatures of hot and cold fluids are t_{h1} and t_{c1} respectively. Set up an expression in terms of the t_{c1} and t_{h2} for the ratio of the area of the counter flow heat exchanger to that of parallel flow heat exchanger which will give the same hot fluid outlet temperature t_{h2}. Determine this ratio if $t_{h1} = 150°C$, $t_{c1} = 30°C$ and $t_{h2} = 90°C$.*

Solution : Given,

$$m_h\, c_{ph} = 0.5\, m_c\, c_{pc}\ ;\ C_h = 0.5\, C_c$$

Obviously

$$C_{min} = C_h = 0.5\, C_c$$

and $\quad C_c = C_{max} = 2C_h$

The effectiveness of a heat exchanger is given by

$$\epsilon = \frac{C_h(t_{h1} - t_{h2})}{C_{min}(t_{h1} - t_{c1})}$$

$$= \frac{C_c(t_{c2} - t_{c1})}{C_{min}(t_{h1} - t_{c1})}$$

$$= \frac{(t_{h1} - t_{h2})}{(t_{h1} - t_{c1})} = \frac{2C_h(t_{c2} - t_{c1})}{C_h(t_{h1} - t_{c1})} = \frac{2(t_{c2} - t_{c1})}{(t_{h1} - t_{c1})}$$

From (i) and (ii) :

$$t_{c2} = t_{c1} + 0.5\,(t_{h1} - t_{h2})$$

This relation is true both for counter flow and parallel flow arrangements. Further, equivalence of hot fluid outlet temperature implies that heat lost by hot fluid is same in both the arrangements. Again, overall coefficient of heat transfer is not dependent on the direction of flow. Accordingly we may write

$$Q = UA_p\,(\theta_m)_p = UA_c\,(\theta_m)_c$$

$$\therefore \frac{A_c}{A_p} = \frac{(\theta_m)_p}{(\theta_m)_c} \qquad ...(i)$$

(i) Parallel flow arrangement

$$\theta_1 = t_{h1} - t_{c1}$$

$$\theta_2 = t_{h2} - t_{c2}$$

$$= t_{h2} - [t_{c1} + 0.5\,(t_{h1} - t_{h2})]$$

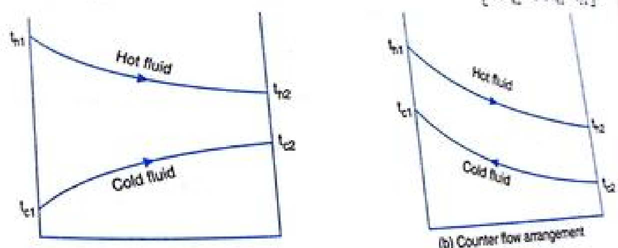
$$= 1.5\, t_{h2} - 0.5\, t_{h1} - t_{c1}$$

$$\therefore \theta_1 - \theta_2 = t_{h1} - t_{c1} - [1.5\, t_{h2} - 0.5\, t_{h1} - t_{c1}]$$

$$= 1.5\,(t_{h1} - t_{h2})$$

$$(\theta_m)_p = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{1.5(t_{h1} - t_{h2})}{\log_e\left[\frac{(t_{h1} - t_{c1})}{1.5 t_{h2} - 0.5 t_{h1} - t_{c1}}\right]}$$



(a) Parallel flow arrangement

(b) Counter flow arrangement

Fig. 14.31.

(b) **Counter flow arrangement**

$$\theta_1 = t_{h1} - t_{c2}$$
$$= t_{h1} - [t_{c1} + 0.5\,(t_{h1} - t_{h2})]$$
$$= 0.5\,t_{h1} + 0.5\,t_{h2} - t_{c1}$$
$$\theta_2 = t_{h2} - t_{c1}$$
$$\therefore \theta_1 - \theta_2 = (0.5\,t_{h1} + 0.5\,t_{h2} - t_{c1}) - (t_{h2} - t_{c1})$$
$$= 0.5\,(t_{h1} - t_{h2})$$
$$(\theta_m)_c = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$
$$= \frac{0.5\,(t_{h1} - t_{h2})}{\log_e\left[\frac{0.5\,(t_{h1} + t_{h2}) - t_{c1}}{(t_{h2} - t_{c1})}\right]}$$

Substituting the values of $(\theta_m)_p$ and $(\theta_m)_c$ in expression (a)

$$\frac{A_c}{A_p} = \frac{1.5(t_{h1} - t_{h2})}{\log_e\left[\frac{(t_{h1} - t_{c1})}{1.5t_{h2} - 0.5t_{h1} - t_{c1}}\right]} \times \frac{\log_e\left[\frac{0.5(t_{h1} + t_{h2}) - t_{c1}}{t_{h2} - t_{c1}}\right]}{0.5(t_{h1} - t_{h2})}$$

$$= 3\left[\frac{\log_e\left[\frac{0.5(t_{h1} + t_{h2}) - t_{c1}}{(t_{h2} - t_{c1})}\right]}{\log_e\left[\frac{t_{h1} - t_{c1}}{1.5t_{h2} - 0.5t_{h1} - t_{c1}}\right]}\right]$$

which is the required expression.

Substituting the data given :

$$t_{h1} = 150°\text{C},\ t_{c1} = 30°\text{C and } t_{h2} = 90°\text{C}$$

$$\frac{A_c}{A_p} = 3\left[\frac{\log_e\left[\frac{0.5(150 + 90) - 30}{90 - 30}\right]}{\log_e\left[\frac{(150 - 30)}{1.5 \times 90 - 0.5 \times 150 - 30}\right]}\right]$$

$$= 3\left[\frac{\log_e(90/60)}{\log_e(120/30)}\right]$$

$$= 3 \times \frac{0.4054}{1.3863} = \mathbf{0.877}$$

EXAMPLE 14.58.

List the basic considerations in the design of heat exchangers.

A heat exchanger has been designed for specified total gas flow rate m and the mass flow rate per unit area G flowing through the tubes in a single pass. The vapour condensing on the shell side remains at constant pressure. Presuming that G, m, heat exchange rate Q and other physical properties are constant, explore the effect of increase in tube diameter d on each of the following : (i) number of tubes needed (ii) heat transfer coefficient obtained (iii) total heat transfer surface needed and (iv) length of tubes required.

Solution : Some basic considerations in the design of heat exchangers are :

(i) heat transfer requirement, *i.e.*, the capacity to accomplish the required heat exchange rate

(ii) high overall heat transfer coefficient; minimum inside and outside fouling resistances

(iii) physical size (space requirement) and adoptability to existing set up

(iv) minimum pressure drop and friction losses, otherwise the cost of pumping the fluid through pipes and tubes would become quite high

(v) high weight and low initial and maintenance costs

(b) Both m and G are constant for the gas flow :

$$\frac{m}{G} = \text{area of flow} = \text{constant}$$

$$\therefore \frac{\pi}{4}\, d^2 n = \text{constant}$$

Apparently if the tube diameter d increases, the required number of tubes would become less.

(ii) Nusselt number Nu

$$= \frac{hd}{k} = 0.023\,(Re)^{0.8} \times (Pr)^{0.4}$$

Now, $Re = \dfrac{\rho V d}{\mu} = \dfrac{Gd}{\mu}$ and $Pr = \dfrac{\mu c_p}{k}$

$\therefore$ Convective coefficient, h

$$= \frac{k}{d} \times 0.023 \times \left(\frac{Gd}{\mu}\right)^{0.8} \left(\frac{\mu c_p}{k}\right)^{0.4}$$

Since, k, G and μ are constant, h varies as $(1/d)^{0.2}$. Thus h decreases with increase in d

(iii) Heat exchange rate, $Q = UA\theta_m$

Q and θ_m are constant and therefore,

$$UA = \text{constant}$$

The overall heat transfer coefficient U is directly dependent upon the convective coefficient h. Therefore variation of U with d will follow the same trend, *i.e.*, U will drop if d increases. Naturally then the total heat transfer surface A increases with increase in tube diameter.

(iv) Surface area $A = n\,\pi\,d\,l$. Let the values corresponding to two different diameters be indicated by subscripts 1 and 2.

$$\frac{A_1}{A_2} = \frac{n_1 \pi\, d_1 l_1}{n_2 \pi\, d_2 l_2} = \frac{Q_1/U_1\theta_{m_1}}{Q_2/U_2\theta_{m_2}}$$

$$= \frac{U_2}{U_1} = \frac{h_2}{h_1} = \left(\frac{d_1}{d_2}\right)^{0.2}$$

or $$\frac{l_1}{l_2} = \frac{d_2}{d_1} \times \frac{n_2}{n_1} \times \left(\frac{d_1}{d_2}\right)^{0.2}$$

Further,

$$n_1 = \frac{4 \times \text{flow area}}{\pi d_1^2}$$

and $$n_2 = \frac{4 \times \text{flow area}}{\pi d_2^2}$$

or $$\frac{n_2}{n_1} = \left(\frac{d_1}{d_2}\right)^2$$

$$\therefore \quad \frac{l_1}{l_2} = \left(\frac{d_2}{d_1}\right) \times \left(\frac{d_1}{d_2}\right)^2 \times \left(\frac{d_1}{d_2}\right)^{0.2}$$
$$= \left(\frac{d_1}{d_2}\right)^{1.2}$$

Obviously the tube length increases with increase in tube diameter.

SALIENT POINTS

1. A heat exchanger is a device in which two fluid streams, one hot and another cold, are brought into thermal contact in order to affect transfer of heat from the hot to the cold fluid.
2. Heat exchangers are classified on the basis of
   A. Nature of heat exchange process
      (i) Direct contact or open heat exchangers
      (ii) Indirect contact heat exchangers which include the regenerators and recuperators or surface exchangers
   B. Relative direction of fluid motion
      (i) parallel flow or unidirectional flow
      (ii) counter flow and
      (iii) cross flow
   C. Design and constructional features
      (i) concentric
      (ii) sheel and tube
      (iii) multishell and tube passes, and
      (iv) compact heat exchangers
   D. Physical state of fluids
      (i) condensers, and
      (ii) evaporators
3. Logarithmic mean temperature difference (LMTD) is defined as that constant temperature difference which would give the same rate of the heat transfer as actually occurs under variable conditions of temperature difference. Both for parallel and counter flow double-pipe heat exchanger, LMTD (θ_m) is

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

where $\theta_1 = t_{h1} - t_{c1}$ and $\theta_2 = t_{h2} - t_{c2}$. The subscripts h and c refer to hot and cold fluids respectively, and the subscripts 1 and 2 refer to two ends of the heat exchanger.

The heat transfer rate is expressed by the relation

$$Q = U\,A\,\theta_m$$

where U is the overall heat transfer coefficient.

4. The effective temperature difference would equal θ_1 and θ_2 in case the temperature differences on either ends of the heat exchanger are equal. Further

(i) LMTD for counter flow unit is greater than that of a parallel system and accordingly the counter flow heat exchanger can transfer more heat than a similar parallel flow heat exchanger. Conversely, a counter flow heat exchanger needs a smaller heating surface for the same rate of heat transfer.

(ii) When one of the fluid remains at constant temperature during the flow passage (condenser and evaporator), both the counter flow and parallel flow arrangements give the same values for LMTD and heating surface area for a specified load.

(iii) When the temperature differences at the two ends of the heat exchanger do not differ by more than a factor of two, the arithmetic mean temperature difference,

$$AMTD = \frac{\theta_1 - \theta_2}{2}, \text{ approximately equals}$$

the LMTD.

5. The effect of multi-tubes, several shell passes or cross flow in an arrangement is considered by identifying a correction factor F such that

$$Q = F\, U\, A\, \theta_m$$

where θ_m is LMTD for a counter flow arrangement. Correction factors for several common arrangements are available in the form of charts.

6. Fouling refers to the formation of scale or deposit on a heat transfer surface. The thermal coating of the deposits is much less than that of tube wall and therefore it offers significant resistance to heat transfer.

The heat transfer resistance offered by scale is called the fouling factor. The fouling factor is zero for a new heat exchanger and it increases with time.

7. The heat exchanger effectiveness ϵ is defined as

$$\epsilon = \frac{\text{actual heat transfer}}{\text{maximum possible heat transfer}}$$

The maximum possible rate of heat transfer is obtained in a counter flow heat exchanger of infinite heat transfer area

$$(\epsilon)_{\text{parallel flow}} = \frac{1 - \exp\left[-NTU(1+R)\right]}{1+R}$$

$$(\epsilon)_{\text{counter flow}} = \frac{1 - \exp\left[-NTU(1-R)\right]}{1 - R\exp\left[-NTU(1-R)\right]}$$

where R is capacity ratio $= \frac{C_{min}}{C_{max}}$ and C defines capacity rate of a fluid flowing in the heat exchanger; C is given by the product of mass and specific heat.

NTU is number of transfer units; a dimensionless parameter defiend as

$$NTU = \frac{UA}{C_{min}}$$

The number of transfer units is a measure of the size of the exchanger.

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. (a) What is meant by a heat exchanger? State the utility and application of heat exchangers.
   (b) Point out the different criterion that form the basis for the classification of heat exchangers.
2. (a) Define the term overall heat transfer coefficient.
   (b) What is a heat exchanger? How heat exchangers are classified?
   (c) Sketch a shell and tube type heat exchanger.
3. (a) Discuss the importance of heat exchangers for industrial use.
   (b) What are the basics for the classification of heat exchangers?
   (c) Sketch a two-shell pass, four tube pass, reversed current heat exchanger. Label the different parts.
4. Sketch and point out any salient feature of
   (i) a shell-and-tube type heat exchanger
   (ii) a multi-pass heat exchanger

5. Define fouling in relation to heat exchangers. What leads to fouling?
   What is meant by fouling factor? How does it affect the performance of a heat exchanger?
6. Methyl alcohol flowing in the inner pipe of a double pipe exchanger is cooled with water flowing in jacket. The inner pipe has 33.4 mm outside diameter and is 3.38 mm thick. The thermal conductivity of steel is 45 W/m-deg. What is the overall heat transfer coefficient based on outside area of inner pipe?

Data	W/m²-deg
Alcohol coefficient	1025
Water coefficient	1420
Inside scale heat transfer coefficient	4535
Outside scale heat transfer coefficient	2265

(Ans. 367 W/m²-deg)

7. Derive an expression for the overall heat transfer coefficient of a shell and tube exchanger taking into consideration scale formation on the inside surface and film coefficients on the inside and outside surface of the tube. Under what conditions the overall coefficient reduces to $(h_i \times h_o)/(h_o + h_i)$ where h_i and h_o are the inside and outside film coefficients.
8. Show the temperature variation along the length of a heat exchanger when
   (i) hot and cold fluids flow in parallel and counter flow fashion
   (ii) steam condenses on the outside of a condenser tube with water flowing inside the tube as coolant.
   (iii) hot fluid as used for evaporating another liquid.
9. Set up expressions for logarithmic mean temperature difference in the case of a
   (i) parallel flow heat exchanger and
   (ii) counter flow heat exchanger
10. Working in terms of inlet and outlet temperatures of the fluids and overall heat transfer coefficient, develop an expression for the heat transfer from one fluid to another in a conventional (i) parallel flow and (ii) counter flow heat exchanger.
11. When one of fluid undergoes a phase change, the flow directions of the two fluids are immaterial in the calculations of logarithmic mean temperature diffenence.
    Comment on the validity of this statement.
12. What will be the value of logarithmic mean temperature difference when temperature difference θ_i between the two fluids at inlet to heat exchanger equals the temperature difference at exit from the heat exchanger.
13. State the condition when the logarithmic temperature difference approximately equals the arithmatic mean temperature difference.
14. A counter flow lubricating oil cooler of surface area 24 m² has been employed to cool 2.75×10^3 kg/hr of oil (c_p = 2.1 kJ/kg-deg) from 62.5°C at inlet to 48.5°C at discharge. The inlet and outlet temperatures of cooling water are 24°C and 32°C respectively. Make calculations for the overall heat transfer coefficient.
    What will be the required surface area for a parallel flow device having the same capacity under identical conditions?
    (Ans. 36 W/m²-deg, 23.4 m²)
15. It is desired to heat 450 kg/hr of water from 10°C to 75°C, utilizing the heat of flue gases (c_p = 0.29 kJ/kg-deg). The gases are available with a flow rate of 1800 kg/hr at a temperature of 165°C. If the overall coefficient of heat transfer is 116 W/m²-deg, determine the length of 3 cm tubing required for parallel and counter flow heat exchange.
    (Ans. 43 m, 34.5 m)
16. Two fluids pass through a heat exchanger, the cold fluid is heated from 40°C to 90°C while the hot fluid is cooled from 200°C to 100°C (a) What percentage saving in surface area is made by using a counter flow exchanger instead of a parallel flow? (b) What are the temperatures of the two fluids at the middle of the exchanger for parallel flow arrangement?
    (Ans. 34.5%, 60°C and 141.3°C)
17. Give an expression for the logarithmic mean temperature difference in the case of a counter flow double tube heat exchanger when both the fluids have the same value for the heat capacity. Further show that for steam condensing in the annulus of the tube heat exchanger, the temperature at one-half of the way along the exchanger is same whether the coolant is flowing in parallel or counter flow.
18. Steam condensing in the annular space of double pipe heat exchanger heats water as passes through the inner pipe. When fir[st] placed in service, 500 kg/hr of water w[...]

heated from 35°C to 90°C by the steam condensing at 120°C. After several weeks of operation, it was found that the 500 kg/hr of water could be heated only from 35°C to 75°C by the condensing steam. Assuming that dirt deposits on the tube has caused the change, compute the fouling factor that has developed. The condenser has 0.9 m^2 of heat transfer surface.

19. A two-pass surface condenser receives 27500 kg/hr of steam at a pressure of 0.046 bar and dryness fraction 0.92. All the steam is condensed to 27.5°C by cooling water which enters the condenser at 13.5°C and leaves at 24.5°C. The condenser tubes are 1.83 cm outside diameter and 1 mm thick and water flows through these tubes with a speed of 1.5 m/s. Assuming that overall coefficient of heat transmission is 3490 W/m^2-deg, determine: (*i*) surface area required (*ii*) length and number of tubes in each pass.

(Ans. 603.5 m^2, 4.45 m, 1165)

20. What type of heat exchanger will require the least area for conditions of heating a cold fluid from 70°C to 125°C while cooling a hot fluid from 150°C to 95°C? Explore the following configurations
(*i*) counter current flow (*ii*) two-four multipass (*iii*) four-eight multipass (*iv*) cross flow.
Assume same overall heat transfer coefficient for all exchangers.
(Conventional counter flow requires the least area. However, the economic and space considerations may cause the selection of a multipass or cross flow exchanger as being most desirable).

21. Explain the meaning of terms : heat capacity ratio, effectiveness and number of heat transfer units as applied to heat exchangers.

22. (*a*) In what situation, the effectiveness approach to heat exchanger calculations has advantages over the log-mean temperature difference approach?
(*b*) What is meant by the term minimum fluid ?

23. Define effectiveness, minimum fluid and number of transfer units in the context of heat exchanges.

24. Derive the relationship between the effectiveness and the number of transfer units for a counter flow heat exchanger.

25. The temperature of one fluid in parallel flow double pipe heat exchanger is given to be very much less than the temperature of the other fluid. From basic principles, show that the effectiveness of the exchanger approximately works out to be

$$\epsilon = 1 - \exp(-NTU)$$

26. In a gas turbine recuperator, the exhaust gases after expansion in the turbine are used to heat the compressed air so that the capacity ratio is very close to unity. Show that under this stipulation,

$\epsilon = 1 - \exp(-NTU)$ for counter flow

$\epsilon = \frac{1}{2}[1 - \exp(-2NTU)]$ for parallel flow

27. In a tubular counter current heat exchanger, 0.3 kg/s of water is heated from 40°C to 80°C by hot flue gases (c_p = 1.0 kJ/kgK) which enter at 200°C and leave at 100°C. The overall heat transfer coefficient is anticipated to be 0.2 kW/m^2K. Workout the required surface area both by the LMTD approach and the effectiveness NTU method.

(Ans. 2.86 m^2)

28. Water enters a counter flow double pipe heat exchanger at 12°C temperature and flow rate of 0.16 kg/s. The water is to cool alcohol (c_p = 2520 J/kgK) from 75°C to 35°C. The convection coefficient between alcohol and the tube wall is 340 W/m^2-deg and that between the tube wall and water is 225 W/m^2-deg. Presuming tube to be thin, make calculations for the capacity ratio, the effectiveness and the heat transfer area required.

(Ans. 0.75, 0.635, 5.31 m^2)

29. An oil with a mean specific heat of 2.5 kJ/kgK is to be cooled from 110°C to 30°C in a single pass counter flow heat exchanger. The coolant is water which enters at 20°C and leaves at 80°C. The exchanger has an overall heat transfer coefficient of 1.5 kW/m^2 K. If the water flow rate is 1500 kg/hr, determine the surface area required and the amount of water which can be cooled. What would be the exit temperature of the fluids when the water flow rate is decreased to 1000 kg/hr for the same oil flow rate? Use *NTU*-method.

30. A counter flow heat exchanger operates using river water as the coolant. During the winter water is available at 7.5°C. The exchanger is to be designed to cool 2000 kg/hr of oil (c_p = 2.5 kJ/kg K) entering at 205°C to 100°C. Specify the water flow rate and the *NTU* required to achieve the design. Will the design work in the summer when the river water is at 18.5°C?

B. Fill in the blanks with appropriate word/words :

1. A .... is a device that transfers heat from one fluid to another.
2. In a .... heat exchanger, one fluid flows through the inside tubes while the other fluid flows through the sheell.
3. A heat exchanger is of .... when the two fluids flowing along the heat transfer surface move at right angles to each other.
4. A 1 - 2 heat exchanger means .... pass on shell side and .... on tube side.
5. Radiator of an automobile engine is a .... type of heat exchange.
6. The .... denotes the heat transfer surface getting coated with deposits or scale.
7. The additional resistance to heat transfer due to scale formation on the surface of tubes of a heat exchanger is called ....
8. The effect of multi tubes, several shell passes or cross flow on the logarithmic mean temperature difference is considered by identifying a ....
9. The heat exchanger .... is defined as the ratio of the actual rate of heat transfer in a given heat exchanger to the maximum possible rate of heat transfer.
10. In heat exchanger design, the ratio of overall conductance to minimum capacity rate is called the ....

Answers : 1. heat exchanger; 2. shell-and-tube; 3. cross flow; 4. single, double; 5. cross flow; 6. fouling; 7. fouling resistance; 8. corrector factor; 9. effectiveness; 10. number of transfer units

C. Multiple choice questions :

1. Pick the odd one out
(*a*) open feed water heaters
(*b*) jet condensers
(*c*) de-super heater
(*d*) surface condenser

2. The steam condenser in a thermal power plant is a heat exchanger of the type
(*a*) direct contact (*b*) regenerator
(*c*) recuperator (*d*) none of these

3. The normal automobile radiator is a heat exchanger of the type
(*a*) direct contact (*b*) parallel flow
(*c*) counter flow (*d*) cross flow

4. Choose the correct statement with respect to a counter flow heat exchanger :
(*a*) both the fluids at inlet are in their coldest state
(*b*) both the fluids at exit are in their hottest state
(*c*) both the fluids at inlet are in their hottest state
(*d*) one fluid is hottest and the other is coldest at inlet

5. In a double pipe parallel flow heat exchanger, there occurs condensation of saturated steam over the inner tube. Subsequently, the entrance and exit connections of the cooling medium are interchanged. The ratio of steam condensation
(*a*) will increase
(*b*) will decrease
(*c*) will remain unchanged
(*d*) may increase or decrease depending upon saturated temperature of steam and inlet temperature of cooling medium

6. The requirement of transfer of a large heat is usually met by
(*a*) increasing the length of tube
(*b*) decreasing the diameter of tube
(*c*) increasing the number of tubes
(*d*) having multiple tube or shell passes

7. The shell of a tubular heat exchanger is provided with expansion bellows to
(*a*) facilitate increase in length of boiler shell
(*b*) impart structural strength to exchanger
(*c*) account for uneven expansion of shell and tube bundles
(*d*) reduce the pressure drop

8. In a heat exchanger with one fluid evaporating or condensing, the surface area required is least in

(a) parallel flow
(b) counter flow
(c) cross flow
(d) same in all the above arrangements

9. For evaporators and condensers, for the given conditions, the logarithmic mean temperature difference for parallel flow is .... that for counter flow
(a) equal to
(b) greater than
(c) smaller than
(d) any one of the above

10. In a counter flow heat exchange, cold fluid enters at 30°C and leaves at 50°C, wheras the hot fluid enters at 150 °C and leaves at 130°C. The mean temperature difference for this case is
(a) 20°C
(b) 80°C
(c) 100°C
(d) indeterminate

11. Consider the following statements :
In a shell and tube heat exchanger, baffles are provided on the shell side to
1. prevent the stagnation of shell side fluid
2. improve heat transfer
3. provide support for tubes
4. prevent fouling of tubes.
Which of these statements are correct ?
(a) 1 and 2
(b) 2 and 3
(c) 1, 2 and 3
(d) 1, 2, 3 sand 4

12. Multipass heat exchangers are used to
(a) reduce the pressure drop
(b) get a compact unit
(c) obtain high heat transfer coefficient
(d) facilitate very large temperature drop through the tube wall

13. In a heat exchanger, the hot liquid enters with a temperature of 180°C and leaves at 160°C. The cooling fluid enters at 30°C and leaves at 110°C. The capacity ratio of the heat exchanger is
(a) 0.25
(b) 1.5
(c) 0.33
(d) 0.2

14. The effectiveness of heat exchange is given by
(a) $\dfrac{t_{c_1} - t_{c_2}}{t_{h_1} - t_{c_1}}$
(b) $\dfrac{t_{h_2} - t_{h_1}}{t_{c_2} - t_{h_1}}$
(c) $\dfrac{t_{h_1} - t_{h_2}}{t_{h_1} - t_{c_1}}$
(d) $\dfrac{t_{c_1} - t_{c_2}}{t_{h_1} - t_{c_1}}$

where t_{c_1} and t_{c_2} are the temperatures of cold fluid at entry and exit respectively; t_{h_1} and t_{h_2} are temperatures of hot fluid at entry and exit point, and the cold fluid has lower heat capacity rate compared to the hot fluid.

15. A heat exchanger with heat transfer surface area A and overall heat transfer coefficient U handles two fluids of heat capacities C_{max} and C_{min}. The parameter NTU (number of transfer units) used in the analysis of heat exchanger is specified as
(a) $\dfrac{AU}{C_{min}}$
(b) AUC_{min}
(c) $\dfrac{U}{AC_{min}}$
(d) $\dfrac{AC_{min}}{U}$

16. A cross flow type air heater has an area of 50 cm^2. The overall transfer coefficeint is 100 W/m^2K and heat capacity of both hot and cold stream is 1000 W/K. The value of NTU is
(a) 1000
(b) 500
(c) 5
(d) 0.2

17. In a balanced counter flow heat exchanger with $m_h c_h = m_c c_c$, the NTU is equal to unity. What is the effectiveness of heat exchanger?
(a) 0.5
(b) 1.5
(c) 0.33
(d) 0.2

18. During the process of boiling and condensation, only a phase change takes place and one fluid remains at constant temperature throughout the heat exchanger. In terms of number of transfer units (NTU), the effectiveness of such an exchanger would be
(a) $\dfrac{NTU}{1+NTU}$
(b) $1 - exp\,(-NTU)$
(c) $\dfrac{1-exp(-2NTU)}{2}$

(d) cannot be worked out as the heat capacities are not known

19. The equation of effectiveness $\epsilon = 1 - e^{-NTU}$ for a heat exchanger is valid in case of
(a) boiler and condenser for parallel flow
(b) boiler and condenser for counter flow
(c) boiler and condenser for both parallel and counter flow
(d) gas turbine for both parallel and counter flow

20. After expansion from a gas turbine, the hot exhaust gases are used to heat the compressed air from a compressor with the help of a cross flow heat exchanger of 0.8 effectiveness. What is the number of transfer units of the heat exchanger?
(a) 2
(b) 4
(c) 8
(d) 16

21. ϵ-NTU method is particularly useful in thermal design of heat echangers when
(a) the outlet temperatures of the fluid streams are not known as a priori
(b) the outlet temperature of the fluid streams in known as a priori
(c) the outlet temperature of the hot fluid stream is known but that of cold fluid stream is not known as priori
(d) inlet temperature of the fluid streams are not known as a priori

22. Which of the following terms is not associated with heat exchangers ?
(a) fouling
(b) Mc Adam's correction factor
(c) NTU
(d) capacity ratio

Answers :

1. (d)	2. (c)	3. (d)	4. (c)	5. (c)
6. (d)	7. (c)	8. (d)	9. (a)	10. (c)
11. (b)	12. (c)	13. (a)	14. (d)	15. (a)
16. (b)	17. (a)	18. (b)	19. (c)	20. (b)
21. (a)	22. (b)			

HINTS AND COMMENTS

3(d):

In a surface condenser, the heat transfer occurs between the fluid streams without mixing or physical contact with each other. All other devices are direct contact or open heat exchangers in which the energy transfer between the hot and cold fluids is brought about through complete physical mixing; there is simultaneous transfer of heat and mass.

9(a):

Since temperature of one of the fluid remains constant during the flow passage, both the parallel flow and counter flow arrangemens would give the same values for log mean temperature difference and heating surface area for a specified load.

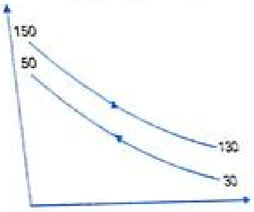
10(d):

$$\theta_1 = 150 - 50 = 100°C$$
$$\theta_2 = 130 - 30 = 100°C$$

Mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e \frac{\theta_1}{\theta_2}}$$

$$= \frac{0}{\log_e 1} = \frac{0}{0} = \text{indeterminate}$$



14.32.

However, the effective temperature difference is taken to be equal to θ_1 or θ_2 in case the temperature differences on either side of a heat exchanger are equal.

16(c):

$$NTU = \frac{AU}{C_{min}}$$
$$= \frac{50 \times 100}{1000} = 5$$

[illegible]

Since capacity ratio $\frac{C_c}{C_h}$

$$= \frac{m_c c_{pc}}{m_h c_{ph}} = 1$$

the effectiveness is given by

$$\varepsilon = \frac{NTU}{NTU+1}$$

$$= \frac{1}{1+1} = 0.5$$

[illegible]

Since capacity ratio is stated to be unity, the effectiveness is

$$\varepsilon = \frac{NTU}{NTU+1}$$

or $$0.8 = \frac{NTU}{NTU+1}$$

Solution gives: NTU = 4

[illegible]

The M_c Adam's correction factor is associated with laminar film condensation on a vertical plate.

❑❑❑

Chapter 15

Mass Transfer

Learning objectives : A study of the subject matter included in this chapter will make the readers familiar with

- *phenomenon of diffusion and mass transfer*
- *mass concentration and molar concentration in a multi-component mixture*
- *Fick's law of diffusion and steady state diffusion through a stagnant gas medium*
- *know how of non-dimensional parameters related to mass transfer and their physical significance*

When a system contains two or more components whose concentrations vary from point to point, there is a natural tendency for mass to be transferred, minimising the concentration differences within the system. The transport of one constituent from a region of higher concentration to that of lower concentration is called ***mass transfer.***

Consider the process of water evaporation from a water pool into an air stream flowing over the water surface. The concentration of water molecules would be higher just above the liquid surface compared to that in the main portion of air stream. Higher concentration means more molecules per unit volume. This concentration difference or gradient would result in the transport of water molecules from the place where its concentration is more towards the main air stream. Apparently the diffusing substance moves from a position of higher concentration to that of lower concentration so that the concentration differences within the system are reduced to minimum.

Mass transfer appears in many practical applications of chemical and mechanical engineering. ***Some examples of industrial importance of mass transfer are :***

- humidification of air in a cooling tower
- evaporation of petrol in the carburetor of a petrol engine
- evaporation of liquid ammonia in the atmosphere of hydrogen in electrolux refrigerator
- dispersion of oxides of sulphur (pollutants) from a power plant discharge of neutron in a nuclear reactor
- estimation of depth to which carbon will penetrate in a mild steel specimen during the act of carburising

Some experiences of mass transfer in our day-to-day life are :

— initial dissolution and subsequent uniform spreading of a lump of sugar added to a cup of tea

— initial diffussion (to the suface) and subsequent evaporation (into the atmosphere) of water retained into the wet clothes or a log of wood

— diffusion of smoke (from a tall chimney) into the surrounding atmosphere

— pleasant fragrance presented by a perfume which is imparted through out the surrounding atmosphere

Attempt has been made in this chapter to provide a general introduction to the mechanism of mass transfer and present some simple relations which may be used to calculate mass diffusion.

15.1. MASS TRANSFER PROCESSES: CLASSIFICATION

The mass transfer operations can be classifed into the following categories;

(i) Diffusion mass transfer ; molecular or eddy diffusion: The molecular diffusion is the transport of matter on a microscopic level as a result of diffusion from a region of high concentration to a region of low concentration in a system of a mixture of liquids or gases. The diffusion mass transfer occurs when a substance diffuses through a layer of stagnant fluid. It is independent of any convection within the system.

The molecular diffusion is further categorised into ;

— Ordinary diffusion resulting from concentration gradient; the diffusing substance moves from a position of high concentration to one of low concentration

— Thermal diffusion which may occur by virtue of temperature gradient

— Pressure diffusion relulting from hydrostatic pressure differences that provide the driving potential

— Forced diffusion which results from the action of external forces

The eddy diffusion occurs when one of the diffusing fluids is in turbulent motion. The eddying motion greatly increases the speed of mass transfer as it is in addition to molecular diffusion.

(ii) Convective mass transfer : free or forced: Mass transfer due to convection involves transfer between a moving fluid and a surface, or between two relatively immiscible moving fluids. The convective mass transfer depends both on the transport properties and on the dynamic (laminar or turbulent) characteristics of the flowing fluid.

(iii) Mass transfer by change of phase: Mass transfer also takes place whenever there is a change from one phase to another. Quite often, the mass transfer is by the simultaneous action of diffusion and convective mass transfer. For example :

- hot gases escaping from a chimney rise by convection and then diffuse into air above the chimney
- mixing of water vapour with air during evaporation of water from the lake surface is partly due to convection and partly due to diffusion.

Again, when water boils in an open air, there is first transfer of mass from liquid to vapour state (change in phase). The vapour mass from the liquid interface is next transferred to the open air both by convection as well as by diffusion.

We generally do not use the term mass transfer to describe gross or bulk fluid motion due to mechanical work such as motion of air induced by a fan, or motion of water being forced through a pipe line.

15.2. CONCENTRATIONS, VELOCITIES AND FLUXES

Some of the terms associated with different species in a multi-component mixture are defined below for better understanding of the phenomenon of mass transfer.

(a) Concentrations : The mass concentration or mass density ρ_a of species A in a multi-component mixture is defined as the mass of species A per unit volume of the mixture.

$$\rho_a = \frac{\text{mass of species A}}{\text{volume of the mixture}} \quad ...(i)$$

The mass concentration is quite often represented by the symbol C also. In that case $\rho_a = C_a$.

The *molar concentration or molar density* represents the number of moles of species A per unit volume of the mixture.

$$n_a = \frac{\text{number of molecules of species A}}{\text{volume of the mixture}} \quad ...(ii)$$

Since the number of moles equal the mass divided by the molecular weight,

$$n_a = \frac{\text{mass of species A}}{\text{volume of the mixture}} \times \frac{1}{\text{mol wt of species A}}$$

$$= \frac{\rho_a}{M_a} \quad ...(iii)$$

The *mass fraction* $\rho_a^{\cdot}$ prescribes the ratio of mass concentration of species A to the total mass density of the mixture

$$\rho_a^{\cdot} = \frac{\rho_a}{\rho} = \frac{C_a}{C} \quad ...(iv)$$

The *mole fraction* $n_a^{\cdot}$ prescribes the ratio of number of moles of species A to the total number of moles of the mixture.

$$n_a^{\cdot} = \frac{n_a}{n_a + n_b} = \frac{n_a}{n} \quad ...(v)$$

where n is the molar density of the mixture, *i.e.*, number of moles in the mixture per unit volume.

By summation of fraction over species,

$$\Sigma \rho^{\cdot} = \rho_a^{\cdot} + \rho_b^{\cdot}$$

$$= \frac{\rho_a}{\rho} + \frac{\rho_b}{\rho} = \frac{\rho_a + \rho_b}{\rho} = 1 \quad ...(vi)$$

$$\text{and } \Sigma n^{\cdot} = n_a^{\cdot} + n_b^{\cdot}$$

$$= \frac{n_a}{n} + \frac{n_b}{n} = \frac{n_a + n_b}{n} = 1 \quad ...(vii)$$

Further,

$$M = \frac{\rho}{n} = \frac{\rho_a + \rho_b}{n}$$

$$= \frac{n_a M_a + n_b M_b}{n}$$

$$= n_a^{\cdot} M_a + n_b^{\cdot} M_b \quad ...(viii)$$

Thus the molecular weight of the mixture can be worked out by adding the molecular weights of the individual species in proportion to their mole fractions.

For a gaseous phase, the concentrations are usually expressed in terms of partial pressures. Invoking the perfect gas law,

$$p_a V = N_a G T$$

the molar concentration is

$$n_a = \frac{N_a}{V} = \frac{p_a}{GT} \quad ...(ix)$$

where p_a is the pressure of species A in the mixture, N_a is the total number of moles of species A in the gas volume V, T is the absolute temperature and G is the universal gas constant.

(G = 8314 Nm/kg mol K)

Likewise the mole fraction for ideal gas is

$$n_a^{\cdot} = \frac{n_a}{n} = \frac{p_a/GT}{p/GT} = \frac{p_a}{p} \quad ...(x)$$

where p is the total pressure exerted by the gas mixture; $p = p_a + p_b$

(b) **Velocities** : Each component/species of a multi-component mixture has a different mobility rate, and the bulk velocity of the mixture is computed either on mass-average or molar-average basis. For a binary mixture of species A and B,

The mass-average velocity V_{mass} is defined as

$$V_{mass} = \frac{\rho_a V_a + \rho_b V_b}{\rho_a + \rho_b}$$

$$= \frac{\rho_a V_a + \rho_b V_b}{\rho}$$

$$= \rho_a^{\cdot} V_a + \rho_b^{\cdot} V_b \quad ...(xi)$$

The molar-average velocity V_{molar} is defined as

$$V_{molar} = \frac{n_a V_a + n_b V_b}{n_a + n_b}$$

$$= \frac{n_a V_a + n_b}{n}$$

$= n_a^{\cdot} V_a + n_b^{\cdot} V_b$ ...(xii)

(c) *Fluxes* : The existence of different velocities and concentrations causes flux of mass transfer. For species A of the multi-component mixture,

absolute flux $= \rho_a V_a$

bulk motion flux $= \rho_a V_{mass}$

diffusion flux $= m_a/A$

The quantity m_a/A represents mass flow per unit area per unit time.

The absolute flux of any constituent as seen by a stationary observer equals the sum of diffusion flux and the bulk motion flux. That is,

$$\rho_a V_a = \frac{m_a}{A} + \rho_a V_{mass}$$

Accordingly the diffusion flux is,

$$\frac{m_a}{A} = \rho_a (V_a - V_{mass}) \quad ...(xiii)$$

The diffusion flux is thus proportional to the difference in the velocity of component and the bulk velocity. This velocity difference is called the ***diffusion velocity***.

$\therefore$ Mass-diffusion velocity of species

$A = V_a - V_{mass}$

Similarly on molar basis ;

molar-diffusion velocity $= V_a - V_{molar}$

diffusion flux $= n_a (V_a - V_{molar})$ ...(xiv)

EXAMPLE 15.1

A binary mixture of oxygen and nitrogen with partial pressures in the ratio 0.21 and 0.79 is contained in a vessel at 300 K. If the total pressure of the mixture is 1×10^5 N/m^2, make calculations for the molar concentration, the mass density, the molar fraction and the mass fraction of each species. Further, proceed to calculate the average velocity of the mixture.

Solution : Invoking the relation for molar concentration $n = p/GT$, the molar concentration for oxygen and nitrogen are

$$n_{o_2} = \frac{0.21 \times 10^5}{8314 \times 300}$$

$= 8.42 \times 10^{-3}$ kg mol/m^3

$$n_{N_2} = \frac{0.79 \times 10^5}{8314 \times 300}$$

$= 31.67 \times 10^{-3}$ kg mol/m^3

From the relation $\rho = Mn$, the mass densities of oxygen and nirogen are:

$\rho_{o_2} = 32 \times (8.42 \times 10^{-3})$

$= 0.269$ kg/m^3

$\rho_{N_2} = 28 \times (31.67 \times 10^{-3})$

$= 0.887$ kg/m^3

Overall mass density ρ

$= 0.269 + 0.867 = 1.156$ kg/m^3

The mass fractions of oxygen and nitrogen are:

$$\dot{\rho}_{o2} = \frac{\rho_{o_2}}{\rho} = \frac{0.269}{1.156} = 0.233$$

$$\dot{\rho}_{N_2} = \frac{0.887}{1.156} = 0.767$$

The molar fractions are equal to the partial pressure fractions.

$n_{o2}^{\cdot} = 0.21$ and $n_{N_2}^{\cdot} = 0.79$

Obviously the molar fractions are not the same as mass fractions,

Average molecular weight

$= \Sigma n_i^{\cdot} M_i$

$= (0.21 \times 32) + (0.79 \times 28) = 28.84$

15.3. FICK'S LAW

Consider a chamber in which two different gas species, B and C, at the same temperature and pressure are initially separated by a partition. The left compartment has a high concentration of gas B (dark circles) where as the right compartment is rich in gas C (white circles). Higher concentration means more molecules per unit volume. When the partition wall is removed there occurs a driving potential which tends to cause the concentration difference to equalize or become uniform. The molecules escape from the zone of higher concentration to travel towards the zone of lower concentration. Apparently both

the species are transported by diffusion and gradually mix with each other.

Fig. 15.1 represents the situation at a certain instant shortly after the removal of partition. The concentration of species B decreases with increasing x while the concentration of C increases with x. Mass diffusion is in the direction of decreasing concentration. Evidently there is net transfer of species B to the right and of species C to the left. After a sufficiently long period, equilibrium conditions prevail, *i.e.*, uniform concentration of species B and C is achieved and then the mass diffusion ceases.

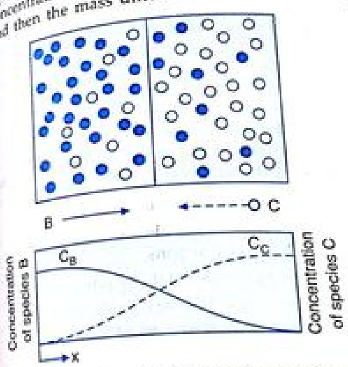


Fig. 15.1. Mass transfer by diffusion in a binary gas mixture

Experimental evidence indicates that molecular diffusion is governed by the empirical relation suggested by Fick :

$$N_b = \frac{m_b}{A} = -D_{bc} \frac{dC_b}{dx} \quad ...(15.1)$$

where,

(m_b/A) = mass flow per unit time per unit area

A = area through which mass is flowing

N_b = mass flux of species B; amount of species B that is transferred per unit time and per unit area perpendicular to the direction of transfer, kg/s-m^2 or kg mol/s-m^2

C_b = concentration or molecules per unit volume of species B

dC_b/dx = concentration gradient for species B and this acts as driving potential

D_{bc} = diffusion coefficient or diffusivity for binary mixture of species B and C

The dependence of diffusion on concentration profile has been illustrated in Fig. 15.2. The concentration of species B is higher on the left side of the dotted plane and apparently the mass flow of species B occurs towards the right.

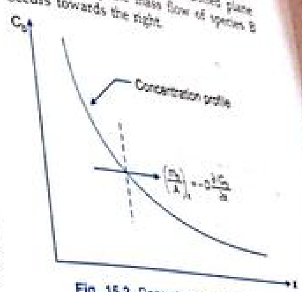


Fig. 15.2. Dependence of diffusion on concentration profile

The diffusion rate for species C would be:

$$N_c = \frac{m_c}{A} = -D_{cb} \frac{dC_c}{dx} \quad ...(15.2)$$

The –ve sign in equations 15.1 and 15.2 accounts for the fact that diffusion takes place in the direction opposite to that of increasing concentration. The diffusion coefficient D_{bc} or D_{cb} is dependent upon the temperature, pressure and nature of the components of the system.

The Fick's law of diffusion as prescribed by equations 15.1 and 15.2 is analogous to the Fourier law of heat conduction

$$\frac{Q}{A} = -k\frac{dt}{dx}$$

and Newton's law of viscosity

$$\tau = \mu\frac{du}{dy}$$

Apparently the Fourier equation describes the transport of heat energy due to temperature gradient, the shear equation describes the transport of momentum due to velocity gradient and the Fick's law describes the mass transport due to concentration gradient.

The dimensions of diffusion coefficient D as described by equation 15.1 are

$$D = -\frac{N}{(\partial C/\partial x)}$$

$$= \frac{(M/L^2T)}{[(M/L^3)\times(1/L)]}$$

$$= \frac{L^2}{T} = m^2/s$$

The units of mass diffusion coefficient are thus seen to be identical with those of thermal diffusivity α and kinematic visosity ν. Diffusion coefficient is thus a transport property of the fluid.

From characteristic equation applied to species B,

$$p_b = \rho_b R_b T = \rho_b \frac{GT}{M_b}$$

where p_b is the partial pressure of species B and M_b is the molecular weight. The mass density ρ_b represents the mass concentration C_b as used in the Fick's law.

$$C_b = \rho_b = \frac{p_b M_b}{M_b} \qquad ...(15.3)$$

Substituting the value of C_b in equation 15.1, the Fick's law of diffusion for component B into constituent C may be written as

$$N_b = \frac{m_b}{A} = -D_{bc}\frac{d}{dx}\left(\frac{p_b M_b}{GT}\right)$$

$$= -D_{bc}\frac{M_b}{GT}\frac{dp_b}{dx} \qquad ...(15.4)$$

Likewise, the diffusion of component C into consituent B would be

$$N_c = \frac{m_c}{A} = -D_{cb}\frac{M_c}{GT}\frac{dp_c}{dx} \qquad ...(15.5)$$

Equations 15.4 and 15.5 are essentially valid only for isothermal (constant temperature) diffusion.

Some aspects of Fick's law of diffusion are :

(*i*) Fick's law connot be derived from first principles, *i.e.*, it is a generalisation based on experimental evidence.

(*ii*) Fick's law is valid for all matter regardless of its state; solid, liquid or gas. Mass transfer is strongly influenced by molecular spacing and as such diffusion occurs more readily in gases than in liquids and more readily in liquids than in solids.

(*iii*) A diffusion substance moves in the direction of decreasing concentration. The difference in concentration in a diffusing process is similar to the difference in temperature in a heat flow process.

(*iv*) Besides concentration gradient, the mass diffusion may occur as the result of a temperature gradient, a pressure gradient or an external force. While applying Fick's law it is to be presumed that these additional effects are either not present or are too small.

(*v*) In general, the diffusivity or diffusion coefficient D dpends upon temperature, pressure and nature of the component of the system. However, for ideal gases and dilute liquids, the diffusivity coefficient can be presumed to remain practically constant for a given range of pressure and temperature.

Empirical relations for the diffusion coefficient of gases have been developed from the concept of kinetic theory of gases and the most general expression is of the form :

$$D = 0.0043\frac{T^{3/2}}{p_t\left(V_b^{1/3}+V_c^{1/3}\right)^2} \times \left(\frac{1}{M_b}+\frac{1}{M_c}\right)^{1/2} \qquad ...(15.6)$$

Table 15.1. Molecular weight and molecular volumes of gases

Gas	Molecular weight	Molecular volume at normal boiling point
Air	29	29.89 cm³/gm mole
Ammonia NH_3	17	25.81
Carbon dioxide CO_2	44	34.00
Carbon monoxide CO	28	30.71
Hydrogen H_2	2	14.28
Nitrogen N_2	28	31.20
Oxygen O_2	32	25.63
Sulphur dioxide SO_2	64	44.79

where,

p_t and T = total pressure in atmosphere and absolute temperature of the binary gaseous system, K

M_b and M_c = molecular weights of the gas species

V_b and V_c = molecular volumes of constituent species at normal boiling points cm³/gm-mol

Table 15.2. Mass diffusivity of gases and vapours through air at 25°C and 1 atm

Substance	Diffusivity, D cm²/s	Schmidt number Sc = ν/D
Ammonia	0.28	0.78
Carbon dioxide	0.164	0.94
Hydrogen	0.410	0.22
Oxygen	0.206	0.75
Water	0.256	0.60
Methanol	0.159	0.97
Ethyl alcohol	0.119	1.30
Acetic acid	0.133	1.16
Benzene	0.088	1.76
Toluene	0.084	1.84

Obviously the diffusion coefficient for gases depends upon pressure, temperature and other molecular properties of the diffusing gases. The molecular weight and molecular volumes of different gases have been listed in Table 15.1 and this data suffices to workout the diffusion of binary gaseous mixtures. Some values of diffusion coefficients for diffusions of certain substances through air are listed with Schmidt numbers in Table 15.2. These values can be extended to other pressures and temperatures by using the equations

$$\frac{D_1}{D_2} = \left(\frac{T_1}{T_2}\right)^{3/2}\left(\frac{p_2}{p_1}\right)$$

For steady state diffusion through a non-diffusing, multi-component mixture of constant composition, an effective diffusivity is calculated from :

$$D = \frac{1}{\frac{n_b'}{D_{ab}}+\frac{n_c'}{D_{ac}}+\frac{n_d'}{D_{ad}}} \qquad ...(15.7)$$

where,

$n_b', n_c', n_d', ...$ = mole fraction compositions of the mixture on a free basis

$D_{ab}, D_{ac}, D_{ad} ......$ = diffusivities of species A through B, C, D ....

Another equation suggested for the diffusion coefficient for gas pairs of non-polar, non-reacting molecules is of the form

$$D_{bc} = \frac{0.001858\,T^{3/2}\left[\frac{1}{M_b}+\frac{1}{M_c}\right]^{1/2}}{P(\sigma_{ab})^2\,\Omega} \qquad ...(15.7a)$$

where,

D_{bc} = mass diffusivity of gas species B diffusing through another gas species C, cm²/s

T = absolute temperature, K

M_b, M_c = molecular weights of gas species B and C respectively

P = absolute pressure in atmospheres

σ_{ab} = collision diameter in °A

Ω = collision integral, a dimensionless function of the temperature and of the inter molecular potential field for one molecule of B and one molecule of C

The values of collision integral Ω have been compiled as a function of kE/ϵ_{ab} where k is the Boltzman constant (1.38×10^{-6} ergs/K) and ϵ_{ab} is the energy of molecular interaction for the binary system BC. For a binary system composed of non-polar molecule pairs,

$$\sigma_{ab} = \frac{\sigma_a + \sigma_b}{2} \quad \text{and} \quad \epsilon_{ab} = \sqrt{\epsilon_a \, \epsilon_b}$$

Equation 15.7*a* also suggests the dependence of diffusivity on pressure, temperature and the components which comprise the system.

The diffusion cofficient for dilute liquids is estimated from the following semi-empirical relation:

$$F = \frac{T}{D_{ab}\,\mu_b} \qquad \text{...(15.8)}$$

where, T = absolute temperature, K

D_{ab} = diffusivity of solute A through a solvent B, m²/s

μ_b = viscosity of the solvent B, centipoises and

F = a function of the molal volume of solute A, K s/cm² centipoises

Charts are available to determine the value of *F* for different values of solute molal volume and solvent factor (a ratio of the value of *F* in the solvent of the value *F* for diffusion in water at constant molal volume)

Some values of mass diffusivity of liquids are given, together with the Schmidt numbers, in Table 15.3. They are obtained partly by estimation and partly through experiments. Diffusivity of most of the organic and inorganic substances in the solvents like water, alcohol, benzene etc. at room temperature lies in the range 0.6×10^{-6} to 2.5×10^{-5} cm²/s. Low values of diffusion coefficient for liquids indicate that diffusion in liquids occurs at a much slower rate than in gases.

Table 15.3. Mass diffusivity of liquids at 20°C (solvent water)

Substance	Diffusivity, $D \times 10^{-5}$ cm²/s	Schmidt number $Sc = \nu/D$
Oxygen	1.80	558
Carbon dioxide	1.77	559
Ammonia	1.76	570
Hydrogen	5.13	196
Chlorine	1.22	824
Hydrochloric acid	2.64	381
Sulphuric acid	1.73	580
Acetic acid	0.88	1140
Ethanol	1.00	1005
Urea	8.06	946
Glucose	0.60	—

EXAMPLE 15.2

Express Fick's law of diffusion in terms of mass and mole fractions.

(b) Develop a relation expressing the equivalence of diffusion coefficients in a binary system.

Solution : In a binary mixture of species B and C, the diffusion rate of species B as prescribed by Fick's law is given by

$$\frac{m_b}{A} = -D_{bc}\frac{\partial C_b}{\partial x} = -D_{bc}\frac{\partial \rho_b}{\partial x} \qquad \text{...(i)}$$

where C_b denotes the mass concentration of species B per unit volume of the mixture, (*i.e.*, ρ_a).

Further $\rho_b = \rho$ and $\rho_b^{*} = \rho C_b^{*}$ where C_b^{*} is the mass fraction and ρ is the mixture density. That gives

$$\frac{m_b}{A} = -D_{bc}\frac{\partial}{\partial x}(\rho C_b^{*})$$

$$= -\rho D_{bc}\frac{\partial C_b^{*}}{\partial x} \qquad \text{...(ii)}$$

The above expression based on mass fractions is generally adopted whilst dealing with diffusion of liquids.

In terms of molar-concentrations, the Fick's law can be written as

$$\frac{N_b \times M_b}{A} = -D_{bc}\frac{\partial}{\partial x}(n_b M_b)$$

$$\text{or} \quad \frac{N_b}{A} = -D_{bc}\frac{\partial n_b}{\partial x} \qquad \text{...(iii)}$$

where N_b/A is the molal flux, *i.e.*, diffusion rate in terms of number of moles of species B per unit time per unit area. Further $n_b = n n_b^{*}$ where n is the number of moles in the mixture and n_b^{*} is the mole fraction.

$$\therefore \quad \frac{N_b}{A} = -D_{bc}\frac{\partial}{\partial x}(n\, n_b^{*})$$

$$= -n\, D_{bc}\frac{\partial n_b^{*}}{\partial x} \qquad \text{...(iv)}$$

Expressions (*iii*) and (*iv*) based on molar density n_b and mole fraction n^{*}_b are found generally convenient while dealing with diffusion of gases.

(b) The molar diffusion flux of species B into C and that of C into B are given by

$$\frac{N_b}{A} = -n\, D_{bc}\frac{\partial n_b^{*}}{\partial x}$$

$$\text{and} \quad \frac{N_c}{A} = -n\, D_{cb}\frac{\partial n_c^{*}}{\partial x}$$

Upon adding these identities,

$$\frac{N_b + N_c}{A} = -n\, D_{bc}\frac{\partial n_b^{*}}{\partial x} - n\, D_{cb}\frac{\partial n_c^{*}}{\partial x} \qquad \text{...(v)}$$

Invoking the relation : $n_b^{*} + n_c^{*} = 1$, one gets

$$\frac{\partial n_b^{*}}{\partial x} = -\frac{\partial n_c^{*}}{\partial x}$$

Expression (*v*) may then be rewritten as

$$\frac{N_b + N_c}{A} = n(D_{bc} - D_{cb})\frac{\partial n_c^{*}}{\partial x}$$

Since n is constant throughout the system during the process, the total number of molecules crossing any place must be zero. Therefore

$$N_b + N_c = 0$$

For the above relation to hold good

$$D_{bc} - D_{cb} = 0$$

$$\text{or} \quad D_{bc} = D_{cb} \qquad \text{...(vi)}$$

Thus the diffusion coefficient for the diffusion of B in C is equal to the diffusion coefficient for the diffusion of C in B. This relation is referred to as ***equivalence of diffusion coefficients***.

EXAMPLE 15.3

Estimate the diffusion coefficient for ammonia in air at 25°C temperature and one atmospheric pressure.

For ammonia :

molecular weight = 17

and molecular volume = 25.81cm³/gm mole

For air :

molecular weight = 29

and molecular volume = 29.89 cm³/gm mole

Solution : The diffusion coefficent for binary gaseous mixtures is worked out from the relation:

$$D = 0.0043\,\frac{T^{3/2}}{P_t\left(V_b^{1/3} + V_c^{1/3}\right)^2}\left(\frac{1}{M_b} + \frac{1}{M_c}\right)^{1/2}$$

Inserting the appropriate values in consistent units

$$D = 0.0043\,\frac{(273+25)^{3/2}}{1 \times \left(25.81^{1/3} + 29.89^{1/3}\right)^2} \times \left(\frac{1}{17} + \frac{1}{29}\right)^{1/2}$$

$$= 0.0043 \times \frac{5144.27}{(2.92 + 3.07)^2} \times (0.0588 + 0.0345)^{1/2}$$

$$= 0.0043 \times \frac{5144.27}{35.88} \times 0.305$$

$$= 0.1880 \text{ cm}^2/\text{s}$$

EXAMPLE 15.4

State the equations that have been suggested to estimate the diffusion coefficient of a gas through another non-diffusing and non-reacting gas. Give the units of each term appearing in these equations.

Evaluate the diffusion coefficient of carbon dioxide in air at 20°C temperature and one atmospheric pressure. The following data is given:

Carbon dioxide :

$$\sigma = 3.996 \text{ Å} \quad \text{and} \quad \frac{\epsilon}{k} = 190 \text{ K}$$

$$\text{Air : } \sigma = 3.167 \text{ Å} \quad \text{and} \quad \frac{\epsilon}{k} = 97 \text{ K}$$

and for the estimation of collision integral Ω, we have

kT/ϵ	2.00	2.10	2.20	2.30
Ω	1.075	1.057	1.041	1.026

Solution : The diffusion coefficient can be estimated by use of the equation.

$$D_{ab} = \frac{0.001858\, T^{3/2}\left[\frac{1}{M_A} + \frac{1}{M_B}\right]^{1/2}}{P(\sigma_{ab})^2\, \Omega}$$

The various parameters for this equation are evaluated as follows :

$$\sigma_{ab} = \frac{\sigma_a + \sigma_b}{2}$$

$$= \frac{3.996 + 3.167}{2} = 3.806 \text{ Å}$$

$$\frac{\epsilon_{ab}}{k} = \left(\frac{\epsilon_a}{k} \times \frac{\epsilon_b}{k}\right)^{1/2}$$

$$= (190 \times 97)^{1/2} = 135.76$$

$$\frac{\epsilon_{ab}}{kT} = \frac{135.76}{(273+20)} = 0.463$$

$$\frac{kT}{\epsilon_{ab}} = \frac{1}{0.463} = 2.16$$

Corresponding to $kT/\epsilon_{ab} = 2.16$, the collision integral Ω becomes :

$$= 1.057 - \frac{(1.057 - 1.041)}{(2.20 - 2.10)} \times (2.16 - 2.10)$$

$$= 1.0474$$

Further,

$$M_{co_2} = 44 \quad \text{and} \quad M_{air} = 29$$

Inserting these values into the equation we obtain :

$$D = \frac{0.001858 \times 293^{3/2}\left[\frac{1}{44} + \frac{1}{29}\right]^{1/2}}{1 \times (3.806)^2 \times 1.0474}$$

$$= \mathbf{0.147\ cm^2/s}$$

EXAMPLE 15.5

Estimate the diffusion coefficient of carbon monoxide through air in which the mole fractions of each constituent are :

$$O_2 = 0.18 ; \quad N_2 = 0.72 ; \quad CO = 0.10$$

The gas mixture is at 300 K and 2 atmosphere total pressure. The diffusivity values are:

$$D_{CO} = 18.5 \times 10^{-6} \text{ m}^2/\text{s at } 273 \text{ K, } 1 \text{ atm}$$

$$D_{CN} = 19.2 \times 10^{-6} \text{ m}^2/\text{s at } 288 \text{ K, } 1 \text{ atm}$$

The subscripts C, O and N refer to carbon monoxide, oxygen and nitrogen respectively.

Solution : The given diffusivity values are corrected for pressure and temperature differences by using the relation :

$$\frac{D_1}{D_2} = \left(\frac{T_1}{T_2}\right)^{3/2}\left(\frac{P_2}{P_1}\right)$$

At the conditions of the gas mixture (P = 2 atm and T = 300 K), we will have :

$$D_{CO} = \left(\frac{300}{273}\right)^{3/2} \times \left(\frac{1}{2}\right) \times 18.5 \times 10^{-6}$$

$$= 10.65 \times 10^{-6} \text{ m}^2/\text{s}$$

$$D_{CN} = \left(\frac{300}{288}\right)^{3/2} \times \left(\frac{1}{2}\right) \times 19.2 \times 10^{-6}$$

$$= 10.20 \times 10^{-6} \text{ m}^2/\text{s}$$

The composition of oxygen and nitrogen on carbon monoxide free basis are :

$$n_o^{\cdot} = \frac{0.18}{1 - 0.10} = 0.20$$

$$n_N^{\cdot} = \frac{0.72}{1 - 0.10} = 0.80$$

The effective diffusivity of carbon monoxide through air of given composition is then obtained from the relation

$$D = \frac{1}{\frac{n_o^{\cdot}}{D_{co}} + \frac{n_N^{\cdot}}{D_{CN}}}$$

$$= \frac{1}{\frac{0.20}{10.65 \times 10^{-6}} + \frac{0.80}{10.20 \times 10^{-6}}}$$

$$= \mathbf{10.29 \times 10^{-6}\ m^2/s}$$

15.4 GENERAL EQUATION OF MASS DIFFUSION IN STATIONARY MEDIA

Consider mass balance of species B diffusing through a control volume ($dx \times dy \times dz$) in a solid or stationary fluid medium C.

Along x-direction;

$$\text{mass influx at the left face} = \left(\frac{m_b}{A}\right) dy\, dz$$

mass efflux at the right face

$$= \left[\frac{m_b}{A} + \frac{\partial}{\partial x}\left(\frac{m_b}{A}\right) dx\right] dy\, dz$$

Accumulation of mass of species B in the elemental volume due to its mass diffusion in the x-direction is given by the difference between the mass influx and the mass efflux. Therefore mass accumulation of species B due to its diffusion in x-direction is

$$= \left(\frac{m_b}{A}\right) dy\, dz - \left[\frac{m_b}{A} + \frac{\partial}{\partial x}\left(\frac{m_b}{A}\right) dx\right] dy\, dz$$

$$= -\frac{\partial}{\partial x}\left(\frac{m_b}{A}\right) dx\, dy\, dz$$

Likewise the mass accumulation of species B due to its diffusion in y and z - directions will be

$$= -\frac{\partial}{\partial y}\left(\frac{m_b}{A}\right) dx\, dy\, dz$$

and

$$= -\frac{\partial}{\partial z}\left(\frac{m_b}{A}\right) dx\, dy\, dz$$

∴ Total or net accumulation of mass of species B

$$= -\left[\frac{\partial}{\partial x}\left(\frac{m_b}{A}\right) + \frac{\partial}{\partial y}\left(\frac{m_b}{A}\right) + \frac{\partial}{\partial z}\left(\frac{m_b}{A}\right)\right] dx\, dy\, dz \quad ...(15.9)$$

If q_b is the mass of species B generated per unit volume, then the total mass of species B generated in the control volume equals

$$= q_b\, dx\, dy\, dz \quad ...(15.10)$$

The total mass of species B accumulated in the control volume due to mass diffusion along the corrdinate axis (Eq. 15.9) and the mass generated within the control volume (Eq. 15.10) serves to increase the mass concentration of species B. This increase is reflected by the time rate of change in mass oconcentration of species B in the control volume and is given by

$$= \frac{\partial C_b}{\partial \tau} dx\, dy\, dz \quad ...(15.11)$$

Thus from mass-balance consideration,

$$-\left[\frac{\partial}{\partial x}\left(\frac{m_b}{A}\right) + \frac{\partial}{\partial y}\left(\frac{m_b}{A}\right) + \frac{\partial}{\partial z}\left(\frac{m_b}{A}\right)\right] dx\, dy\, dz + q_b\, dx\, dy\, dz$$

$$= \frac{\partial C_b}{\partial \tau} dx\, dy\, dz$$

Dividing both sides by $dx\, dy\, dz$,

$$-\left[\frac{\partial}{\partial x}\left(\frac{m_b}{A}\right) + \frac{\partial}{\partial y}\left(\frac{m_b}{A}\right) + \frac{\partial}{\partial z}\left(\frac{m_b}{A}\right)\right] + q_b$$

$$= \frac{\partial C_b}{\partial \tau}$$

For an isotropic medium, the diffusion coefficient D is same in all the three-directions. The above expression may then be written as

$$\frac{\partial C_b}{\partial \tau} = \frac{\partial}{\partial x}\left(D\frac{\partial C_b}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial C_b}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial C_b}{\partial z}\right)+q_b$$

Treating diffusion coefficient D as constant and presuming no internal mass generation of species B, we get

$$\frac{\partial C_b}{\partial \tau} = D\left(\frac{\partial^2 C_b}{\partial x^2}+\frac{\partial^2 C_b}{\partial y^2}+\frac{\partial^2 C_b}{\partial z^2}\right) \quad ...(15.12)$$

which is the ***general mass diffusion equation.*** This equation is similar to heat conduction equation and its solution can be obtained by applying the relevant boundary conditions. A few typical boundary conditions are listed below :

(*i*) Specified boundary concentration

$$C_b = C_{bo} \quad \text{at} \quad x = 0$$

(*ii*) Impermeable surface at boundary

$$\frac{\partial C_b}{\partial x} = 0 \quad \text{at} \quad x = 0$$

(*iii*) Specified mass flux at a surface

$$\frac{m_b}{A} = -D\frac{\partial C_b}{dx} \quad \text{at} \quad x = 0$$

(*iv*) Specified mass transfer coefficient (convection) at a surface

$$N_b = h_m\,(C_{bs} - C_{b\infty})$$

where h_m is the convective mass transfer coefficient, C_{bs} is concentration in the fluid adjacent to the surface and $C_{b\infty}$ is the bulk concentration in the fluid stream.

15.5. STEADY STATE DIFFUSION THROUGH A PLAIN MEMBRANE

Consider mass diffusion of fluid B through a plain membrane whose thickness δ is small in comparison with other dimensions. At the opposite wall faces, the mass concentration of the fluid are C_{b1} and C_{b2} respectively.

Under stipulations of the system with steady state, one-dimensional conditions and no chemical reactions, the general mass diffusion equation 15.12 reduces to

$$\frac{d^2 C_b}{dx^2} = 0$$

Upon integration

$$\frac{dC_b}{dx} = C_1$$

and $\quad C_b = C_1\,x + C_2 \quad ...(15.13)$

From the prescribed boundary conditions

$$C_b = C_{b1} \quad \text{at} \quad x = 0$$

and $\quad C_b = C_{b2}$ at $x = \delta$

the constants take the values

$$C_2 = C_{b1} \quad \text{and} \quad C_1 = \frac{C_{b2} - C_{b1}}{\delta}$$

Substituting these values of C_1 and C_2 in equation 15.13, the general expression for concentration becomes :

$$C_b = (C_{b2} - C_{b1})\frac{x}{\delta}+C_{b1}$$

Further the mass transfer rate is

$$\frac{m_b}{A} = -D_b\frac{\partial C_b}{\partial x}$$

$$= -D_b\frac{\partial}{\partial x}\left[(C_{b2}-C_{b1})\frac{x}{\delta}+C_{b1}\right]$$

$$= -D_b\frac{(C_{b2}-C_{b1})}{\delta}$$

$$= \frac{D_b}{\delta}(C_{b1}-C_{b2}) \quad ...(15.14)$$

The above relation could also be set up directly from Fick's law of diffusion. Upon separating the variables and integrating, we obtain.

$$\frac{m_b}{A}\int_0^{\delta} dx = -D_b\int_{C_{b1}}^{C_{b2}} dC_b$$

This is upon the presumption that diffusivity is independent of concentration and that the rate of diffusion is constant.

$$\frac{m_b}{A}\,\delta = D_b\,(C_{b1} - C_{b2})$$

or $$m_b = \frac{C_{b1}-C_{b2}}{\dfrac{\delta}{D_b\,A}}$$

$$= \frac{\text{concentration potential}}{\text{diffusion resistance}} \quad ...(15.15)$$

The above relation does suggest that the conduction heat transfer and diffusion mass transfer are analogous.

The convective film resistance on a solid surface in terms of convective mass transfer coefficient will be

$$R_{film} = \frac{1}{h_m\,A_s}$$

These expressions for diffusion resistance and the convective film resistance can be conveniently used to solve problems on composite planes.

Quite often, the species concentration at the gas-solid interface is prescribed in terms of the partial pressure of the gas adjoining the interface and a solubility factors S.

Concentration $C_b = S\,p_a$

Data is available for values of solubility factor S for several gas-solid combinations.

For steady state conditions with no homogeneous reactions and one-dimensional diffusion, the diffusion rate in the radial direction of a cylindrical system of inner and outer radii of r_1 and r_2 respectively and length l is

$$m_b = \frac{D_b\,(C_{b1}-C_{b2})}{\Delta x}A_m \quad ...(15.16)$$

where, $\quad \Delta x = (r_2 - r_1)$

and $$A_m = \frac{2\pi l\,(r_2 - r_1)}{\log_e \dfrac{r_2}{r_1}}$$

The appropriate form of the mass diffusion rate through a spherical system of inner and outer radii of r_1 and r_2 respectively is

$$m_b = \frac{D_b\,(C_{b1}-C_{b2})}{\Delta x}A_m \quad ...(15.17)$$

where, $\quad \Delta x = (r_2 - r_1) \quad$ and $\quad A_m = 4\,\pi\,r_1\,r_2$

EXAMPLE 15.6

A rectangular container having steel walls of 8 mm thickness stores gaseous hydrogen at elevated pressure. The molar concentration of hydrogen in the steel at the inner and outer surfaces of the wall are approximated to be 1.0 kg-mol/m³ and 0.0 kg-mol/m³ respectively. Presuming that the binary diffusion coefficient for hydrogen in steel is 0.24 × 10⁻¹² m²/s, workout the diffusion flux for hydrogen through the steel wall. Point out the assumptions made in the derivation of the relation used by you.

Solution : The molar diffusion flux of hydrogen (h) thorugh the steel wall (s) is prescribed by the Fick's law

$$\frac{m_h}{A} = \frac{D_{hs}\,(C_{h1}-C_{h2})}{(x_2 - x_1)}$$

which has been worked out with the following assumptions :

(*i*) Steady-state conditions

(*ii*) one-dimensional species diffusion through a plane wall which is approximated as a stationary medium

(*iii*) no chemical reaction of the diffusing substance in the solid wall

Inserting the appropriate data in consistent units :

$$N_h = \frac{m_h}{A} = \frac{0.24\times 10^{-12}\,(1-0)}{0.008}$$

$$= 3.0 \times 10^{-11} \text{ kg-mol/s-m}^2$$

Since the molecular weight of hydrogen is 2 kg/kg-mol, the mass flux of hydrogen is:

$$= 2 \times 3.0 \times 10^{-11} = 6.0 \text{ kg/s-m}^2$$

EXAMPLE 15.7

A plastic membrane 0.25 mm thick has hydrogen gas maintained at pressures of 2.5 bar and 1 bar on its opposite sides. The binary diffusion coefficient of hydrogen in the plastic is 8.5×10^{-8} m²/s and the solubility of hydrogen in the membrane is 1.5×10^{-3} kg-mol/m³-bar. Under uniform temperature conditions of 25°C, workout : (a) molar concentrations of hydrogen at the opposite faces of the membrane, (b) molar and mass diffusion flux of hydrogen through the membrane.

Solution : The molar concentrations (C), the partial pressures (p) and the solubility (S) of the diffusing gas are related to each other by the expression,

$$C = S\,p$$

Therefore molar concentrations of hydrogen at the opposite faces of the plastic membrane are :

$$C_{h1} = 1.5 \times 10^{-3} \times 2.5 = 3.75 \times 10^{-3} \text{ kg-mol/m}^3$$

$$C_{h2} = 1.5 \times 10^{-3} \times 1 = 1.5 \times 10^{-3} \text{ kg-mol/m}^3$$

(*b*) The molar diffusion flux of hydrogen through the membrane is worked out from the relation

$$N_h = \frac{m_h}{A} = \frac{D_{hp}\left(C_{h1} - C_{h2}\right)}{(x_2 - x_1)}$$

The subscripts *h* and *p* refer to hydrogen and plastic, respectively

Inserting appropriate values in consistent units,

$$N_h = \frac{8.5 \times 10^{-8}\left(3.75 \times 10^{-3} - 1.5 \times 10^{-3}\right)}{0.25 \times 10^{-3}}$$

$$= 76.5 \times 10^{-8} \text{ kg-mol/s-m}^2$$

Since the molecular weight of hydrogen is 2 kg/kg-mol, the mass flux of hydrogen is

$$= 2 \times 76.5 \times 10^{-8}$$

$$= \mathbf{153 \times 10^{-8} \text{ kg/s-m}^2}$$

EXAMPLE 15.8

Hydrogen gas at 2 bar and 300 K flows through a rubber tubing of 10 mm inside radius and 20 mm outside radius. The diffusivity of hydrogen through rubber is stated to be 0.75 × 10^{-4} m²/hr and the solubility of hydrogen is 0.052 m³ of hydrogen/m³ of rubber at 1 atmosphere. What would be the diffusion loss of hydrogen per metre length of the rubber tubing ? It may be presumed that resistance to diffusion of hydrogen from the outer surface of the tube is negligible.

Solution : The solubility of hydrogen at the operating pressure of 2 atmosphere is :

$$= 2 \times 0.052$$

$$= 0.104 \text{ m}^3/\text{m}^3 \text{ of rubber}$$

Then from the characteristic gas equation

$$pV = m\,RT$$

where the gas constant *R* for hydrogen is 4240 J/kg K

$$\therefore \; 2 \times 10^5 \times 0.104 = m \times 4240 \times 300$$

$$m = \frac{2 \times 10^5 \times 0.104}{4240 \times 300}$$

$$= 0.01635 \text{ kg/m}^3 \text{ of rubber}$$

Therefore, mass concentration of hydrogen at the inner surface of the pipe is 0.01635 kg/m³. We can approximate the mass concen-tration to be zero at the pipe surface as resistance to diffusion is stated to be negligible at that surface. Thus

$$C_{h1} = 0.01635 \text{ kg/m}^3$$

and $C_{h2} = 0.0$

The diffusion flux through a cylindrical system is given by

$$m = \frac{D\left(C_{h1} - C_{h2}\right)}{\Delta x} A_m$$

where, $\Delta x = (r_2 - r_1)$

$$= (20 - 10) \times 10^{-3} = 10 \times 10^{-3} \text{ m}$$

$$A_m = \frac{2\pi l\,(r_2 - r_1)}{\log_e \frac{r_2}{r_1}}$$

$$= \frac{2\pi \times 1 \times (10 \times 10^{-3})}{\log_e \frac{20}{10}}$$

$$= 9.06 \times 10^{-2} \text{ m}^2$$

$$\therefore \quad m = \frac{0.75 \times 10^{-4} \times (0.01635 - 0.0)}{10 \times 10^{-3}} \times (9.06 \times 10^{-2})$$

$$= \mathbf{1.11 \times 10^{-5} \text{ kg/hr}}$$

EXAMPLE 15.9

The air pressure in a tyre tube of surface area 0.5 m² and wall thickness of 0.01m is approximated to drop from 2 bar to 1.99 bar in a period of 5-days. The solubility of air in rubber is 0.07m³ of air/m³ of rubber at 1 bar. Estimate the diffusivity of air in rubber at the operating temperature of 300 K if the volume of air in the tube is 0.025 m³.

Solution : Initial mass of air in the tube,

$$m_i = \frac{P_i V}{RT} = \frac{2 \times 10^5 \times 0.025}{287 \times 300}$$

$$= 0.05807 \text{ kg}$$

Final mass of air in the tube

$$m_f = \frac{P_f V}{RT} = \frac{1.99 \times 10^5 \times 0.025}{287 \times 300}$$

$$= 0.05778 \text{ kg}$$

Mass of air escaped is equal to

$$m_i - m_f = 0.05807 - 0.05778$$

$$= 2.9 \times 10^{4} \text{ kg}$$

Therefore, the mass flux equals

$$\frac{m_a}{A} = \frac{\text{mass escaped}}{\text{time elapsed} \times \text{area}}$$

$$= \frac{2.9 \times 10^{-4}}{(5 \times 24 \times 3600) \times 0.5}$$

$$= 1.342 \times 10^{-9} \text{ kg/s-m}^2$$

The solubility of air at the mean operating pressure of (2 + 1.99)/2 = 1.995 bar is

$$= 1.995 \times 0.07$$

$$= 0.1396 \text{ m}^3 \text{ of air/m}^3 \text{ of rubber}$$

The air which escapes to atmosphere will be at 1 bar pressure and its solubility will remain at 0.07 m³ of air/m³ of rubber.

The corresponding mass concentrations are worked out from the characteristic equation,

$$C_{a1} = \frac{PV}{RT} = \frac{1.995 \times 10^5 \times 0.1396}{287 \times 300}$$

$$= 0.3234 \text{ kg/m}^3$$

$$C_{a2} = \frac{1 \times 10^5 \times 0.07}{287 \times 300} = 0.0813 \text{ kg/m}^3$$

The diffusion flux of air through rubber is :

$$\frac{m_a}{A} = \frac{D_a\left(C_{a1} - C_{a2}\right)}{(x_2 - x_1)}$$

$$\therefore \; 1.342 \times 10^{-9} = \frac{D_a(0.3234 - 0.0813)}{0.01}$$

Therefore, diffusivity of air in rubber is

$$D_a = \frac{1.342 \times 10^{-9} \times 0.01}{(0.3234 - 0.0813)}$$

$$= 0.55 \times 10^{-10} \text{ m}^2/\text{s}$$

15.6. EQUIMOLAL DIFFUSION

Equimolal diffusion between species B and C of a binary gas system implies that each molecule of component B is replaced by each molecule of constituent C and vice-versa. Invoking Fick's law, molar diffusion rates of species B and C are given by:

$$N_b = \frac{m_b}{M_b} = -D_{bc}\frac{A}{GT}\frac{dp_b}{dx}$$

$$N_c = \frac{m_c}{M_c} = -D_{cb}\frac{A}{GT}\frac{dp_c}{dx} \quad ...(15.16)$$

where p_b and p_c are the partial pressures, N_b and N_c are the molar diffusion rates.

Reference Fig. 15.3, the component B is shown to be diffusing in the direction of its drop in concentration and the constituent C is diffusing in the opposite direction. Distillation operations form good example of this process and the concentration at any point in the gas mixture remains constant with time.

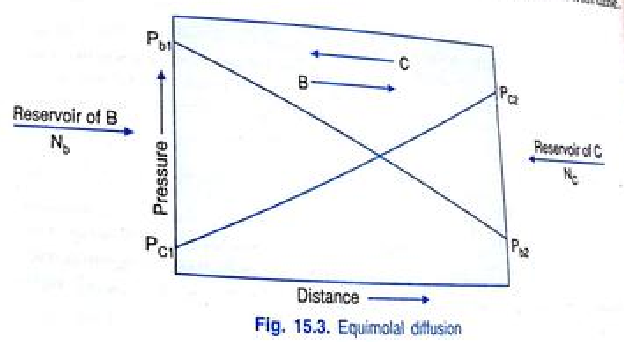


Fig. 15.3. Equimolal diffusion

From Dalton's law of partial pressures, the total pressure p_t of the system equals the sum of the partial pressures of the constituents.

$$p_t = p_b + p_c$$

Differentiating with respect to x,

$$\frac{dp_t}{dx} = \frac{dp_b}{dx} + \frac{dp_c}{dx}$$

Under steady conditions, the total pressure of the system remains constant. Therefore

$$\frac{dp_t}{dx} = 0 \text{ and so } \frac{dp_b}{dx} = -\frac{dp_c}{dx} \quad ...(i)$$

Further, when the two species diffuse simultaneously in opposite directions but at a constant rate of given number of molecules per unit time, then both N_b and N_c are numerically equal.

$$N_b = -N_c$$

$$-D_{bc}\frac{A}{GT}\frac{dp_b}{dx} = D_{cb}\frac{A}{GT}\frac{dp_c}{dx}$$

Replacing dp_c/dx in terms of dp_b/dx as given by expression (i),

$$-D_{bc}\frac{A}{GT}\frac{dp_b}{dx} = -D_{cb}\frac{A}{GT}\frac{dp_c}{dx}$$

or $\quad D_{bc} = D_{cb} = D \quad ...(15.19)$

Thus for equi-molal diffusion, the diffusion coefficient D_{bc} for diffusion of gas component B into gas constituent C is equal to the diffusion coefficient D_{cb} for diffusion of gas constituent C into gas component B. The value of this diffusion coefficient for a binary mixture of gases can be calculated by using the equation 15.6.

Thus if the diffusion coefficient D_{bc} is assumed constant, equation 15.18 may be integrated between any two planes to give :

$$N_b = \frac{m_b}{M_b} = -D\frac{A}{GT}\frac{(p_{b_2} - p_{b_1})}{(x_2 - x_1)}$$

$$= D\frac{A}{GT}\frac{(p_{b_1} - p_{b_2})}{(x_2 - x_1)} \quad ...(15.20)$$

where p_{b1} and p_{b2} are partial pressures of gas B at locations x_1 and x_2 of the system.

Equation 15.20 is valid only for equi-molal counter diffusion.

EXAMPLE 15.10

A distillation column containing a mixture of benzene and toluene is at a pressure of 1 atmosphere and 100°C temperature. The liquid and vapour phases contain 30 mol % and 45 mol % of benzene. At 100°C temperature, the vapour pressure of toulene is 70 kN/m² and the diffusivity is 5×10^{-6} m²/s

Workout the rate of interchange of benzene and toulene between the liquid and vapour phases if resistance to mass transfer lies in a film 0.25 mm thick.

Take atmospheric pressure = 101 kN/m² and Universal gas constant G = 8.314 kJ/kg-mol K.

Solution : Let subscripts b and t refer to benzene and toulene respectively. At the liquid plane 1, the partial pressure of toulene is

p_{t_1} = molar concentration × vapour pressure

$= (1 - 0.3) \times 70 = 49$ kN/m²

and at the vapour plane 2,

$p_{t_2} = (1 - 0.45) \times 101 = 55$ kN/m²

For the equi-molal diffusion, the molar diffusion flux of toulene is;

$$N_t = \frac{m_t}{A} = \frac{D}{GT}\frac{(p_{t_1} - p_{t_2})}{(x_2 - x_1)}$$

$$= \frac{5 \times 10^{-6}}{8.314 \times 373} \times \frac{(49 - 55)}{0.25 \times 10^{-3}}$$

$$= -38.69 \times 10^{-6} \text{ kg-mol/s m}^2$$

The negative sign indicates that transfer of toulene is from vapour to liquid. Benzene will diffuse in the opposite direction at the same rate.

15.7. DIFFUSION OF WATER VAPOURS THROUGH AIR

Consider evaporation of water contained in a tube or tank and its subsequent diffusion through the stagnant layer of air over it.

Assumptions :

(*i*) the system is under steady state and isothermal conditions

(*ii*) the total pressure within the tube is constant and equal to the outside pressure

(*iii*) both air and water vapours behave as perfect gases

(*iv*) air which flows across the open end of the tube or tank has negligible solubility in water, and

(*v*) the slight movement of air over the top of the tube does not bring about any change in the concentration profile of air. However, the movement is just sufficient to carry the water vapours which diffuse to that point.

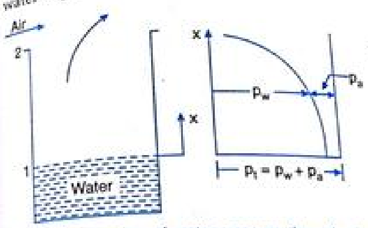


Fig. 15.4. Diffusion of water vapours through air

The water evaporates and diffuses upward through the air. Under stipulations of steady state, the upward movement must be balanced by a downward diffusion of air so that concentration at any location from the water surface remains constant,

Downward mass diffusion of air,

$$m_a = -D\frac{A\,M_a}{GT}\frac{dp_a}{dx} \quad ...(15.21)$$

where A is cross-sectional area of the tube and dp_a/dx is the partial pressure gradient of air

Since there can be no net mass movement of air downward at the surface of water, there will be a bulk mass movement upward with a velocity just large enough to compensate for the mass diffusion of air downward.

Bulk mass transfer of air is equal to

$$-\rho_a A v = -p_a \frac{M_a}{RT} A v$$

where v is the bulk mass velocity upward $\quad ...(15.22)$

Combination of equations 15.21 and 15.22 gives :

$$v = \frac{D}{p_a}\frac{dp_a}{dx} \quad ...(15.23)$$

The mass transfer of water is due to (*i*) upward mass diffusion of water and (*ii*) water vapours carried upward along with upward bulk movement of moving air. Thus the total mass transfer of water vapour is

$$-D\frac{A}{GT}M_w\frac{dp_w}{dx} + \frac{p_w M_w}{GT}vA \quad ...(15.24)$$

Substituting the value of bulk mass velocity v form equation 15.23,

$$(m_w)_{total} = -D\frac{A}{GT}M_w\frac{dp_w}{dx} + \frac{p_w M_w}{GT}\frac{D}{p_a}\frac{dp_a}{dx}A \quad ...(15.25)$$

From Dalton's law of partial pressures,

$$p_t = p_a + p_w$$

$$\frac{dp_t}{dx} = \frac{dp_a}{dx} + \frac{dp_w}{dx}$$

Since total pressure of the system remains constant within the tube $dp_t/dx = 0$ and therefore

$$\frac{dp_a}{dx} = -\frac{dp_w}{dx}$$

Equation 15.25 may then be rewritten as;

$$(m_w)_{total} = -\frac{DA}{GT}M_w\frac{dp_w}{dx} - p_w\frac{M_w A}{GT}\frac{D}{p_a}\frac{dp_w}{dx}$$

$$= -\frac{DA}{GT}M_w\frac{dp_w}{dx}\left\{1 + \frac{p_w}{p_a}\right\}$$

$$= -\frac{DA}{GT}M_w\frac{dp_w}{dx}\left\{\frac{p_a + p_w}{p_a}\right\}$$

$$= -\frac{DA}{GT} M_w \frac{dp_w}{dx} \left\{ \frac{p_t}{p_t + p_w} \right\} \quad ...(15.26)$$

Equation 15.26 is referred to as ***Stefan's law*** *for diffusion of an ideal gaseous component through a practically stagnant and ideal constituent of the binary system.*

The expression 15.26 for Stefan's law may be integrated between the planes x_1 and x_2:

$$m_w \int_{x_1}^{x_2} dx = -\frac{DA}{GT} M_w p_t \int_{p_{w_1}}^{p_{w_2}} \frac{dp_w}{p_t - p_w}$$

$$m_w (x_2 - x_1) = \frac{DA}{GT} M_w p_t \int_{p_{w_1}}^{p_{w_2}} \frac{dp_w}{p_w - p_t}$$

$$= \frac{DA}{GT} M_w p_t \log_e \frac{p_{w_2} - p_t}{p_{w_1} - p_t}$$

$$= \frac{DA}{GT} M_w p_t \log_e \frac{p_t - p_{w_2}}{p_t - p_w}$$

$$\therefore (m_w)_{total} = \frac{DA}{GT} \frac{M_w p_t}{(x_2 - x_1)} \log_e \frac{p_{a_2}}{p_{a_1}} \quad ...(15.27)$$

Introducing the concept of log mean partial pressure of air (LMPA);

$$LMPA = \frac{p_{a_2} - p_{a_1}}{\log_e \dfrac{p_{a_2}}{p_{a_1}}}$$

or $$\log_e \frac{p_{a_2}}{p_{a_1}} = \frac{p_{a_2} - p_{a_1}}{LMPA}$$

The equation 15.27 can be recast as

$$(m_w)_{total} = \frac{DA}{GT} \frac{M_w p_t}{(x_2 - x_1)} \frac{(p_{a_2} - p_{a_1})}{LMPA}$$

$$= \frac{DA}{GT} \frac{M_w p_t}{(x_2 - x_1)} \frac{(p_{w_1} - p_{w_2})}{LMPA} \quad ...(15.28)$$

When the partial pressure of water vapour does not change appreciably compared with the total pressure of water vapour air mixture, then the mass of water vapours can be calculated (without appreciable error) by using arithmetic mean partial pressure of air $(p_{a1} + p_{a2})/2$ instead of log mean partial pressure.

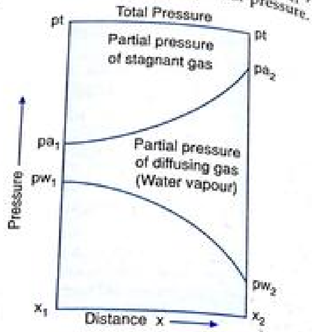


Fig. 15.5. Partial pressures of diffusing of gas (water vapour) and stagnant gas (air)

It follows from equation 15.27 that at any point x between x_1 and x

$$m_w (x - x_1) = \frac{DA}{GT} M_w p_t \log_e \frac{p_t - p_w}{p_t - p_{w_1}}$$

or $$p_w = p_t - (p_t - p_{w_1}) \times \exp\left[\frac{m_w}{p_t M_w} (x - x_1) \frac{GT}{DA} \right]$$

and for the stagnant gas (air)

$$p_a = p_t - p_w$$

$$= (p_t - p_{w_1}) \exp\left[\frac{M_w}{p_t - M_w} (x - x_1) \frac{GT}{DA} \right]$$

The variation of pressure p_w and p_a in the direction of x has been plotted in Fig. 15.5. These curves also represent the variation of concentration in the direction of x.

EXAMPLE 15.11

Derive an expression for the steady state diffusion of a gas A through another stagnant gas B.

Oxygen is diffusing through stagnant carbon monoxide at 0°C and 760 mm of Hg pressure under steady state conditions. The partial pressure of oxygen at two planes 0.3 cm apart is 100 mm of Hg and 25 mm of Hg respectively. Calculate the rate of diffusion of oxygen in gm-moles through cm² area. It may be presumed that:

Diffusivity of oxygen in carbon monoxide $= 0.185$ cm²/s

Gas constant $R = 82.06$ cm² atm/gm mole K

Solution : Let subscripts o and c refer to oxygen and carbon monoxide.

Partial pressures of oxygen on the given planes are :

$$p_{o_1} = \frac{100}{760} = 0.1316 \text{ atm}$$

$$p_{o_2} = \frac{25}{760} = 0.329 \text{ atm}$$

Partial pressures of carbon monoxide are :

$$p_{c_1} = 1 - 0.1316 = 0.8684 \text{ atm}$$

$$p_{c_2} = 1 - 0.0329 = 0.9671 \text{ atm}$$

Log mean partial pressure for non-diffusing carbon monoxide is given by:

$$LMPC = \frac{p_{c_1} - p_{c_2}}{\log_e \dfrac{p_{c_1}}{p_{c_2}}}$$

$$= \frac{0.8684 - 0.9671}{\log_e \dfrac{0.8684}{0.9671}}$$

$$= 0.9173 \text{ atm}$$

The diffusion rate of oxygen is given by,

$$N_{o_2} = \frac{DA}{GT} \times \frac{p_t}{(x_2 - x_1)} \times \left(\frac{p_{o_1} - p_{o_2}}{LMPC} \right)$$

$$= \frac{0.185 \times 1}{82.06 \times 273} \times \frac{1}{0.3} \frac{(0.1316 - 0.0329)}{0.9173}$$

$$= \mathbf{2.962 \times 10^{-6}} \textbf{ gm-mol/s}$$

Alternatively,

$$N_{o_2} = \frac{DA}{GT} \times \frac{p_t}{(x_2 - x_1)} \times \log_e \frac{p_{c_2}}{p_{c_1}}$$

$$= \frac{0.185 \times 1}{82.06 \times 273} \times \frac{1}{0.3} \times \log_e \frac{0.9671}{0.8684}$$

$$= \mathbf{2.963 \times 10^{-6}} \textbf{ gm-mol/s}$$

EXAMPLE 15.12

Estimate the diffusion coefficient of carbon tetrachloride into air from the following data recorded in a Stefan-tube experiment with carbon tetrachloride and oxygen:

Diameter of tube and its length above liquid surface : 1 cm and 15 cm respectively

Temperature and pressure maintained : 0°C and 760 mm of mercury

Evaporation of carbon tetrachloride : 0.03 gm in 10 hour

Vapour pressure of carbon tetrachloride : 33 mm of mercury

Solution : The molecular weights of carbon tetrachloride M_c and oxygen M_o are :

$$M_c = 12 + 4\,(31.5)$$

$$= 154 \text{ and } M_o = 32$$

Partial pressures of these elements at bottom and top are :

$$p_{c_1} = 33 \text{ mm of Hg} ; p_{c_2} = 0$$

$$p_{o_1} = 760 - 33 = 727 \text{ mm of Hg}$$

$$p_{o_2} = 760 \text{ mm of Hg}$$

and $p_t = 1 \times 10^5$ N/m²

Further,

$$m = \frac{0.03 \times 10^{-3}}{10 \times 3600}$$

$$= 8.333 \times 10^{-10} \text{ kg/s}$$

$$= \frac{8.333 \times 10^{-10}}{154}$$

$$= 5.41 \times 10^{-12} \text{ kg-mol/s}$$

$$A = \frac{\pi}{4} (0.01)^2 = 7.854 \times 10^{-5}$$

$$(x_2 - x_1) = 15 \text{ cm} = 0.15 \text{ m}$$

Inserting these values in the diff equation,

$$m = \frac{DA}{GT} \frac{p_t}{(x_2 - x_1)} \log_e \frac{p_{o_2}}{p_{o_1}}$$

$$5.41 \times 10^{-12} = \frac{D \times 7.854 \times 10^{-5}}{8314 \times 273} \times \frac{10^5}{0.15} \times \log_e \frac{760}{727}$$

$$= 1.024 \times 10^{-6}\, D$$

∴ Diffusion coefficient of carbon tetrachloride in air

$$D = \frac{5.41 \times 10^{-12}}{1.024 \times 10^{-6}}$$

$$= \mathbf{0.528 \times 10^{-5}\ m^2/s}$$

EXAMPLE 15.13

An open tank, 6 mm in diameter, contains 1 mm deep layer of benzene (Mol wt = 78) at its bottom. The vapour pressure of benzene in the tank is 13.15 kN/m² and its diffusion takes place through a stagnant air film 2.5 mm thick. At the operating temperature of 20°C, the diffusivity of benzene in the tank is 8.0 × 10⁻⁶ m²/s. If the benzene has a density of 880 kg/m³, calculate the time taken for the entire benzene to evaporate. Take atmospheric pressure as 101.3 kN/m² and neglect any resistance to diffusion of benzene beyond the air film.

Solution : Let subscripts b and a refer to benzene and air respectively.

Partial pressures of benzene and air at the two levels are :

$$p_{b_1} = 13.5\ \text{kN/m}^2$$

and $p_{b_2} = 0.0\ \text{kN/m}^2$

$$p_{a_1} = p_t - p_{b_1} = 101.3 - 13.5 = 87.8\ \text{kN/m}^2$$

$$p_{a_2} = p_t - p_{b_2} = 101.3 - 0.0 = 101.3\ \text{kN/m}^2$$

The diffusion rate of benzene is given by

$$N_b = \frac{DA}{GT} \frac{p_t}{(x_2 - x_1)} \log_e \frac{p_{a_2}}{p_{a_1}}$$

$$= \frac{8.0 \times 10^{-6} \times \frac{\pi}{4}(6)^2}{8.314 \times 293} \times \frac{101.3}{0.0025} \times \log_e \frac{101.3}{87.8}$$

$$= 5.378 \times 10^{-4}\ \text{kg-mol/s}$$

$$= (5.378 \times 10^{-4}) \times 78$$

$$= 0.04195\ \text{kg/s}$$

Mass of benzene which is to be evaporated,

$$= \frac{\pi}{4}(6)^2 \times 0.001 \times 880$$

$$= 24.87\ \text{kg}$$

∴ Time needed for evaporation

$$= \frac{24.87}{0.04195} = \mathbf{592.85\ s}$$

EXAMPLE 15.14

Water is available at the bottom of a well which is 2.5 m in diameter and 5 m deep. Estimate its diffusion rate into dry atmospheric air at 25°C. The diffusion coefficient for the system is approximated to be 0.0925 m²/hr and the atmospheric pressure is 1.032 bar.

Solution : The partial pressure of water vapour at the water surface ($x = 0$) corresponds to the saturation pressure of water vapour at 25°C.

For steam tables;

$$p_{w_1} = 0.0317\ \text{bar}$$

$$\therefore \quad p_{a_1} = p_t - p_{w_1} = 1.032 - 0.0317 = 1.0003\ \text{bar}$$

Since dry air does not contain any water vapour at the top ($x = 5$ m),

$$p_{w_2} = 0.0$$

$$p_{a_2} = 1.032 - 0.0 = 1.032\ \text{bar}$$

Diffusion rate of water vapour is given by,

$$m_w = \frac{DA}{GT} \frac{p_t}{(x_2 - x_1)} \log_e \frac{p_{a_2}}{p_{a_1}}$$

Inserting the appropriate values in consistent units,

$$m_w = \frac{0.0925 \times \left[\frac{\pi}{4} \times 2.5^2\right]}{8314 \times (273 + 25)} \times \frac{1.032 \times 10^5}{(5.0 - 0.0)} \log_e \frac{1.032}{1.0003}$$

$$= 1.179 \times 10^{-4}\ \text{kg-mol/hr}$$

$$= 1.178 \times 10^{-4} \times 18$$

$$= \mathbf{2.122 \times 10^{-3}\ kg/hr}$$

EXAMPLE 15.15

Due to accidental opening of a valve, the water has been spilled out on the floor of an industrial plant. The water layer is 1.25 mm and at 25°C temperature. The temperature and pressure of air are 25°C and 1 atm (= 1.032 bar) respectively. Make calculations for the time required to completely evaporate the water layer if evaporation takes place through an air film of 6 cm thickness. Assume the following data :

Diffusion coefficient of water into air

$$= 0.26 \times 10^{-4}\ m^2/s$$

Absolute humidity of air

$$= 2\ \text{gm per kg of air}$$

Solution : The partial pressure of water corresponds to the saturation pressure of water at 25°C.

From steam tables :

$$p_{w_1} = 0.0317\ \text{bar}$$

$$\therefore \quad p_{a_1} = p_t - p_{w_1} = 1.032 - 0.0317 = 1.0003\ \text{bar}$$

The partial pressure of water at the surface of water, p_{w_2}, can be worked out by using the following expression for the specific humidity :

Specific humidity

$$= \frac{0.622\, p_{w_2}}{p_t - p_{w_2}}$$

$$2 \times 10^{-3} = \frac{0.622 \times p_{w_2}}{1.032 - p_{w_2}}$$

$$0.00206 - 2 \times 10^{-3} p_{w_2} = 0.622\ p_{w_2}$$

$$p_{w_2} = \frac{0.00206}{0.624} = 0.0033\ \text{bar}$$

$$\therefore \quad p_{a_2} = p_t - p_{w_2} = 1.032 - 0.0033 = 1.0287\ \text{bar}$$

Diffusion rate of water is given by

$$m_w = \frac{DA}{GT} \frac{p_t}{(x_2 - x_1)} \log_e \frac{p_{a_2}}{p_{a_1}}$$

Inserting the appropriate values in consistent units,

$$m_w = \frac{(0.26 \times 10^{-4})}{8314 \times 298} \times 1 \times \frac{1.032 \times 10^5}{0.006} \log_e \frac{1.0287}{1.0003}$$

$$= 5.05 \times 10^{-6}\ \text{kg mol/m}^2\ \text{s}$$

$$= (5.05 \times 10^{-6}) \times 18$$

$$= 90.9 \times 10^{-6}\ \text{kg/m}^2\ \text{s}$$

Total amount of water to be evaporated per m² of area

$$= \left[(1.25 \times 10^{-3}) \times 1\right] \times 1000$$

$$= 1.25\ \text{kg/m}^2$$

∴ Time required

$$= \frac{1.25}{90.9 \times 10^{-6}}$$

$$= 0.01375 \times 10^6\ \text{s} = \mathbf{3.82\ hr}$$

EXAMPLE 15.16

Nitrogen (Mol wt = 28) is diffusing under steady state conditions through a non-diffusing gaseous mixture which has the following composition by volume :

$$C_2H_4 = 30\%\ ;\ C_2H_6 = 20\%\ ;\ C_4H_{10} = 50\%.$$

At the operating temperature of 25°C and a total pressure of 100 kN/m², the partial pressures of nitrogen at two planes 10 mm apart are 13.5 and 6.65 kN/m² respectively. Workout the diffusion rate of nitrogen across the two planes.

The diffusivities of nitrogen through C_2H_4, C_2H_6 and C_4H_{10} are 16.5 × 10⁻⁶, 14.8 × 10⁻⁶ and 9.5 × 10⁻⁶ m²/s respectively.

Solution : For steady state diffusion through a non-diffusing multi-component mixture of constant composition, the effective diffusivity is calculated from :

$$D = \frac{1}{\frac{\dot{n}_b}{D_{ab}} + \frac{\dot{n}_c}{D_{ac}} + \frac{\dot{n}_d}{D_{ad}}}$$

where,

$n_b^{\cdot}, n_c^{\cdot}, n_d^{\cdot}, \ldots$ = mole fraction compositions of the mixture on a free basis

$D_{ab}, D_{ac}, D_{ad} \ldots$ = Diffusivities of species A through B, C, D .....

Therefore the effective diffusivity of nitrogen through gaseous mixture of given composition is

$$D = \frac{1}{\dfrac{0.3}{16.5\times10^{-6}} + \dfrac{0.20}{14.8\times10^{-6}} + \dfrac{0.50}{9.5\times10^{-6}}}$$

$= 11.86 \times 10^{-6}$ m²/s

Let subscripts n and m refer to nitrogen and gaseous mixture respectively.

The diffusion rate of nitrogen is,

$$m_n = \frac{DA}{GT}\frac{p_t}{(x_2 - x_1)}\log_e \frac{p_{m_2}}{p_{m_1}}$$

For the given case :

$$p_{n_1} = 13.5\ \text{kN/m}^2 ;\ p_{n_2} = 6.65\ \text{kN/m}^2$$

$$p_{m_1} = p_t - p_{n_1} = 100 - 13.5 = 86.5\ \text{kN/m}^2$$

$$p_{m_2} = p_t - p_{n_2} = 100 - 6.65 = 93.35\ \text{kN/m}^2$$

$$\therefore \quad m_n = \frac{11.86\times 1}{8.314\times 298}\times\frac{100}{0.01}\log_e\frac{93.35}{86.5}$$

$= 3.646 \times 10^{-6}$ kg-mol/s

$= (3.646 \times 10^{-6}) \times 28$

$= \mathbf{1.021 \times 10^{-4}}$ **kg/s**

15.8. MASS TRANSFER COEFFICIENT

Recapitulate that for one-dimensional steady-state diffusion of a fluid (gas or liquid) across a layer of thickness $(x_2 - x_1)$, the mass diffusion is given by:

$$m_b = \frac{DA\left(C_{b_1} - C_{b_2}\right)}{(x_2 - x_1)}$$

where C_{b_1}, C_{b_2} denote the fluid concentrations at the two faces.

Analogous to convective heat transfer ($Q = h\ A\ \Delta t$), the mass diffusion may be rewritten as

$$m_b = \frac{DA\left(C_{b_1} - C_{b_2}\right)}{(x_2 - x_1)} = h_{mc}\,A\left(C_{b_1} - C_{b_2}\right)$$

The mass transfer coefficient (based on concentration differences) is then defined as

$$h_{mc} = \frac{D}{(x_2 - x_1)} \qquad \text{...(15.29)}$$

The mass transfer coefficient can be expressed in terms of partial pressure differences also. For example, the mass diffusion in equi-molar counter diffusion may be written as :

$$m_b = DA\frac{M_b}{GT}\frac{(p_{b_1} - p_{b_2})}{(x_2 - x_1)}$$

$$= \frac{D}{(x_2 - x_1)}\frac{M_b}{GT}A\left(p_{b_1} - p_{b_2}\right)$$

$$= h_{mc}\frac{M_b}{GT}A\left(p_{b_1} - p_{b_2}\right)$$

$$= h_{mp}\,A\left(p_{b_1} - p_{b_2}\right)$$

Thus the relationship between mass transfer-coefficient based upon pressure (h_{mp}) and concentration (h_{mc}) differences may be written as

$$h_{mp} = h_{mc}\frac{M_b}{GT} = h_{mc}\frac{1}{RT}$$

For diffusion of water vapour through a layer of stagnant air, we had

$$m_w = \frac{DA}{GT}\frac{M_w\,p_t}{(x_2 - x_1)}\log_e\frac{p_t - p_{w_2}}{p_t - p_{w_1}}$$

$$= h_{mp}\,A\left(p_{w_1} - p_{w_2}\right)$$

Therefore the mas transfer coefficient h_{mp} hased upon pressure differences works out to be

$$h_{mp} = \frac{D\,p_t}{(x_2 - x_1)\left(p_{w_1} - p_{w_2}\right)}\frac{M_w}{GT}\log_e\frac{p_t - p_{w_2}}{p_t - p_{w_1}} \qquad \text{...(15.30)}$$

The corresponding expression for mass transfer coefficient h_{mc} based upon concentration difference would be

$$h_{mc} = \frac{D\,p_t}{(x_2 - x_1)\left(p_{w_1} - p_{w_2}\right)}\log_e\frac{p_t - p_{w_2}}{p_t - p_{w_1}} \qquad \text{...(15.31)}$$

15.9. CONVECTIVE MASS TRANSER

Molecular diffusion mass transfer is analogous to conduction heat transfer. During molecular diffusion, the bulk velocities are insignificant and only diffusion velocities are considered. Convective mass transfer is identical to heat transfer by convection. During convective mass transfer, the bulk velocities are significant *i.e.*, both the species in a binary mixture are moving with appreciable velocities. Like heat convection, the mass transfer by convection can take place under free and forced conditions. In free (natural) convective mass transfer, the buoyancy force causing circulation results from the difference in densities of vapour-gas mixture of varying compositions. Evaporation of alcohol is an example of free convective mass transfer. In forced convective mass transfer, an external agency such as pump or fan imparts forced circulation to the species. The evaporation of water from an ocean when air flows past it constitutes forced convective mass transfer. The convective mass transfer may be further associated with laminar and turbulent flow conditions.

Convective mass transfer involves the transportation of material between a boundary surface and a moving fluid or between two immiscible moving fluids. It is prescribed by the relation

$$\frac{m_b}{A} = h_m\,(\Delta C_b)$$

where h_m is the mass transfer coefficient of species B and m_b/A is the mass flux which occurs in the direction of decreasing concentration.

15.10. NON-DIMENSIONAL CORRELATIONS FOR CONVECTIVE MASS TRANSPORT

(*i*) Recall that for the boundary layer development on a flat plate, the equations for conservation of momentum and energy are :

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\text{and} \quad u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \alpha\frac{\partial^2 t}{\partial y^2}$$

In analogy with these, the concenration equation may be written as

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} \qquad \text{...(15.32)}$$

where C is the concentration of the component which is diffusing through the boundary layer.

- The dimensionless ratio ν/α defines the *Prandtl number* and it forms the connecting link between the velocity and temperature profiles. These profiles become identical when $Pr = \nu/\alpha = 1$
- The dimensionless ratio ν/D defines the *Schmidt number*, $Sc = \nu/D$, and it forms the connecting link between the velocity and concentration profiles. These profiles will have the same shape when the Schmidt number equals unity
- The dimensionless ratio α/D defines the *Lewis number*, $Le = \alpha/D$, and it forms the connecting link between the temperature and concentration profiles. When Lewis number equals unity, these two profiles are identical.

Obviously the solution for velocity, temperature and concentration boundary layers will be same if

$$Pr = Sc = Le$$

and all the three boundary layers coincide.

(*ii*) A close similarity existing between the governing equations for heat, mass and momentum suggests that the empirical correlations for mass transfer coefficent h_m

are similar to those for the heat transfer coefficient. For example, corresponding to the expression

$$Nu = \frac{h\,l}{k} = f\,(Re\,,\,Pr)$$

for heat transfer coefficient, we have the following expression for convective mass transfer coefficient.

$$Sh = \frac{h_m\,l}{D} = f\,(Re\,,\,Sc) \qquad ...(15.33)$$

where Sh is a non-dimensional mass transfer number called the *Sherwood number*. The mass transfer coefficient under forced flow condition as prescribed by equation 15.33 can be worked out by the method of dimensional analysis. The physical quantities affecting mass transfer by forced convection with their dimensions in MLT-system of unit are:

Variable	h_m	l	ρ	μ	V	D
Dimension	LT^{-1}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L^2T^{-1}

It can be premised that the functional relationship is

$$f\,(D,\,\rho,\,l,\,h_m,\,\mu,\,V) = 0$$

There are six physical quantities and three fundamental units, hence (6-3) or 3-π terms. We choose fluid density ρ, fluid diffusivity D and characteristic lenght l as the core group (repeated variables) with unknown exponents and establish the π-terms as follows:

$$\pi_1 = D^a\,\rho^b\,l^c\,h_m \qquad ...(i)$$

$$M^0\,L^0\,T^0 = (L^2T^{-1})^a\,(ML^{-3})^b\,(L)^c\,(LT^{-1})$$

Equating the exponents of fundamental dimensions on both sides:

$M:\quad 0 = b$

$L:\quad 0 = 2a - 3b + c + 1$

$T:\quad 0 = -a - 1$

Solution gives : $a = -1$, $b = 0$ and $c = 1$

Substituting these values in expression (*i*),

$$\pi_1 = D^{-1} \times \rho^0 \times l^1 \times h_m = \frac{h_m\,l}{D}$$

Following the same procedure, we would obtain :

$$\pi_2 = D^a\,\rho^b\,l^c\,\mu = \frac{\mu}{\rho D}$$

$$\pi_3 = D^a\,\rho^b\,l^c\,V = \frac{V\,l}{\rho D}$$

We can also define,

$$\pi_4 = \frac{\pi_3}{\pi_2} = \frac{Vl}{D} \times \frac{\rho D}{\mu} = \frac{\rho V l}{\mu} = Re$$

(Reynolds number)

As per Buckingham statement, the functional relation can be written as

$$\pi_1 = f\,(\pi_3\,,\,\pi_4)$$

or $\quad Sh = f\,(Sc,\,Re)$

Similar non-dimensional relations can also be set up for the natural (free) convective mass transfer coefficient. The parameters considered are

$$h_m = \{D,\,\rho,\,l,\,\mu,\,(g\,\Delta\rho)\}$$

where $(g\,\Delta\rho)$ is the buoyant force whose dimension is $(ML^{-2}T^{-2})$

$$\therefore f\,\{D,\,l,\,\mu,\,h_m,\,\rho,\,(g\,\Delta\rho)\} = 0$$

Adopting the Buckingham π-theorem, the following non-dimensional groups can be obtained

$$\pi_1 = \frac{h_m\,l}{D} = Sh \text{ (Sherwood number)}$$

$$\pi_2 = \frac{\mu}{\rho D} = Sc \text{ (Schmidt number)}$$

$$\pi_3 = \frac{l^3\,g\,\Delta\rho}{\mu D}$$

$$\pi_4 = \frac{\pi_3}{\pi_2} = \frac{l^3\,g\,\Delta\rho}{\mu D} \times \frac{\rho D}{\mu} = \frac{\rho l^3\,g\,\Delta\rho}{\mu^2}$$

$$= Gr \text{ (Mass Grashof number)}$$

Obviously then the functional relationship for free convective mass transfer can be written as

$$\pi_1 = f\,(\pi_2,\,\pi_4)$$

or $\quad Sh = f\,(Sc,\,Gr) \qquad ...(15.34)$

(*iii*) The following empirical relations based on analogy with heat transfer have been suggested for mass transfer coefficient under different flow conditions:

For laminar and turbulent boundary layer flows past a flat plate oriented at zero angle of incidence, the local mass transfer coefficent is

$$Sh_x = \frac{h_m\,x}{D} = 0.332\,(Re_x)^{0.5}\,(Sc)^{0.33}$$

$$\text{and} \quad Sh_x = \frac{h_m\,x}{D} = 0.0298\,(Re_x)^{0.8}\,(Sc)^{0.33} \qquad ...(15.35)$$

The average mass transfer coefficient can be worked out by integrating the above expressions for the plate length. The corresponding results are.

$$\overline{Sh} = 0.664\,(Re_1)^{0.5}\,(Sc)^{0.33}$$

$$\text{and} \quad \overline{Sh} = 0.036\,(Re)^{0.8}\,(Sc)^{0.33} \qquad ...(15.36)$$

For mixed boundary layer conditions with transition occuring at a ciritical Reynolds number of $Re_c = 5 \times 10^5$, the correlations for heat and mass transfer are :

$$\overline{Nu} = (0.036\,Re_1^{\,0.8} - 836)\,pr^{0.33}$$

$$\overline{Sh} = (0.036\,Re_1^{\,0.8} - 836)\,Sc^{0.33} \qquad ...(15.37)$$

For forced mass convection through a tube, the Sherwood number is prescribed by the following empirical correlations suggested by Gilliand :

$$Sh = 0.023\,(Re)^{0.83}\,(Sc)^{0.44} \qquad ...(15.38)$$

which is applied when

$$2000 < Re < 35000 \text{ and } 0.6 < Sc < 2.5$$

For natural mass convection through a tube, we adopt, the following Steinberger and Treybol relation.

$$Sh = 2 + 0.57\,(Gr\,Sc)^{0.25}$$

when $Gr\,Sc < 10^8$

$$= 2 + 0.025\,(Gr\,Sc)^{0.33}\,(Sc)^{0.245}$$

when $Gr\,Sc > 10^8 \qquad ...(15.39)$

(*iv*) The mass transfer coefficient can be expressed in terms of friction factor by extending the Reynolds and Colburns analogies for flow through pipes

Recalling the Reynolds analogy for heat transfer (section 12.10)

$$\frac{Nu}{Re\,Pr} = St = \frac{h}{\rho c_p V} = \frac{C_f}{2}$$

The corresponding formulation for mass transfer is

$$\frac{Sh}{Re\,Sc} = St_m = \frac{h_m}{V} = \frac{C_f}{2} \qquad ...(15.40)$$

The Reynolds analogy is valid only when Pr and Sc = 1. When Pr and Sc are different from one, we adopt the Colburn analogy which is of the form :

$$j_H = St\,(Pr)^{2/3} = \frac{C_f}{2} \quad 0.5 < Pr < 50$$

$$\text{and} \quad j_M = St_m\,(Sc)^{2/3}$$

$$= \frac{C_f}{2} \quad 0.6 < Sc < 3000 \qquad ...(15.41)$$

where j_m and j_M are Colburn factors for heat and mass transfer respectively.

When the above identities are combined with the non-dimensional heat transfer coefficient for turbulent flow through pipes,

$$Nu = 0.023\,(Re)^{0.8}\,(Pr)^{0.33}$$

We get :

$$j_H = \frac{C_f}{2} = 0.023\,(Re)^{-0.2}$$

$$j_M = \frac{C_f}{2} = 0.023\,(Re)^{-0.17} \qquad ...(15.42)$$

Further it may be recalled that for flow over smooth flat plates

$$\frac{C_f}{2} = \frac{0.332}{(Re)^{0.5}} \text{ (laminar flow)}$$

$$\text{and} \quad \frac{C_f}{2} = \frac{0.0296}{(Re)^{0.2}} \text{ (turbulent flow)}$$

(*v*) When the friction coefficient C_f is eliminated from Colburn analogies for heat and mass transfer, we get:

$$j_H = j_M$$

$$\text{or} \quad St\,(Pr)^{2/3} = St_m\,(Sc)^{2/3}$$

$$\text{or} \quad \frac{h}{\rho c_p V}\,(Pr)^{2/3} = \frac{h_m}{V}\,(Sc)^{2/3}$$

$$\text{or} \quad \frac{h}{h_m} = \rho c_p \left(\frac{Sc}{Pr}\right)^{2/3}$$

$$= \rho c_p \left(\frac{\alpha}{D}\right)^{2/3}$$

$$= \rho c_p (Le)^{2/3} \quad \text{...(15.43)}$$

which prescribes the relationship between heat and mass transfer coefficients. This correlation, referred to as Lewis relation, is useful in the solution of problems involving simultaneous transport of heat and mass.

EXAMPLE 15.17

Air at 30°C temperature flows at 45 m/s past a wet flat plate 0.5 m long. Make calculations for the mass transfer coefficient of water vapour in air. Assume that the water vapour content of air initially is negligible and take the following thermo-physical properties of air:

$$D = 0.256 \times 10^{-4}\ m^2/s$$
$$\mu = 1.86 \times 10^{-5}\ kg/m\text{-}s$$
$$c_p = 1.005\ kJ/kg°C$$
$$P_r = 0.701 \quad \text{and} \quad \rho = 1.165\ kg/m^3$$

Solution : Schmidt number Sc

$$= \frac{\mu}{\rho D} = \frac{1.86 \times 10^{-5}}{1.165 \times 0.256 \times 10^{-4}}$$
$$= 0.623$$

Reynolds number Re

$$= \frac{Vl\rho}{\mu} = \frac{45 \times 0.5 \times 1.165}{1.86 \times 10^{-5}}$$
$$= 14.09 \times 10^5$$

Obviously flow is turbulent and therefore,

$$j_M = \frac{h_m}{V}(Sc)^{2/3} = 0.0296(Re)^{-0.2}$$

∴ Mass transfer coefficient of water vapour,

$$h_m = \frac{0.0296\,(Re)^{-0.2} \times V}{(Sc)^{0.667}}$$

$$= \frac{0.0296 \times \left(14.09 \times 10^5\right)^{-0.2} \times 45}{(0.623)^{0.667}}$$

$$= \mathbf{0.1076\ m/s}$$

EXAMPLE 15.18

Air at 1 atm and 25°C, containing small quantities of iodine, flows with a velocity of 5.25 m/s inside a 3 cm diameter tube. Determine mass transfer coefficient for iodine transfer from the air stream to the weak surface. Assume the following thermo-physical properties of air.

$$D = 0.82 \times 10^{-5}\ m^2/s$$
$$\nu = 15.5 \times 10^{-6}\ m^2/s$$

Solution : Schmidt number,

$$Sc = \frac{\mu}{\rho D} = \frac{\nu}{D} = \frac{15.5 \times 10^{-6}}{0.82 \times 10^{-5}} = 1.89$$

Reynolds number,

$$Re = \frac{Vd\rho}{\mu} = \frac{Vd}{\nu}$$

$$= \frac{5.25 \times 0.03}{15.5 \times 10^{-6}} = 10161$$

Obviously the flow is turbulent and therefore,

$$Sh = 0.023\ (Re)^{0.83}\ (Sc)^{0.44}$$
$$= 0.023\ (10161)^{0.83}\ (1.89)^{0.44}$$
$$= 64.44$$

Now, $Sh = \dfrac{h_m d}{D}$

∴ Mass transfer coefficient of iodine,

$$h_m = \frac{Sh\,D}{d} = \frac{64.44 \times 0.82 \times 10^{4}}{0.03}$$
$$= \mathbf{0.0176\ m/s}$$

EXAMPLE 15.19

Air at 20°C flows past a tray full of water with a velocity of 2.5 m/s. Calculate the evaporation rate of water if temperature on the water surface is 15°C. The tray measures 25 cm along the flow direction and its width is 40 cm.

The moving air has a total pressure of 1.01 bar and the partial pressure of water associated with it is 0.0075 bar. Take the following physical properties of air :

Density $\rho = 1.205\ kg/m^3$
kinematic viscosity $\nu = 15.06 \times 10^{-6}\ m^2/s$
and diffusivity $D = 0.15\ m^2/hr$

Solution : The flow Reynolds number is,

$$Re = \frac{\rho Vl}{\mu} = \frac{Vl}{\nu} = \frac{2.5 \times 0.25}{15.06 \times 10^{-6}}$$
$$= 0.416 \times 10^5$$

As $Re < 5 \times 10^5$, the flow over the plate is laminar in character. For laminar flow, the mass transfer coefficient (h_{mc}) based on concentration differences is calculated from the following empirical relation :

$$Sh = \frac{h_m l}{D} = 0.664\,(Re)^{0.5}\,(Sc)^{0.33}$$

$$Sc = \frac{\nu}{D} = \frac{15.06 \times 10^{-6} \times 3600}{0.15}$$
$$= 0.3614$$

$$\therefore \frac{h_m \times 0.25}{0.15} = 0.664\ (0.416 \times 10^5)^{0.5}\ (0.3614)^{0.33}$$
$$= 0.664 \times 203.96 \times 0.7146$$
$$= 96.794$$

$$h_m = \frac{96.794 \times 0.15}{0.25} = 58.076\ m/hr$$

Mass transfer coefficient based on pressure difference is,

$$h_{mp} = h_m \times \frac{M}{GT} = h_m \times \frac{1}{RT}$$
$$= 5.076 \times \frac{1}{287 \times (15 + 273)}$$
$$= 7.026 \times 10^{-4}\ m/hr$$

Mass diffusion of water,

$$m_w = h_{mp}\,A\,(p_{w_1} - p_{w_2})$$

where, p_{w_1} = partial pressure of water over the water surface
= saturation pressure of water at 15°
= 0.57 bar

and p_{w_2} = partial pressure of water vapour associated with air
= 0.0075 bar

$$\therefore \quad m_w = 7.026 \times 10^{-4} \times (0.25 \times 0.4) \times (0.017 - 0.0075) \times 105$$
$$= 0.0667\ kg/hr$$

SALIENT POINTS

1. Diffusion is the movement of a chemical species from a region of high concentration to a region of low concentration.
   The process of transfer of mass as a result of concentration difference is called mass transfer.
2. Modes of mass transfer are : diffusion, convection and change of phase.
3. For species A in a multicomponent mixture
   (*i*) Mass concentration or mass density ρ_a
   $$= \frac{\text{mass of species } A}{\text{volume of the mixture}}$$
   (*ii*) Molar concentration or molar density n_a
   $$= \frac{\text{number of molecules of species } A}{\text{volume of the mixture}}$$
   (*iii*) Mass fraction ρ_a^*
   $$= \frac{\text{mass concentration of species } A}{\text{total mass density of the mixture}}$$
   $$= \frac{\rho_a}{\rho}$$
   (*iv*) Mole fraction n_a^*
   $$= \frac{\text{number of moles of species } A}{\text{total number of moles of the mixture}}$$
   $$= \frac{n_a}{n}$$
4. For a binary mixure of species A and B
   (*i*) Mass average velocity V_{mass}
   $$= \frac{\rho_a V_a + \rho_b V_b}{\rho_a + \rho_b}$$
   $$= \frac{\rho_a V_a + \rho_b V_b}{\rho} = \rho_a^* V_a + \rho_b^* V_b$$
   (*ii*) Molar average velocity V_{molar}
   $$= \frac{n_a V_a + n_b V_b}{n_a + n_b}$$
   $$= \frac{n_a V_a + n_b V_b}{n} = n_a^* V_a + n_b^* V_b$$
5. Base on experiments, the molecular diffusio[n] is governed by Fick's law which is express[ed] as

$$N_b = \frac{m_b}{A} = -D_{bc}\frac{dC_b}{dx}$$

where $\frac{m_b}{A}$ is mass flux, *i.e.*, mass flow of species *B* per unit time per unit area, $\frac{dC_b}{dx}$ is concentration gradient of species *B* and this acts as the driving potential.

The parameter D_{bc} is the diffusion coefficient or diffusivity for the binary mixture of species *B* and *C*. The units of diffusion coefficient are m^2/s.

Liquid mass diffusivities are considerably smaller than those for gases. Diffusion in solids is even slower than that in liquids.

Fick's law of diffusion is analogous to Newton's law of viscosity and Fourier's law of heat conduction.

6. Equimolal diffusion between species *B* and *C* of a binary gas implies that each molecule of component *B* is replaced by each molecule of constituent *C* and *vice-versa*.

7. The diffusion rates of water vapours through air is given by Stefan's law prescribed as

$$(m_w)_{total} = -\frac{DA}{GT} M_w \frac{dp_w}{dx} \frac{p_t}{p_t + p_w}$$

where *G* is the universal gas constant and M_w is the molecular mass of water.

8. Convective mass transfer involves the transportation of material between a boundary surface and a moving fluid or between two immisible fluids, and it is prescribed by the relation

$$\frac{m_b}{A} = h_m (\Delta C_b)$$

where h_m is the mass transfer coefficient of species *B* and $\frac{m_b}{A}$ is the mass flux that occurs in the direction of decreasing concentration.

The dimensionless groups pertinent to convective mass transfer and their significant aspects are:

(*i*) Prandtl number

$$P_r = \frac{\nu}{\alpha} = \frac{\text{kinamatic viscosity}}{\text{thermal diffusivity}}$$

forms the connecting link between the velocity and temperature profiles; these profiles become identical when P_r equals unity.

(*ii*) Schmidt number,

$$S_c = \frac{\nu}{D} = \frac{\text{kinamatic viscosity}}{\text{diffusion coefficient}}$$

forms the connecting link between the velocity and concentration profiles; these profiles show identical behaviour when S_c equals unity.

(*iii*) Lewis number,

$$L_e = \frac{\alpha}{D} = \frac{\text{thermal diffusivity}}{\text{diffusion coefficient}}$$

forms the connecting link between the temperature and concentration profiles; these profiles become identical when L_e equals unity.

10. Corresponding to expression $Nu = \frac{hl}{k} = f(R_e, P_r)$ for heat transfer, we have the following expression for convective mass transfer

$$Sh = \frac{h_m l}{D} = f(R_e, S_c)$$

where *Sh* is a non-dimensional mass transfer number called Sherwood number.

The following empirical correlations have been suggested for local mass transfer coefficient for laminar and turbulent boundary layer flows past a flat plate.

$$Sh_x = \frac{h_m x}{D} = 0.332\,(R_{ex})^{0.5}\,(Sc)^{0.33} \quad \text{... laminar}$$

$$Sh_x = \frac{h_m x}{D} = 0.0298\,(R_{ex})^{0.8}\,(Sc)^{0.33} \quad \text{... turbulent}$$

The corresponding expressions for average mass transfer coefficient are

$$\overline{Sh} = 0.664\,(Re)^{0.5}\,(Sc)^{0.33}$$

and $$\overline{Sh} = 0.036\,(Re)^{0.8}\,(Sc)^{0.33}$$

REVIEW QUESTIONS

A. Conceptual and conventional questions:

1. What is meant by diffusion? Name the property of the molecules of a species that leads to its diffusion.
2. Define the process of mass transfer and list some industrial applications where mass transfer is involved.
3. (*a*) State and explain the different modes of mass transfer.
   (*b*) Distinguish between molecular diffusion, thermal diffusion, pressure diffusion and forced diffusion.
4. Define and distinguish between:
   (*i*) mass flux and molar flux
   (*ii*) local mass average velocity and local molar average velocity
5. Distinguish between molecular diffusion and eddy diffusion and indicate which takes place predominantly in the following situations:
   (*i*) atomisation of an oil droplet in a diesel engine cylinder
   (*ii*) vaporisation of petrol in a carburetor
   (*iii*) diffusion welding of metals
   (*iv*) water cooling in spray ponds
   (*v*) air pollution due to car exhaust
6. State Fick's law of diffusion. Define the various symbols used and give their units.
   Show the similarity of this law to Fourier equation for conduction and Newtons' equation for shear stress. Express Fick's law in terms of partial pressures for diffusion of component A into component B and of component B into component A.
7. (*i*) State Fick's law of diffusion? What is the driving force for mass transfer?
   (*ii*) Define diffusivity, diffusional resistance and mass transfer coefficient. Write the units for each term.
8. What do you understand by the term rate of diffusion ? Show in which case the rate of diffusion is more :
   (*i*) uni-component,
   (*ii*) equimolal.
   Gas A is diffusing through the non-diffusing gas mixture of B and C in the volume ratio 2 : 1. The partial pressure of A at two planes 0.15 cm apart is 100 and 50 mm of Hg respectively. The diffusivities of gas A through B and C are stated to be [illegible] cm^2/s and 0.184 cm^2/s respectively. If the system operates at a total pressure of 1 atm and 0°C temperature, calculate the rate of diffusion of gas A in gm-moles/cm^2 s.
9. (*a*) Deduce equations for flux of diffusion (moles/hr/unit area) at steady state :
   (*i*) When component A is diffusing through a non-diffusing component B.
   (*ii*) When equimolar counter diffusion has taken place between components A and B.
   (*b*) Equimolal counter current diffusion occurs between components A and B under steady state conditions at 0°C and 760 mm of Hg pressure. The partial pressure of component A at two planes, 0.2 cm apart, is 100 and 50 mm of Hg respectively. Calculate the rate of diffusion of A if its diffusivity is stated to be 0.165 cm^2/s.
10. Derive an expression for diffusion on one gas through a stagnant gas in terms of logarithmic mean partial pressure. Consider temperature and pressure of the system to be constant.
11. Write out the momentum, energy and concentration equations and bring out their similarities. Proceed to define Prandtl, Schmidt and Lewis numbers.
12. Air at temperature 20°C and relative humidity 40% flows past a water surface at an average velocity of 1.2 m/s. The length of water surface parallel to the direction of air flow is 20 cm, and the average surface temperature of water is stated to be 15°C.
    The system has a total pressure of 1.033 bar; partial pressure of saturated water vapour at 20°C and 15°C temperature is 0.025 bar and 0.0175 bar respectively.
    Take diffusion coefficient D = 0.26 cm^2/s and kinematic viscosity of air $\nu = 14.8 \times 10^{-6}$ m/s.
13. Show that the total mass of water vapour diffused from a water column to the air passing over the water container is given by :

$$(m_w)_{total} = \frac{DA}{GT}\frac{M_w p_t}{(x_2 - x_1)}\log_e\left(\frac{p_t - p_{w_2}}{p_t - p_{w_1}}\right)$$

where D is diffusion coefficient, A is cro sectional area of water column, G is unive

gas constant, T is absolute temperature, M_w is molecular weight of water vapour, $(x_2 - x_1)$ is height of container above the water level, p_t is the total pressure, p_{w1} is the partial pressure of water vapour at the water surface and p_{w2} is the partial pressure of water vapour at the top of container.

14. Water evaporates from the surface of a lake. Derive Stefan's equation for the rate of evaporation. State the assumptions made. Estimate the diffusion rate of water from the bottom of a test tube 1.5 cm in diameter and 15 cm long into dry atmospheric air at 25°C. Take diffusion coefficient $D = 0.256$ cm^2/s.

B. Fill in the blanks with appropriate word/words :

1. The movement of a chemical species from a region of high concentration to a region of low concentration is called ....
2. The mass of species i that passes through a unit area per unit time is called ....
3. The process of mass transfer continues so long as there is .... in a system or mixture.
4. .... occurs when a substance diffuses through a layer of stagnant fluid.
5. The .... diffusion occurs when one of the diffusing fluids is in turbulent motion.
6. The .... represents the number of moles of a species per unit volume of the mixture.
7. Fick's law of diffusion is analogous to .... law of viscosity and .... law of heat conduction.

The units of mass diffusion coefficient are ....

.... diffusion between species A and B of a binary gas mixture refers to the isothermal diffusion process in which each molecule of component A is replaced by each molecule of constituent B and vice versa.

The velocity and concentration profiles show identical behaviour when .... is unity.

...swers : 1. diffusion; 2. mass flux; 3. ...ncentration difference; 4. molecular diffusion; ...eddy; 6. molar concentration or molar mass; ...Newton's, Fourier's; 8. m^2/s; 9. equimolar ...nter; 10. Schmidt number.

C. Multiple choice questions :

1. Consider the following statements:
   1. mass transfer refers to mass in transit due to species concentration gradient in a mixture
   2. there must be a mixture of two or more species for mass transfer to occur
   3. the species concentration gradient is the driving potential for mass transfer
   4. mass transfer by diffusion in analogous to heat transfer by conduction.

   Which of these statements are correct?

   (a) 1, 2 and 3 (b) 1, 2 and 4
   (c) 2, 3 and 4 (d) 1, 2, 3 and 4

2. Schmidt number is the ratio of
   (a) product of mass transfer coefficient and diameter to diffusivity of fluid
   (b) kinematic viscosity to thermal diffusivity of fluid
   (c) kinematic viscosity to diffusion coefficient of fluid
   (d) thermal diffusivity to diffusion coefficient of fluid

3. In a mass transfer process of diffusion of hot smoke in cold air in a power plant, the temperature profile and concentration profile will become identical when
   (a) Prandtl-number = 1
   (b) Nusset number = 1
   (c) Lewis number = 1
   (d) Schmidt number = 1

4. If heat and mass transfer take place simultaneously, then the ratio of heat transfer coefficient to mass transfer coefficient is a function of the ratio of
   (a) Schmidt and Reynolds numbers
   (b) Schmidt and Prandtl numbers
   (c) Nusselt and Lewis numbers
   (d) Reynolds and Lewis numbers

5. Match List-I with List-II and select the correct answer using the codes given below the list:

List - I (Phenomenon)	List - II (Number/law/factor)
A. Schmidt number	1. $\frac{k}{\rho c_p D}$
B. Thermal diffusivity	2. $\frac{h_m L}{D}$
C. Lewis number	3. $\frac{\mu}{\rho D}$
D. Sherwood number	4. $\frac{k}{\rho c_p}$

Codes :	A	B	C	D
(a)	4	3	2	1
(b)	4	3	1	2
(c)	3	4	2	1
(d)	3	4	1	2

6. Match the sets

Set-A	Set-B
(a) Fourier number	(i) forced convection
(b) Fourier law	(ii) free convection
(c) Grashof number	(iii) conduction
(d) Wien displacement law	(iv) transient heat flow
(e) Stanton number	(v) radiation

7. Match column-A with column-B

Column-A	Column-B
(a) Thermal diffusivity	(i) $\frac{\alpha\tau}{l^2}$
(b) Fourier number	(ii) $\frac{hd}{k}$
(c) Critical radius of insulation	(iii) $\frac{m c_p}{kl}$
(d) Graetz number	(iv) $\frac{k}{\rho c_p}$
(e) Nusselt number	(v) $\frac{2k}{h_o}$

8. Match the sets

Set-A	Set-B
(a) Reciprocity theorem	(i) free convection
(b) Biot number	(ii) forced convection
(c) Stanton number	(iii) boiling
(d) Grashof number	(iv) transient heat conduction
(e) Lieidon-frost effect	(v) radiation heat transfer

9. Match List-I with List-II and select the correct answer using the codes given below the lists (Notations have their usual meanings)

List-I	List-II
A. Fin	1. $\frac{UA}{C_{min}}$
B. Heat exchanger	2. $\frac{x}{2\sqrt{\alpha t}}$
C. Transient conduction	3. $\sqrt{\frac{hP}{kA}}$
D. Heisler chart	4. $\frac{hl}{k}$

Codes :	A	B	C	D
(a)	3	1	2	4
(b)	2	1	3	4
(c)	3	4	2	1
(d)	2	4	3	1

10. Match List-I with List-II and select the correct answer using the codes given below the lists.

List-I	List-II
A. Radiation heat transfer	1. Biot number
B. Conduction heat transfer	2. View factor
C. Forced convection	3. Fourier's law
D. Transient heat flow	4. Stanton number

Codes :	A	B	C	D
(a)	4	3	2	1
(b)	2	1	4	3
(c)	4	1	2	3
(d)	2	3	4	1

11. Match List-I with List-II and select the correct answer using the codes given below the lists.

List-I	List-II
A. Heat exchangers	1. View factor
B. Turbulent flow	2. Effectiveness
C. Free convection	3. Nusselt number
D. Radiation heat transfer	4. Eddy diffusivity

Codes :	A	B	C	D
(a)	3	1	2	4
(b)	2	4	3	1
(c)	3	4	2	1
(d)	2	1	3	4

12. Match List-I with List-II and select the correct answer using the codes given below the lists.

List-I	List-II
A. Grashof number	1. Mass transfer
B. Stanton number	2. Radiation
C. Sherwood number	3. Free convection
D. Reciprocity theorem	4. Forced convection

Codes :	A	B	C	D
(a)	4	3	1	2
(b)	3	4	1	2
(c)	4	3	2	1
(d)	3	4	2	1

Answers :

1. (*d*) 2. (*c*) 3. (*c*) 4. (*b*) 5. [illegible]
6. (*a-iv, b-iii, c-ii, d-v, e-i*)
7. (*a-iv, b-i, c-v, d-iii, e-ii*)
8. (*a-v, b-iv, c-ii, d-i, e-iii*) 9. (*a*) 10. [illegible]
11. (*b*) 12. (*b*)

Appendix-A

Conversion Factors : Some of the more important and frequently required conversion factors between SI, Metric and British system of units are given below :

(*i*) *Length and area*

1 m = 1000 mm = 100 cm = 39.37 in = 3.2802 feet
1 mm = 1000 microns (μ)
1 micron = 10^4 cm 10^4 Å (Angstrom)
1 km = 0.6213 mile
1 m^2 = 1550 in^2 = 10.764 ft²

(*ii*) *Volume and volume flow rate*

1 litre = 1000 cc = 61.024 in^3
1 m^3 = 35.315 ft^3 = 6.102 × 10^4 in^3 = 264.17 gal
1 m^3/s = 1.2713 × 105 ft^3/hr = 1.5850 × 104 gal/min

(*iii*) *Mass, mass flow rate and density*

1 kg = 2.205 lb = 0.01 quintal
= 0.001 ton = 6.852 × 10^{-2} slugs
1 kg/cc = 62.43 lb/ft^3
1 kg/m^3 = 0.6243 lb/ft^3
1 kg/s = 7936.6 lb/hr

(*iv*) *Acceleration and frequency*

1 m/s^2 = 3.28 ft/s^2
1 cycle/s = 1 Hz

(*v*) *Force*

1 N = 10^5 dyne = 0.012 kg_f = 0.2248 lb_f
= 7.256 poundal (pdl)

(*vi*) *Pressure*

1 N/m^2 = 1 Pascal = 10^{-5} bar
= 0.1019 kg_f/m^2 = 0.1019 × 10^{-4} kg_f/cm^2
= 1.4505 × 10^{-4} lb_f/in^2 = 0.02088 lb_f/ft^2
= 4.015 × 10^{-3} in of water

= 10.198×10^{-3} cm of water
= 2.953×10^{-4} in of Hg
= 7.50×10^{-4} cm of Hg
= 1 standard atmosphere

1.0133×10^5 N/m² = 1.01325 bar = 101.325 N/m²
1 atm = 760 mm of Hg
1 atm = 750 mm of Hg
1 bar = 736 mm of Hg
1 ata (kg_f/cm^2)

(vii) Dynamic viscosity

1 gm/cm s (poise) = 0.1 kg/ms = 360 kg/m hr
= 1 dyne s/cm³
= 100 cp (centi poises)
= 2419.1 lb/ft hr

1 Ns/m² = 5.8016×10^{-6} kg_f/hr
= 1 kg/ms = 0.1019 kg_f/m^2

(viii) Diffusivity and kinematic viscosity

1 m²/s = 104 cm²/s = 3600 m²/hr
= 3.875×10^4 ft²/hr

1 cm²/s (stoke) = 104 m²/s = 0.36 m².hr = 3.875 ft²/hr

(ix) Work and energy

1 J = 1 Nm = 1 kg m²/s² = 1 Ws
= 10^7 erg = 0.7376 ft lbr
= 0.2390×10^{-3} kcal
= 2.779×10^{-7} kW hr
= 3.725×10^{-7} HP hr

(x) Power

1 W = 1 J/s = 0.86 kcal/hr
1 HP = 75 m kgf/s
= 0.1757 kcal/s
= 735.3 W

(xi) Temperature

K = (5/9)° R
= 5/9 (°F + 460)
= °C + 273

Temperature difference 1 K = 1° C

(xii) Heat transfer rate, heat flux and heat generation rate

1 W = 0.86 kcal/hr = 0.239 cal/s
= 3.414 BTU/hr

1 W/m² = 0.86 kcal/m² hr
= 0.239×10^{-4} cal/s cm²
= 0.317 BTU/hr ft²

1 W/m³ = 0.09665 BTU/hr ft³

1 ton of refrigeration = 50 kcal/min
= 3.517 kW
= 200 BTU/min

(xiii) Latent heat, specific heat and calorific value

1 kJ/kg = 0.2389 kcal/kg
= 0.4299 BTU/lb

1 KJ/kg K = 0.2389 BTU/lb °F
= 0.2389 kcal/kg K

(xiv) Thermal conductivity and heat transfer coefficient

1 W/m K = 2.390×10^{-3} cal/m s K
= 0.8598 kcal/m hr K
= 0.5778 BTU/ft hr °F

1 W/m² = 0.8598 kcal/m² hr K
= 2.390×10^{-5} cal/cm² sK
= 0.176 BTU/ft² hr °F

1 K/W = 0.5275 °F/hr BTU
= 1.163 °K/kcal

Appendix-B

Table B.1. Physical Properties of Water (Liquid)

t °C	ρ kg/m³	c_p kJ/kgK	$k \times 10^2$ W/mK	$\alpha \times 10^4$ m²/hr	$\mu \times 10^2$ kg/hr-m	$\nu \times 10^6$ m²/s	Pr —
0	999.9	4.212	55.093	4.71	644.093	1.789	13.67
10	999.7	4.191	57.418	4.94	469.818	1.306	9.54
20	998.2	4.183	59.859	5.16	361.892	1.006	7.02
30	995.7	4.174	61.718	5.35	288.668	0.805	5.42
40	992.2	4.174	63.345	5.51	235.602	0.659	4.31
50	988.1	4.178	64.740	5.65	197.771	0.556	3.54
60	983.2	4.178	65.902	5.78	169.305	0.478	2.98
70	977.8	4.187	66.716	5.87	146.370	0.415	2.55
80	971.8	4.195	67.413	5.96	127.924	0.365	2.21
90	965.3	4.208	67.995	6.03	113.507	0.326	1.95
100	958.4	4.220	68.227	6.09	101.910	0.295	1.75
110	951.0	4.233	68.460	6.13	92.215	0.272	1.60
120	943.2	4.250	68.576	6.16	85.448	0.252	1.47
130	934.8	4.266	68.576	6.29	78.744	0.233	1.36
140	926.1	4.287	68.460	6.21	72.475	0.217	1.26
150	917.0	4.312	68.343	6.22	66.792	0.203	1.17
160	907.0	4.346	68.227	6.23	62.206	0.191	1.10
170	897.3	4.379	67.878	6.22	58.623	0.181	1.05
180	886.9	4.417	63.413	6.20	54.976	0.173	1.00
190	876.0	4.459	66.949	6.17	51.921	0.165	0.96
200	863.0	4.505	66.251	6.14	49.266	0.158	0.93
220	840.3	4.614	64.508	5.99	44.823	0.148	0.89
240	813.6	4.756	62.764	5.84	41.356	0.141	0.87
260	784.3	4.949	60.440	5.61	38.274	0.135	0.87
280	750.7	5.229	57.418	5.27	35.596	0.131	0.90
300	712.5	5.736	53.931	4.75	32.835	0.129	0.97

Table B.2. Physical Properties of Air

t °C	ρ kg/m³	c_p kJ/kgK	$k \times 10^2$ W/mK	$\alpha \times 10^2$ m²/hr	$\mu \times 10^2$ kg/hr-m	$\nu \times 10^6$ m²/s	Pr —
−50	1.584	1.013	2.036	4.57	5.264	9.22	0.728
−40	1.515	1.013	2.115	4.96	5.475	10.04	0.728
−30	1.453	1.013	2.197	5.37	5.645	10.80	0.723
−20	1.395	1.009	2.278	5.38	5.822	12.09	0.716
−10	1.342	1.009	2.360	6.28	5.996	12.43	0.712
0	1.293	1.005	2.441	6.77	6.188	13.28	0.707
10	1.247	1.005	2.511	7.22	6.346	14.16	0.705
20	1.205	1.005	2.592	7.71	6.533	15.06	0.703
30	1.165	1.005	2.673	8.23	6.717	16.00	0.701
40	1.128	1.005	2.755	8.75	6.904	16.96	0.699
50	1.093	1.005	2.824	9.29	7.067	17.95	0.698
60	1.060	1.005	2.894	9.79	7.221	18.97	0.696
70	1.029	1.009	3.045	10.28	7.523	21.09	0.692
80	1.000	1.009	3.045	10.87	7.523	21.09	0.692
90	0.972	1.009	3.127	11.48	7.701	22.10	0.690
100	0.946	1.009	3.208	12.11	7.880	23.13	0.688
120	0.898	1.009	3.336	13.26	8.170	25.45	0.686
140	0.854	1.013	3.487	14.52	8.479	27.80	0.684
160	0.815	1.017	3.638	15.80	8.786	30.08	0.682
180	0.779	1.022	3.778	17.10	9.070	32.49	0.681
200	0.746	1.026	3.929	18.49	9.380	34.85	0.680
250	0.674	1.038	4.266	21.49	10.020	40.61	0.677

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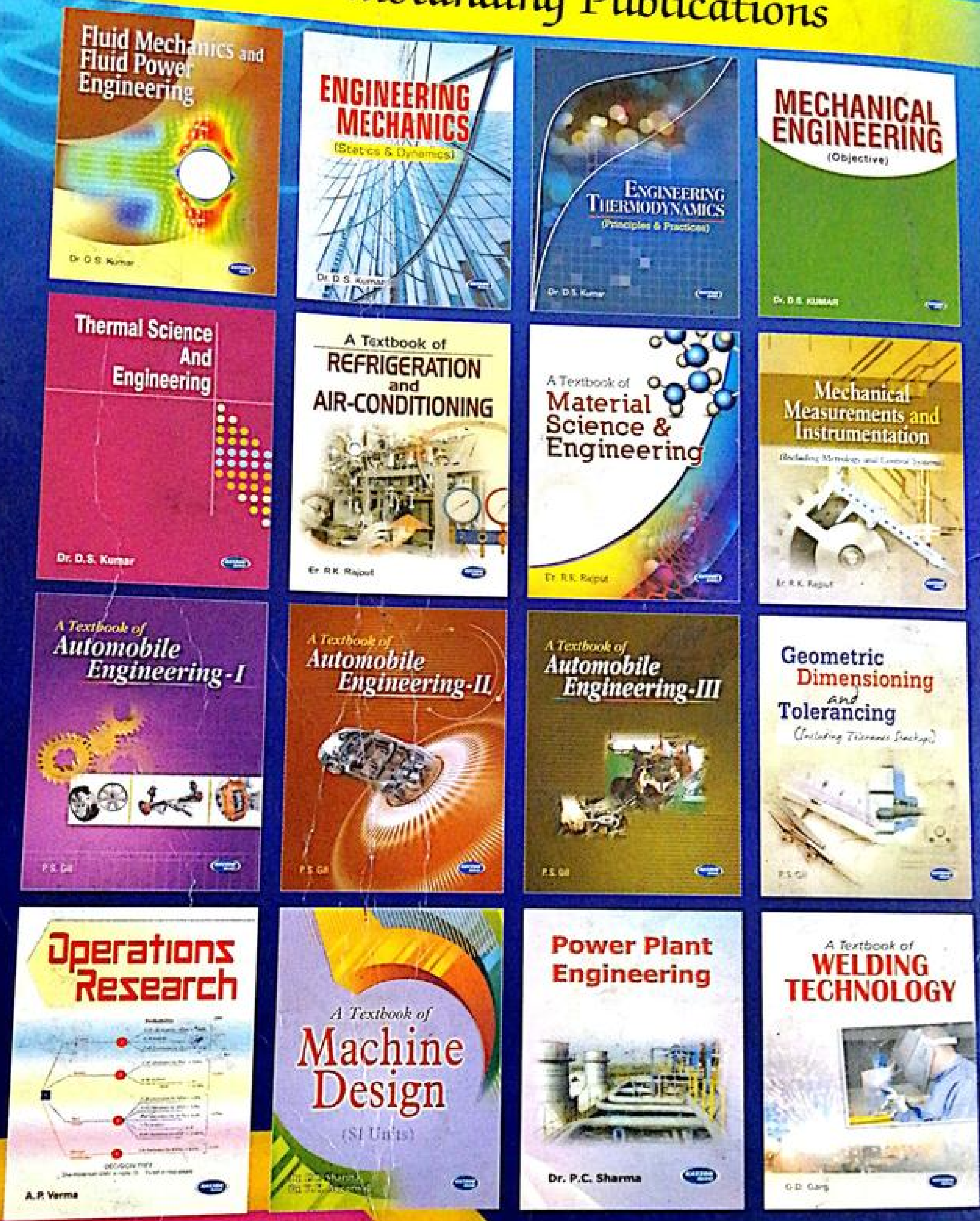

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